Hydroelastic analysis of floating structures with liquid tanks and comparison with experimental tests

Kang-Heon Lee, Seongpil Cho, Ki-Tae Kim, Jin-Gyun Kim, Phill-Seung Lee

1. Introduction

Since the early 2000s, the hydrodynamic analysis of floating liquid storage structures subjected to surface regular waves has been widely studied due to the significant increase in demand for floating production storage and offloading (FPSO) units, floating liquified natural gas (FLNG) units, and other related structures. One of the important design issues is the influence of sloshing in liquid tanks on the dynamic response of floating structures during offloading operations, see Refs. [2,3] for comprehensive reviews of sloshing phenomena and their importance. Related mathematical, numerical, and experimental studies are presented in Refs. [4–19].

Most previous studies mainly addressed the coupling effect between rigid body motions and sloshing. However, the size and the weight of floating liquid storage structures are becoming increasingly greater in tandem with growing market demand and on the basis of their economic benefits. In such floating structural systems, the assumption of rigid body motions is no longer suitable because, as the dimensions of floating structures increase, the overall stiffness decreases, resulting in relatively low resonant frequencies close to the range of excitation frequencies and sloshing resonance frequencies. In spite of the increasing importance of the hydroelastic behavior of floating liquid storage structures, few related studies have been reported [20–22]. Accordingly, a complete mathematical formulation has not been developed and the numerical results have not been verified by experimental studies.

A direct coupling method was first developed for 1D and 2D linear hydroelastic problems [23–26]. The main idea is that the structural and fluid equations are directly coupled to each other and the coupled equations are solved simultaneously.

The solution procedure is consequently simpler than that of the conventional method [4,7,13,14,17], which requires radiation and diffraction analysis procedures to obtain the interaction coefficients. Recently, this method was generalized for a 3D linear hydroelastic analysis of floating structures by Kim et al. [1]. Since the 3D formulation is obtained by consistently linearizing nonlinear solid mechanics equations, all the interaction terms including hydrostatic stiffness are included [1,27,28].

In this study, we extend the direct-coupled formulation developed in Ref. [1,29,30] for a hydroelastic analysis of floating structures with liquid tanks. The structural formulation is based on the updated Lagrangian approach, which is consistently applied to hydrostatic and steady state hydrodynamic analyses. The velocity potential is employed to model both internal and external fluid flows without decomposing them into the diffraction and radiation
potentials. The finite element method is employed for the structure and the internal fluid, and the boundary element method is used for the external fluid. The structural equation is then directly coupled with the fluid equations. The use of the mode superposition method for the discrete structural and internal fluid equations is also introduced to improve the computational efficiency. Of course, all the interaction terms among structural motions, sloshing and water waves are completely included in the formulation. In particular, the initial stress is correctly considered in the geometric stiffness [1,27]. In the fluid formulation, we use the extended boundary integral equations to remove the well-known irregular frequency effect [31–33].

To verify the proposed formulation, various numerical tests including free-vibration, rigid body hydrodynamic and hydroelastic analyses are conducted for a box barge with three rectangular liquid tanks. We then present the 3D hydroelastic experiments performed to verify the proposed formulation. A floating production unit (FPU) model with three rectangular sloshing tanks was designed and fabricated for the experimental tests in an ocean basin. An overall description of the experimental setup and the test model are provided in detail. The measured dynamic responses are compared with the numerical results obtained using the proposed formulation.

We present the mathematical formulation in Section 2 and the numerical procedure in Section 3, and several numerical test results are provided in Section 4. In Section 5, the overall description of the experimental setup is presented and the test results are compared with the numerical results.

2. Mathematical formulation

Fig. 1 shows the problem description considered in this study. It is assumed that the floating structure has a homogeneous, isotropic, and linear elastic material and the fluid flow is incompressible, inviscid, and irrotational and thus the potential flow theory can be used. An incident regular water wave comes continuously from the positive x1 direction with an angle θ and the amplitude is assumed to be small enough to use the linear wave theory. Also, the resulting motions of the floating structure and sloshing in tanks are assumed to be small. All the waves are gravity waves with a zero atmospheric pressure assumption and the surface tension effect is ignored.

The volumes occupied by the floating structure, the internal fluid in tanks, and the external fluid are denoted by $V_S$, $V_{fI}$, and $V_{fE}$, respectively. The surface of the floating structure $S_C$ consists of dry, internal wet, and external wet surfaces, which are denoted by $S_D$, $S_{WI}$, and $S_{WE}$, respectively. The internal fluid is bounded by

---

**Fig. 1.** Problem description: a floating structure with a liquid tank in an incident water wave.

**Fig. 2.** Three equilibrium states.
the internal wet surface $S_{WI}$ and the internal free surface $S_{R}$ and the external fluid is enveloped by the external wet surface $S_{WE}$, the external free surface $S_{P}$, the surface $S_{\infty}$ which is a circular cylinder with a sufficiently large radius $R$, and a flat bottom surface $S_{C}$. The external water depth $h_{E}$ is measured from the flat bottom to the external free surface of calm water. The internal water depth $(h_{I} = h_{I}(x_{1}, x_{2}))$ is the distance from the wet surface ($S_{WI}$) to the free surface ($S_{R}$) at rest in tanks.

The Cartesian coordinate system $(x_{1}, x_{2}, x_{3})$ on the external free surface of calm water is introduced. For clear and compact notation, the subscripts $i$ and $j$, which vary from 1 to 3, are used to express the components of tensors and the Einstein summation convention is adopted.

