Higher order weighted integral stochastic finite element method and simplified first-order application

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Abstract

It is well known that expansion-based stochastic methods are approximate schemes, as they are based on a first or, at most, second order series expansion on the basic variable, e.g. displacement. Therefore, expansion-based stochastic analysis schemes are bound to show small response variability when compared with Monte Carlo simulation (MCS) results, and application of these schemes is limited to stochastic problems with relatively small variability. In order to overcome these general drawbacks of the expansion methods, we suggest a higher-order stochastic field function that can be employed in the expansion-based stochastic analysis scheme of the weighted integral method. We then propose a new weighted integral formulation using the higher-order stochastic field function. The new formulation is not only applicable to stochastic problems with a high degree of uncertainty but also can reproduce the phenomenon of accelerated increase in the response variability when the coefficient of variation of the stochastic field increases, as observed in the MCS. In order to show the validity of the proposed formulation, we provide two numerical examples and the results are discussed in detail.

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1. Introduction

The Monte Carlo simulation (MCS), despite requiring large computational costs and memory, has not merely survived but is a universally employed means of finding solutions for various stochastic problems. Much of its popularity owes to its conceptual simplicity of application (Hurtado and Barbat, 1998). At present, however, there is a strong need for development of an efficient and accurate scheme for random field generation. One reason why the MCS provides an accurate solution for any type of stochastic problem lies in that a numerically generated stochastic field is directly used. In contrast, application of expansion methods, which
employ a first-order approximation, or up to second-order at most, is limited to stochastic systems with small variations in random parameters (Choi and Noh, 1996, 2000; Kaminski, 2001; Noh, 2004, 2005a,b; Shinozuka and Yamazaki, 1995). The error of the expansion methods increases drastically as the coefficient of variation of the stochastic field increases (Chakraborty and Bhattacharyya, 2002; Kaminski, 2001; Noh, 2005a,b, 2006; Yamazaki et al., 1988), making the expansion-based approximate methods inappropriate for practical problems involving large variations of system parameters.

Bearing these points in mind and noting that some ordinary materials such as concrete and moderate soil medium have a coefficient of variation up to or exceeding 0.2, we intend to develop a non-statistical stochastic analysis scheme that is suitable for uncertain media with relatively high degree of uncertainty. As widely accepted, the conventional first-order expansion methods are only applicable to stochastic systems with an uncertainty around 0.1 (Choi and Noh, 1996, 2000; Papadopoulos et al., 2005) Several attempts to develop approximate methods for stochastic systems with higher degree of uncertainty have been reported in the literature: Choi and Noh (2000) introduced the Lagranian reminder in the expansion on the displacement while Shinozuka and Yamazaki (1995) used a second order expansion. Some other analysis techniques presented in the literature (Ghanem and Spanos, 1991; Spanos and Ghanem, 1989) show the dependency of the response variability, to a degree, on the coefficient of variation of stochastic field. Improvement in these trials, however, is restricted only to the mean and variance, without any enhancement of the response COV (coefficient of variation) (Choi and Noh, 2000). Furthermore, the degree of improvement was observed to be relatively low (Shinozuka and Yamazaki, 1995). Ghanem and Spanos (1991), meanwhile, provide the response variability that depends on two expansion parameters, the order of the Neumann expansion \( p \) and the order of Karhunen-Loeve expansion \( M \).

Despite that stochastic procedures can theoretically give an “envelope” that encompasses all possible uncertainties of loads and structural parameters (Schueller, 2001), the cost-effectiveness and applicability of these procedures are not satisfactory for practical engineering applications. This is true for the first-order expansion schemes as well as the higher-order non-statistical schemes and the MCS. Accordingly, there is strong need for a scheme with which accurate statistical moments can be evaluated, even for the problems with moderately large coefficient of variation of stochastic field within a tolerable computational effort.

In this paper, we first overview the first-order weighted integral method, developed by Takada (1990a,b) and applied to various types of structures (Choi and Noh, 1996; Deodatis, 1991; Deodatis and Shinozuka, 1991; Deodatis et al., 1991; Noh, 2004, 2005a,b), and then propose a theoretical formulation employing an infinite series expansion on the basic variable. After examining the characteristics of the MCS and providing a comparison with a theoretical infinite series formulation, we suggest a higher-order stochastic field function. Employing the higher-order stochastic field function, we propose a new weighted integral formulation that is not only applicable to stochastic fields with high degree of uncertainty but also can reproduce the phenomenon of accelerated increase in the response variability as the coefficient of variation of the stochastic field increases. In the formulation, we also address the equivalence between the formulation with an infinite series and the new weighted integral formulation.

The proposed formulation has two main advantages over conventional formulations: (1) a noticeable improvement in the response statistics and outstanding agreement with the MCS, not only in the response variability but also in the mean and standard deviation of the response; and (2) relatively low computation cost when compared with the conventional statistical stochastic procedures. As widely known, the accuracy of the statistical approach of the MCS highly depends on the number of samples involved in the analysis. The proposed scheme, however, can be performed within tolerable computational expense.

2. Conventional weighted integral method and higher-order formulation

2.1. Stochastic field

A spatially random parameter is assumed to have fluctuation that is added to the mean of that parameter. In the case of the elastic modulus, the following equation can be written:

\[
E(x) = E_o + \Delta E(x) = E_o[1 + \phi(x)],
\] (1)
where \( E_o \) is the mean value of the elastic modulus and the fluctuation of the elastic modulus, \( \Delta E \), can be given as a position-dependent scaled mean value. The scaling function is known as a stochastic field function \( f(x) \) that determines the probabilistic feature of the spatially random parameter. In this study, the degree of uncertainty in the stochastic field is assumed to be moderate and having a coefficient of variation of less than 0.25. The stochastic field \( f(x) \) is assumed to be homogeneous with zero mean and to have values in the range of \(-1 + \delta_f < f < 1 - \delta_f\), where \( 0 \leq \delta_f < 1.0 \).

2.2. Conventional weighted integral method