Fig. 2 shows three important states: initial state, hydrostatic equilibrium state, and hydrodynamic equilibrium state. The initial state is a virtual configuration in which the structure does not contact the external and internal fluids. These three states are denoted by the left superscripts $0$, $\tilde{0}$, and $\tilde{\tau}$, respectively. The material point vectors for the floating structure in each state are then expressed by $\tilde{x}_{i}$, $\tilde{0}x_{i}$, and $x_{i}$, respectively. The displacement vectors of the floating structure are defined by

$$
\ddot{0}u_{i} = \dot{x}_{i} - \dot{0}x_{i}, \quad \dot{0}u_{i} = \dot{x}_{i} - 0x_{i}.
$$

(1)

The total pressure fields of the external and internal fluids are defined as

$$
0P_{E} = -\rho_{E}g\tilde{x}_{3}, \quad \tilde{0}P_{E} = -\rho_{E}g\tilde{x}_{3}, \quad \tilde{0}P_{I} = -\rho_{I}g\tilde{x}_{3}, \quad P_{DE} = \frac{1}{2}P_{DE} \tilde{x}_{3} + P_{DE} \tilde{x}_{3}, \quad \tilde{x}_{3} = x_{3} - z_{T},
$$

(2)

where $\rho_{E}$ is the density of the external fluid, $\rho_{I}$ is the density of the internal fluid, $g$ is the gravitational acceleration, $z_{T}$ is the vertical position of the internal free surface, and $P_{DE}$ is the hydrodynamic pressures for the external and internal fluids.

In the following sections, the mathematical formulations of the floating structure, the external fluid, and the internal fluid are briefly derived. The detailed derivation of the formulations for the floating structure and the external fluid can be found in Ref. [1].

### 2.1. Equations for the floating structure

Note that a hydrostatic analysis is an essential procedure to find the hydrostatic equilibrium state referred to the configuration of the initial state. Through a hydrodynamic analysis, we find the hydrodynamic equilibrium state referred to the configuration of the hydrostatic equilibrium state.

The updated Lagrangian formulation [34] is consistently applied to the hydrostatic and hydrodynamic analyses. The equilibrium equations at time $\tau + \Delta \tau$ are

$$
\frac{\tilde{0}u_{i}}{0} = \tilde{0}x_{i} - 0x_{i}, \quad \dot{0}u_{i} = \dot{x}_{i} - 0x_{i}.
$$

After linearizing the principle of virtual work at time $\tau + \Delta \tau$ referred to the configuration at time $\tau$, the following weak form can be obtained

$$
\int_{V_{S}} \left( \dot{t} \rho_{S} \tilde{u}_{i} \dot{0}u_{i} + \int_{V_{S}} C_{ijkl} \tilde{u}_{j} \dot{0}u_{j} \right) dV + \int_{V_{S}} \sigma_{ij} \tilde{u}_{ij} dV + \int_{S_{WE}} \tau_{ij} \tilde{u}_{ij} dS = 0
$$

(4)

where

$$
\tau_{ij} = \frac{\partial \tilde{u}_{i}}{\partial x_{j}} + \frac{\partial \tilde{u}_{j}}{\partial x_{i}} + \frac{\partial \tilde{u}_{i}}{\partial x_{j}}
$$

(5)

Note that an iterative solution scheme like the Newton–Raphson method is required to find the hydrostatic equilibrium state using Eq. (6). Once the hydrostatic equilibrium state is obtained, the right hand side of Eq. (4) vanishes. We then set $\tau = 0$ and $\tau + \Delta \tau = \tau$ in Eq. (4) and invoke a harmonic response with angular frequency $\omega$ to $\dot{0}u_{i} = Re \{i(\omega \tilde{x}_{k})e^{\omega t} \}$. Finally, the following equation is obtained for a hydrodynamic analysis in the steady state:
\[-\omega^2 \int_{V_S} \rho \partial_t \bar{u}_i \partial_t dV + \int_{S_{SW}} C_{ijkl} \partial_{ij} \bar{u}_k \partial_t dS + \int_{S_W} \sigma_{ij} \bar{\eta}_{ij} \partial_t dS\]
\[-\int_{S_{SW}} \rho \varepsilon_{ij} \partial_t \bar{u}_i \partial_t dS - \int_{S_{SW}} \rho \varepsilon_{ij} \partial_t \bar{u}_i dS \]
\[-\int_{S_{SW}} \rho \varepsilon_{ij} \partial_t \bar{u}_i dS - \int_{S_{SW}} \rho \varepsilon_{ij} \partial_t \bar{u}_i dS \]
\[+ \int_{S_{SW}} P_{DE} \partial_t \bar{u}_i dS + \int_{S_{SW}} P_{DE} \partial_t \bar{u}_i dS = 0, \quad (7)\]

where
\[\varepsilon_{ij} = \text{Re} \{ \varepsilon_{ij}(0 \bar{x}_k) e^{i \omega t} \}, \quad \bar{\eta}_{ij} = \text{Re} \{ \eta_{ij}(0 \bar{x}_k) e^{i \omega t} \}, \]
\[\varepsilon_{ij} = \text{Re} \{ Q_{ij}(0 \bar{x}_k) e^{i \omega t} \}, \quad \text{P}_{DE} = \text{Re} \{ P_{DE}(0 \bar{x}_k) e^{i \omega t} \}, \]
\[\varepsilon_{ij} = \text{Re} \{ P_{DE}(0 \bar{x}_k) e^{i \omega t} \}. \quad (8)\]

Since we assume that the motion of the floating structure is small and the change of the wet surface is negligible, Eq. (7) can be solved without an iterative solution scheme.

2.2. Equations for the external fluid

In the steady state, the governing equation and boundary conditions for the external fluid, which are approximated at the configuration of the hydrostatic equilibrium state, are given as
\[\varepsilon_{ij} = \text{Re} \{ \varepsilon_{ij}(0 \bar{x}_k) e^{i \omega t} \}, \quad (9a)\]
\[\frac{\partial^2 \varepsilon_{ij}}{\partial x_i \partial x_j} = 0 \quad \text{in} \quad V_{FE}, \quad (9b)\]
\[\frac{\partial \varepsilon_{ij}}{\partial x_3} = \frac{\omega^2}{\bar{g}} \varepsilon_{ij} \quad \text{on} \quad S_E \quad (x_3 = 0), \quad (9c)\]
\[\frac{\partial \varepsilon_{ij}}{\partial x_3} = 0 \quad \text{on} \quad S_G \quad (x_3 = -h_E), \quad (9d)\]
\[\sqrt{\frac{k}{\bar{g}}} \left( \frac{\partial}{\partial x_3} + jk \right) (\phi - \phi^i) = 0 \quad \text{on} \quad S_{\infty} \quad (R \to \infty), \quad (9e)\]
\[\frac{\partial \varepsilon_{ij}}{\partial n_e} = j \omega u_t n_i \quad \text{on} \quad S_{WE}, \quad (9f)\]

where \(\varepsilon_{ij}\) is the velocity potential for the external fluid, \(k\) is the wave number, \(\phi\) is the velocity potential for an incident wave, Eq. (9c) is the combined free surface boundary condition linearized at \(x_3 = 0\) [35,36], and Eq. (9e) is the Sommerfeld radiation condition [35]. The body boundary condition in Eq. (9f) means that the normal velocities of the structure and the external fluid should be the same on the external wet surface.

The corresponding boundary integral equation is
\[2 \pi \int_{S_{WE}} \varepsilon_{ij} \bar{\phi}_E dS \]
\[= -P.V. \int_{S_{WE}} \frac{\partial G(x_3; \xi_3)}{\partial n_e} \frac{\partial \varepsilon_{ij}(\xi_3)}{\partial n_e} dS_E + 4 \pi \phi^i(x_i) \quad \text{for} \quad x_i \quad \text{on} \quad S_{WE}, \quad (10)\]

where \(P.V.\) refers to the Cauchy principal value, the subscript \(\xi\) denotes that the integral is conducted with respect to the variable \(\xi_3\), and \(G(x_3; \xi_3)\) is the Green's function, which is located at position \(\xi_3\) and generated by a source potential with strength \(-4\pi\) and angular frequency \(\omega\). The detailed procedure to obtain the Green's function in finite and infinite depth cases is described by Wehausen and Laitone [36].

It should be noted that Eq. (10) could result in the irregular frequencies and thus the coefficient matrix in the corresponding discrete equation becomes ill-conditioned near those frequencies. In order to remove the irregular frequency effects, several methods have been developed by Ohmatsu [31], Kleinman [32], and Lee et al. [33]. We here employ the extended boundary integral equations [33].

\[2 \pi \phi^i(x_i) - P.V. \int_{S_{WE} + S_{EXT}} \frac{\partial G(x_3; \xi_3)}{\partial n_e} \phi^i(\xi_3) dS_E \]
\[= -P.V. \int_{S_{WE}} G(x_3; \xi_3) \frac{\partial \phi^i(\xi_3)}{\partial n_e} dS_E + 4 \pi \phi^i(x_i) \quad \text{for} \quad x_i \quad \text{on} \quad S_{WE}, \quad (11a)\]
\[-4 \pi \phi^i \phi^i(x_i) - P.V. \int_{S_{WE} + S_{EXT}} \frac{\partial G(x_3; \xi_3)}{\partial n_e} \phi^i(\xi_3) dS_E \]
\[= -P.V. \int_{S_{WE}} G(x_3; \xi_3) \frac{\partial \phi^i(\xi_3)}{\partial n_e} dS_E + 4 \pi \phi^i(x_i) \quad \text{for} \quad x_i \quad \text{on} \quad S_{WE}, \quad (11b)\]

where \(S_{EXT}\) is the extended external free surface described in Fig. 3 and \(\phi^i\) is the velocity potential on the extended boundary \(S_{EXT}\).

Multiplying test functions \(\varepsilon\) and \(\phi^i\) to Eqs. (11a) and (11b), respectively, and integrating over the external wet surface \(S_{WE}\) and the internal free surface \(S_{EXT}\), the following equations are obtained:
\[2 \pi \int_{S_{WE}} \varepsilon_{ij} \phi^i dS \]
\[-\int_{S_{WE}} P.V. \int_{S_{WE} + S_{EXT}} \left( \frac{\partial G(x_3; \xi_3)}{\partial n_e} \phi^i - G(x_3; \xi_3) \frac{\partial \phi^i(\xi_3)}{\partial n_e} \right) dS_E \phi^i dS_x \]
\[= 4 \pi \int_{S_{WE}} \varepsilon_{ij} \phi^i dS \quad \text{for} \quad x_i \quad \text{on} \quad S_{WE}, \quad (12a)\]
\[-4 \pi \int_{S_{EXT}} \varepsilon_{ij} \phi^i dS \]
\[-\int_{S_{EXT}} P.V. \int_{S_{EXT} + S_{SY}} \left( \frac{\partial G(x_3; \xi_3)}{\partial n_e} \phi^i - G(x_3; \xi_3) \frac{\partial \phi^i(\xi_3)}{\partial n_e} \right) dS_E \phi^i dS_x \]
\[= 4 \pi \int_{S_{EXT}} \varepsilon_{ij} \phi^i dS \quad \text{for} \quad x_i \quad \text{on} \quad S_{EXT}, \quad (12b)\]

2.3. Equations for the internal fluid

In the steady state, the governing equation and boundary conditions for the internal fluid in tanks, which are approximated at the configuration of the hydrostatic equilibrium state, are given as
\[\varepsilon_{ij} = \text{Re} \{ \varepsilon_{ij}(0 \bar{x}_k) e^{i \omega t} \}, \quad (13a)\]
\[\rho \varepsilon_{ij} = \frac{\partial^2 \phi^i}{\partial x_i \partial x_j} = 0 \quad \text{in} \quad V_{FI}, \quad (13b)\]
\[\frac{\partial \phi^i}{\partial x_3} = \frac{\omega^2}{\bar{g}} \phi^i \quad \text{on} \quad S_{FI}(x_3 = z_T), \quad (13c)\]
\[ \frac{\partial \phi_t}{\partial t} = j \omega u_i n_i \quad \text{on } 0 S_{WI}, \]  

(13d)

where \( \phi_t \) is the velocity potential in time domain and \( \phi_i \) are the velocity potentials in the steady state.

By multiplying a test function \( \phi_t \) to Eq. (13b) and integrating over the volume of the internal fluid \( \int V_{FI} \), the following equation can be obtained:

\[ \int_{V_{FI}} \rho_I \frac{\partial \phi_t}{\partial t} \phi_t \, dV = 0. \]  

(14)

After Eq. (14) is integrated by part and the divergence theorem and the boundary condition in Eq. (13c) are applied to Eq. (14), the weak form equation of the internal fluid is obtained

\[ \int_{S_{WE}} \rho_I \left( \frac{\omega^2}{g} \right) \phi_t \phi_t \, dS - \int_{S_{WE}} \rho_I \frac{\partial \phi_t}{\partial n} \phi_t \, dS - \int_{V_{FI}} \rho_I \frac{\partial \phi_t}{\partial x_i} \frac{\partial \phi_t}{\partial x_i} \, dV = 0. \]  

(15)

2.4. Direct-coupled equations

To obtain the direct-coupled equations, Eq. (7) for the floating structure, Eq. (12) for the external fluid, and Eq. (15) for the internal fluid are considered with the interaction conditions in Eqs. (9) and (13d). Using the linearized Bernoulli equations, the hydrodynamic pressures \( P_{DE} \) and \( P_{DI} \) can be expressed as

\[ P_{DE} = -j \omega \rho E \phi_E, \quad P_{DI} = -j \omega \rho E \phi_I. \]  

(16)

Substituting Eq. (16) into Eq. (7), Eq. (9f) into Eq. (12), and Eq. (13d) into Eq. (15), the following coupled equations are obtained:

\[ -j \omega^2 \int_{V_{FI}} 0 \rho_E u_i \bar{u}_i \, dV + \int_{V_{FI}} C_{ijkl} \epsilon_{kl} \epsilon_{ij} \, dV + \int_{V_{FI}} 0 \sigma_{ij} \bar{\epsilon}_{ij} \, dV \]

\[ - \int_{S_{WE}} \rho_E u_i \bar{u}_i \, dS - \int_{S_{WE}} \rho_E u_i \bar{u}_i \, dS \]

\[ - \int_{S_{WE}} \rho E x_{ij} 0 \sigma_{ij} \bar{u}_i \, dS - \int_{S_{WE}} \rho E x_{ij} 0 \sigma_{ij} \bar{u}_i \, dS \]

\[ -j \omega \int_{S_{WE}} \rho E \phi_E 0 \bar{u}_i \, dS - j \omega \int_{S_{WE}} \rho E \phi_I 0 \bar{u}_i \, dS = 0, \]  

(17)

\[ 2\pi \int_{S_{WE}} \phi_E \phi_E \, dS \]

\[ - j \omega \int_{S_{WE} + S_{EXT}} \left( \frac{\partial G}{\partial n} \phi_E - j \omega G u_i n_i \right) \, dS \phi_E \, dS \]

\[ = 4\pi \int_{S_{WE}} \phi_E \phi_E \, dS \quad \text{for } x_i \quad \text{on } 0 S_{WE}, \]  

(18a)

\[ -4\pi \int_{S_{EXT}} \phi_E \phi_E \, dS \]

\[ - j \omega \int_{S_{EXT} + S_{EXT}} \left( \frac{\partial G}{\partial n} \phi_E - j \omega G u_i n_i \right) \, dS \phi_E \, dS \]

\[ = 4\pi \int_{S_{EXT}} \phi_E \phi_E \, dS \quad \text{for } x_i \quad \text{on } 0 S_{EXT}, \]  

(18b)
3. Numerical methods

In this section, the direct-coupled equations are discretized by using the finite and boundary element methods. In addition, for efficient computation, reduced linear equations are introduced by using the mode superposition method. We also extract added mass matrices, the radiated wave damping matrix, and the wave exciting force vector for the 3D hydroelastic analysis of floating structures with liquid tanks.

\[
\int_{\Omega_{S\Omega}} \rho_l \left( \frac{\omega^2}{g} \right) \phi_\Omega \tilde{\phi}_\Omega dS - j\omega \int_{\Gamma_{S\Omega}} \rho_l u_i \tilde{\phi}_\Omega dS - \int_{\Omega_{V\Omega}} \rho_l \frac{\partial \phi_\Omega}{\partial x_i} \frac{\partial \tilde{\phi}_\Omega}{\partial x_i} dV = 0. \tag{19}
\]

3.1. Finite and boundary element discretization

The finite element method is employed for the floating structure and internal fluid in tanks, and the boundary element method is used for the external fluid. The finite and boundary element meshes and the mesh matching scheme are depicted in Fig. 4.

The fields of structural displacements and velocity potentials are interpolated using the nodal displacement vector \(\mathbf{u}\) and the nodal velocity potential vectors \(\phi_i\) and \(\phi_E\) for internal and external

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{solution_diagram.png}
\caption{Solution procedures for the steady state hydrodynamic analysis in the present and conventional formulations.}
\end{figure}
similarly, the boundary element discretization of eq. \(18\) yields
\[
2\pi \int_{S_{WE}} \phi^T_{E} \dot{\phi}_{E} dS - 4\pi \int_{S_{EXT}} \dot{\phi}_{E}^{EXT} \dot{\phi}_{E}^{EXT} dS = \hat{\phi}_{E}^{T} \hat{P}_{E} \hat{\phi}_{E}. \tag{21a}
\]
\[
\int_{S_{WE}} P.V. \int_{S_{WE} + S_{EXT}} \frac{\partial G}{\partial \tilde{x}_{E}} \tilde{\phi}_{E} \tilde{\phi}_{E} dS_{x} dS_{x}
+ \int_{S_{EXT}} P.V. \int_{S_{WE} + S_{EXT}} \frac{\partial G}{\partial \tilde{x}_{E}} \tilde{\phi}_{E}^{EXT} \tilde{\phi}_{E}^{EXT} dS_{x}
= \hat{\phi}_{E}^{T} \hat{F}_{G} \hat{\phi}_{E}. \tag{21b}
\]
\[
j_\omega \int_{S_{WE}} P.V. \int_{S_{WE} + S_{EXT}} G_{ij} \tilde{n}_{i} \tilde{\phi}_{E} \tilde{\phi}_{E} dS_{x}
+ j_\omega \int_{S_{EXT}} P.V. \int_{S_{WE} + S_{EXT}} G_{ij} \tilde{n}_{i} \tilde{\phi}_{E}^{EXT} \tilde{\phi}_{E}^{EXT} dS_{x}
= \hat{\phi}_{E}^{T} j_\omega \hat{F}_{U} \hat{\phi}_{E}. \tag{21c}
\]
\[
4\pi \left( \int_{S_{WE}} \phi^T_{E} \dot{\phi}_{E} dS + \int_{S_{EXT}} \phi^{EXT} \dot{\phi}_{E}^{EXT} dS \right) = \hat{\phi}_{E}^{T} 4\pi \hat{R}^{T}. \tag{21d}
\]
finally, the finite element discretization of eq. \(19\) yields
\[
\int_{S_{SI}} \rho_{E} \left( \frac{\alpha^2}{\omega} \right) \phi_{I} \phi_{I} dS = \hat{\phi}_{I}^{T} \omega^{2} \hat{P}_{M} \hat{\phi}_{I}, \tag{22a}
\]
\[
j_\omega \int_{S_{SI}} \rho_{I} \tilde{n}_{i} \phi_{I} dS = \hat{\phi}_{I}^{T} j_\omega \hat{F}_{U}^{I} \hat{\phi}_{I}, \tag{22b}
\]
\[
\int_{S_{NI}} \rho_{I} \frac{\partial \phi_{I}}{\partial x_{i}} \frac{\partial \phi_{I}}{\partial \tilde{x}_{i}} dS = \hat{\phi}_{I}^{T} \hat{F}^{I} \hat{\phi}_{I}. \tag{22c}
\]
the final discrete coupled equation for the steady state 3d hydroelastic analysis of floating structures with liquid tanks is given by
\[
\begin{bmatrix}
-\omega^{2} S_{M} + S_{CH} & j_\omega S_{D}^{E} & -j_\omega S_{D}^{I} \\
-j_\omega S_{D}^{E} & F_{M}^{E} - F_{G}^{I} & 0 \\
-j_\omega S_{D}^{I} & 0 & -\omega^{2} F_{M}^{I} - F_{K}^{I}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_E \\
\hat{\phi}_I \\
\hat{\phi}_I
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
4\pi \hat{R}^{T}
\end{bmatrix}
\]
with
\[
S_{CH} = S_{KN} - S_{ID}^{E} \quad S_{CH} = S_{ID}^{I} - S_{IN}^{E} - S_{IN}^{I}, \tag{23}
\]
where the matrix \(S_{CH}\) is the complete hydrostatic stiffness of the floating liquid storage structure. The terms \(S_{ID}, S_{IN}, S_{ID},\) and \(S_{IN}\)
are the hydrostatic pressure stiffnesses and \(S_{KN}\) is the geometric stiffness. in particular, the contributions of the internal fluid to the hydrostatic pressure stiffness are \(S_{ID}\) and \(S_{IN}\), and a hydrostatic analysis should be performed in advance to properly obtain the geometric stiffness \(S_{KN}\). note that no artificial damping is considered in eq. \(23\).
Since the extended boundary integral method is applied in the external fluid part of Eq. (23), the resonance phenomena induced by sloshing and structural elasticity can be distinguished from the resonances caused by the irregular frequency effects.

3.2. Reduced equation

We now apply the standard mode superposition method in Eq. (23). First, the following two eigenvalue problems should be solved

\[ S_k \Psi_i = \lambda_i S_M \Psi_i; \quad i = 1, 2, \ldots, N_a \quad \text{for the floating structure}, \]
\[ F_k^e \gamma_i = \mu_i F_M \gamma_i; \quad i = 1, 2, \ldots, M_a \quad \text{for the internal fluid}, \]

where \( N_a \) and \( M_a \) are the numbers of degrees of freedom in the floating structure and internal fluid, respectively, \( \Psi_i \) and \( \gamma_i \) are the eigenvectors which are orthonormalized with respect to the matrices \( S_M \) and \( F_M \), respectively, and \( \lambda_i \) and \( \mu_i \) are the corresponding eigenvalues.

The nodal displacement vector of the floating structures and the nodal potential vector of the internal fluid are approximated as

\[ \bar{u} \approx q_1 \Psi_1 + q_2 \Psi_2 + \cdots + q_{N_a} \Psi_{N_a} = \Psi \bar{q}, \quad \hat{N}_a < N_a, \]
\[ \Phi \approx \gamma_1 \gamma_1 + \gamma_2 \gamma_2 + \cdots + \gamma_{M_a} \gamma_{M_a} = \hat{\gamma} \hat{y}, \quad \hat{M}_a < M_a, \]

in which \( \bar{q} \) and \( y \) are the generalized coordinate vectors.

Substituting Eqs. (25a) and (25b) into Eq. (23) and premultiplying by \( \Psi^e \) and \( \gamma^e \) to the structural and internal fluid parts of Eq. (23), respectively, the following reduced equation is obtained:

\[
\begin{bmatrix}
-j \omega^2 I + \Lambda + S_{CH}^e & -j \omega \Psi D_{D} & -j \omega \Psi D_{E} \\
0 & F_M^e - F_G & 0 \\
0 & 0 & -j \omega \Psi D_{E}
\end{bmatrix}
\begin{bmatrix}
\bar{q} \\
\Phi_D \\
\Phi_E
\end{bmatrix} =
\begin{bmatrix}
0 \\
4 \pi R' \\
0
\end{bmatrix},
\]

where \( I_j = \delta_{ij}, \quad \Lambda = \lambda_j \delta_{ij}, \) and \( \Omega_k = \mu_k \delta_{kl} \) (no summation); \( i, j = 1, 2, \ldots, N_a \) and \( k, l = 1, 2, \ldots, M_a \).

Note that, in Eq. (26), \( S_{CH}^e \) is the complete hydrostatic stiffness in the generalized coordinate \( \Psi^e = \Psi S_{CH}^e \Psi \), and the rigid body hydrodynamic analysis can be conducted when only the rigid body modes of the floating structure are contained in Eq. (25a).

Condensing out the fluid variables in Eq. (26), we can extract added masses, the radiated wave damping matrix, and the wave exciting force vector in the generalized coordinates. Therefore, the present direct-coupled formulation can be linked term-by-term to the conventional formulation [1,24]. The condensed structural equation becomes

\[
(-\omega^2 (I + S_{MA}^G - S_{MA}^E) + j \omega S_{GW}^G + \Lambda + S_{CH}^G) \bar{u} = R_{GW}^G,
\]

where \( S_{MA}^G, S_{MA}^E, S_{GW}^G, \) and \( R_{GW}^G \) are the interaction coefficients, which are defined as follows:

\[
S_{MA}^G = \operatorname{Re} (\Psi^e S_{CH}^G (F_M^e - F_G)^{-1} F_G \Psi^e) \quad \text{added mass (external fluid)},
\]
\[
S_{MA}^E = \Psi^e S_{CH}^G (\omega^2 I - \Omega)^{-1} \Psi^e \quad \text{added mass (internal fluid)},
\]
\[
S_{GW}^G = -\omega \times \operatorname{Im} (\Psi^e S_{CH}^G (F_M^e - F_G)^{-1} F_G \Psi^e) \quad \text{radiated wave damping matrix},
\]
\[
R_{GW}^G = j \omega \Psi^e S_{CH}^G (F_M^e - F_G)^{-1} 4 \pi R' \quad \text{wave exciting force vector}.
\]

In this section, we finally remark on the solution procedures for the present and conventional formulations as depicted in Fig. 5. In the present solution procedure, the displacement of the structure \( (\bar{u}) \) and the velocity potential of the external \( (\Phi^e) \) and internal \( (\Phi) \) fluids are obtained by solving the discrete coupled equations, and one additional step of a modal analysis can be optionally employed to reduce the number of degrees of freedom. The conventional solution procedure requires four solution steps. However, both procedures provide theoretically equivalent solutions and their solution efficiency is also similar.

4. Numerical tests

In this section, various numerical tests are presented for a 3D box barge model: free vibration analyses, a rigid body hydrodynamic analysis, and a hydroelastic analysis. The hydrostatic analysis is performed prior to the dynamic analysis to include the initial stress effect in the hydroelastic analysis. In the hydrostatic and hydrodynamic analyses, the reference configuration is assumed as the hydrostatic equilibrium state calculated for the rigid body case.

Fig. 6(a) presents a 3D box barge with three liquid tanks (length \( L \) is 300 m, width \( W \) is 50 m, and height \( H \) is 30 m) used in the numerical tests. The model consists of three parts: the bottom, side hulls (bow, stern, starboard, and port), and four bulkheads. The thicknesses and material properties are listed in Table 1, and it is ensured that the draft \( d \) is 10 m and the vertical center of gravity (COG) is \(-4 \) m in the rigid body case. For simplicity, all three rectangular
4.1. Free vibration analyses

Free vibration analyses of the box barge and the internal fluid are performed to obtain the natural frequencies and the mode shapes. The finite element model of the box barge is shown in Fig. 6(b). The four elastic dry mode shapes of the box barge that correspond to the first four natural frequencies are presented in Fig. 7. It is observed that the first mode shown in Fig. 7(a) is the torsional mode.

For the internal fluid, the first eight free surface mode shapes and the corresponding natural frequencies (\(\omega_N^i\); \(i\) indicates the free surface mode number) computed here are illustrated in Fig. 8 and

<table>
<thead>
<tr>
<th>Material properties of the box barge model.</th>
<th>Bottom</th>
<th>Side hulls</th>
<th>Bulkheads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness, (t) (m)</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Density, (\rho) (kg/m(^3))</td>
<td>15,000</td>
<td>15,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Young’s modulus, (E) (GPa)</td>
<td>(2.0 \times 10^{11})</td>
<td>(2.0 \times 10^{11})</td>
<td>(2.0 \times 10^{11})</td>
</tr>
<tr>
<td>Poisson’s ratio, (\nu)</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
</tbody>
</table>

The meshes used for the box barge, internal fluid, and external fluid are shown in Fig. 6(b)–(d). A four-node mixed interpolation of tensorial components (MITC) shell finite element [37–40] is used for the floating structure, an eight-node brick element is used for the internal fluid in tanks, and a four-node boundary element is used for the external fluid.
Numerical ($\omega^N_{m,n}$) and analytical ($\omega^A_{m,n}$) results for the natural frequencies (rad/s) of the internal fluid.

<table>
<thead>
<tr>
<th>Numerical results ($\omega^N_{m,n}$)</th>
<th>Analytical results ($\omega^A_{m,n}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.339 ($\omega^N_{1,0}$)</td>
<td>0.339 ($\omega^A_{1,0}$)</td>
</tr>
<tr>
<td>0.588 ($\omega^N_{1,1}$)</td>
<td>0.586 ($\omega^A_{1,1}$)</td>
</tr>
<tr>
<td>0.646 ($\omega^N_{2,0}$)</td>
<td>0.642 ($\omega^A_{2,0}$)</td>
</tr>
<tr>
<td>0.662 ($\omega^N_{2,1}$)</td>
<td>0.659 ($\omega^A_{2,1}$)</td>
</tr>
<tr>
<td>0.827 ($\omega^N_{2,2}$)</td>
<td>0.822 ($\omega^A_{2,2}$)</td>
</tr>
<tr>
<td>0.905 ($\omega^N_{4,0}$)</td>
<td>0.895 ($\omega^A_{4,0}$)</td>
</tr>
<tr>
<td>1.012 ($\omega^N_{4,1}$)</td>
<td>1.003 ($\omega^A_{4,1}$)</td>
</tr>
<tr>
<td>1.038 ($\omega^N_{4,2}$)</td>
<td>1.023 ($\omega^A_{4,2}$)</td>
</tr>
</tbody>
</table>

where the subscripts $m$ and $n$ denote the longitudinal and transverse directions, respectively, and $\omega^A_{m,n}$ is the analytical solution of the sloshing natural frequencies.

4.2. Rigid body hydrodynamic analysis

We perform a rigid body hydrodynamic analysis and the results are compared with the results of WAMIT [41]. In WAMIT, a higher-order method (using the 4th-order B-spline functions) is employed for the external and internal fluids. For the spatial discretization, 60, 20, and 4 panels are used for the box barge in the length, width, and depth directions and 18, 20, and 4 panels are used for each internal fluid in the three tanks.

Figs. 9 and 10 show the response amplitude operators (RAOs) when the incident wave angles $\theta$ are 0°, 90°, and 45°. In the figures, $q_i$ is the generalized coordinates defined with respect to the center of floatation of the box barge. The subscript $i$ varies from 1 to 6, which denote 6 rigid body motions, i.e., surge, sway, heave, roll, pitch, and yaw motions.
All the results of the present formulation and WAMIT are in good agreement. However, unlike the higher-order method used in WAMIT, a bilinear interpolation is employed in the present discretization and thus small differences can be observed near the resonance points.

4.3. Hydroelastic analysis

The hydroelastic analysis of the box barge model is performed with the mode superposition method for efficient computation. The dry modes of the box barge and the sloshing modes of the internal fluid that correspond to the natural frequencies up to $\sqrt{1000}$ rad/s are contained for the reduced equation and the generalized coordinates are constructed with respect to the center of floatation of the box barge.

As well known, the hydrostatic stiffness plays an important role in hydrodynamic analysis of floating structures in both rigid and elastic body cases. In particular, the geometric stiffness term $S_{KN}$ in Eq. (20c) should be carefully considered in hydroelastic analysis because Eq. (20c) requires the initial stress field $\sigma_{ij}$ due to hydrostatic pressures. Therefore, hydrostatic analysis is requisite for hydroelastic analysis.

Fig. 11 shows the vertical displacements calculated in both rigid and elastic body cases. In particular, it is observed that, in the elastic cases, quite different results are obtained when the initial stress effect is not considered. Table 3 shows the differences in hydrostatic stiffness terms. This comparison study demonstrates the importance of the initial stress in hydroelastic analysis. However, the differences would be small for relatively rigid floating structures.

In Fig. 11, there are peaks which arise from many different resonance sources (e.g. floating structure, sloshing, and external waves) and their combinations. It is a hard task to identify the sources of peaks, in particular, when multiple sloshing tanks are considered.

Finally, some diagonal components of the external and internal added masses $S_{E,G}$ and $S_{I,G}$ calculated using Eq. (28) are presented in Figs. 12 and 13. In contrast to the results in Fig. 12, resonance phenomena can be found in Fig. 13.

Fig. 10. RAOs of rigid body motions of the box barge when $\theta = 45^\circ$: (a) surge, sway, and heave, and (b) roll, pitch, and yaw motions.
5. Hydroelastic experiments

To verify the hydroelastic analysis procedure developed in this study, 3D hydroelastic experiments were conducted using a simplified FPU model with three rectangular liquid tanks. In the following, the hydroelastic experiment setup and the numerical model are described. The experimental results are then compared with the numerical results.
5.1. Experimental setup

In this section, we present the overall description of the 3D hydroelastic experiment, including the experimental conditions, structural model, mooring method, wave conditions, and measuring devices.

As shown in Fig. 14, the hydroelastic experiments of the FPU model are carried out in an ocean basin and the
water depth \((h_E)\) is set to 1.5 m. The FPU model is made of polycarbonate. Details of the FPU model are presented in Fig. 15 and Table 4. The lowest elastic mode of the experimental model is a twisting mode \((\omega_{n_{\text{twist}}} = 15.66 \text{ rad/s})\) and the first twenty natural sloshing frequencies are within a range of 4.95–19.22 rad/s.

Fig. 16 illustrates the overall experimental setup. In order to measure the incident wave frequency \((\omega)\), wave length \((\lambda)\), and
Fig. 14. Hydroelastic experiment of the FPU model with three liquid tanks in an ocean basin (15 m × 10 m × 1.5 m).

Table 3
Comparison of the hydrostatic stiffness terms. The subscripts i and j vary from 1 to 10; 1, 2, ..., 6, denote the values corresponding to the six rigid body motions and 7, 8, 9, 10 denote the values corresponding to the first four elastic modes shown in Fig. 6.

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>$S_{GH,ij}$</th>
<th>$S_{GH,ij}^{w/o}$ initial stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 3)</td>
<td>0.98000</td>
<td>0.98000</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>0.42365</td>
<td>0.69250</td>
</tr>
<tr>
<td>(5, 5)</td>
<td>0.85619</td>
<td>0.87069</td>
</tr>
<tr>
<td>(7, 7)</td>
<td>0.41157</td>
<td>0.61645</td>
</tr>
<tr>
<td>(8, 8)</td>
<td>0.54137</td>
<td>0.03300</td>
</tr>
<tr>
<td>(9, 9)</td>
<td>0.46628</td>
<td>0.03144</td>
</tr>
<tr>
<td>(10, 10)</td>
<td>1.81060</td>
<td>0.03505</td>
</tr>
</tbody>
</table>

Table 4
Details of the FPU model.

<table>
<thead>
<tr>
<th>Details of the FPU model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (L)</td>
<td>2.4 m</td>
</tr>
<tr>
<td>Width (W)</td>
<td>0.4 m</td>
</tr>
<tr>
<td>Height (H)</td>
<td>0.2 m</td>
</tr>
<tr>
<td>Thickness (t)</td>
<td>0.003 m</td>
</tr>
<tr>
<td>Draft (d)</td>
<td>0.08 m</td>
</tr>
<tr>
<td>Young’s modulus (E)</td>
<td>2 GPa</td>
</tr>
<tr>
<td>Tank1 $(L_i \times W \times h_i)$</td>
<td>$0.6 \times 0.4 \times 0.1$ m</td>
</tr>
<tr>
<td>Tank2 and 3 $(L_i \times W \times h_i)$</td>
<td>$0.4 \times 0.4 \times 0.1$ m</td>
</tr>
</tbody>
</table>

Fig. 15. FPU model.
Fig. 16. A schematic of the experimental setup: (a) top view, (b) front view, and (c) mooring lines.
Fig. 17. Meshes used for (a) FPU with three rectangular tanks and (b) internal fluid.

Fig. 18. RAOs of the displacements of the FPU model for two incident wave angles ($\theta = 0^\circ$ and $\theta = 90^\circ$).
amplitude \((a)\), one wave probe is installed at the free surface, located a distance of 1.5 m in front of the test model. The wave probe measures the wave elevation during the experiments. The three translations of the floating structure are then measured through four motion capture cameras with infrared reflective (IR) markers. Fig. 16(a) shows the positions of the six IR markers attached on the FPU model.

The incident waves belong to the range of the linear wave theory in deep water condition \(2a/h_{w} < 1.0 \) and \( h_{w}/\lambda < 1.0 \) [42]. The drift of the FPU model due to the incident wave was prevented by mooring the structure upward with four strings. Since the strings should prevent the drift without restraining surge, sway, and heave motions, a small amount of tension is introduced such that the strings are horizontally connected to the structure. That is, the connection angle between the strings and the structure is almost 180°, see Fig. 16(c).

We then performed the 3D hydroelastic experiments considering eight wave frequencies \((\omega = 4.3, 5.3, 6.2, 7.4, 8.4, 9.5, 10.2, 12.2, \) and 15 rad/s) and three different incident wave angles \((\theta = 0°, 45°, \) and 90°).

5.2. Comparison between experimental and numerical results

Rigid body hydrodynamic and hydroelastic analyses are conducted using the numerical method developed in this study. Fig. 17 shows the meshes used for the FPU model with three rectangular tanks. The FPU model and internal fluids in tanks are discretized by 5200 shell elements and 7000 brick elements, respectively. The external wet surface is discretized by 2680 boundary elements.

Figs. 18 and 19 show the RAOs of structural displacements \((u_1, u_2, \) and \( u_3)\) for three incident wave angles \((\theta = 0°, 90°, \) and 45°). The results of the rigid body hydrodynamic and hydroelastic analyses are compared with the experimental results. It is observed that the experimental results are in good agreement with the results of the hydroelastic analysis, especially, when the initial stress effect is considered. Since artificial damping is not considered in the numerical analyses, the numerical results over-predict the peaks. In the heading angle \((\theta = 0°)\), the FPU model moves like a rigid body in both experimental and numerical results due to its relatively large overall bending rigidity.
Fig. 20. Modal responses calculated: (a) RAOs of the modal coordinate \( q_7 \) and phase angle \( (\theta = 45^\circ) \), and (b) mode shape \( (\psi_7) \).

**Fig. 20** presents the RAOs of the modal coordinate \( q_7 \) and phase angle calculated, which correspond to the first elastic mode \( (\psi_7, \text{twisting mode}) \) when the incident wave angle is \( 45^\circ \). **Fig. 21(a)** shows the twisting angles of the FPU model measured in the numerical and experimental results. The largest twisting angle \( (\theta_{\text{twisting}}) \) is observed at 7.4 rad/s in the experimental results.

In **Fig. 22**, we finally present some snapshots of free surface profiles, shape of structural deformation and the corresponding structural displacements when the wave frequency is 7.4 rad/s. It is observed that the sloshing motion is not beyond the linear potential theory and the tendency of free surface profiles in **Fig. 22(a)** agrees well with the numerical results.

**Fig. 21.** Twisting angle \( (\theta_{\text{twisting}}) \) of the FPU model \( (\theta = 45^\circ) \) and wave amplitude \( (a = 0.03 \text{ m}) \): (a) comparison between numerical and experimental results and (b) definition of twisting angle.
6. Conclusions

In this paper, we presented a mathematical formulation and a numerical method for a hydroelastic analysis of floating structures with liquid tanks in the frequency domain, in which the direct-coupling method was employed to couple structural motions, sloshing, and water waves. The extended boundary integral equation was adopted in order to avoid the irregular frequency problem. The proposed formulation includes all the terms required for a linear hydroelastic analysis of floating structures with liquid tanks.

The proposed formulation was verified through a comparison with the analysis results of WAMIT in a rigid body hydrodynamic analysis. The importance of the initial stress was demonstrated through a comparative hydroelastic analysis. In addition, 3D hydroelastic experiments were performed for a FPU model. We also simulated the hydroelastic behavior of the FPU model. The numerical results were compared with experimental results and good agreement between the results was observed.

In future works, it will be valuable to extend the present direct coupled formulation to nonlinear hydroelastic analyses, in which we could deal with the large motions of floating structures and fluid and wet-surface change. Also, it will be an interesting study to identify resonance sources in hydroelastic analysis of floating structures with liquid tanks.

Acknowledgements

This work was supported by the Basic Science Research Programs through the National Research Foundation of Korea (NRF), funded by the Ministry of Education, Science and Technology (No. 2014R1A1A1A05007219 and No. 2014M2B2A9030561).
References


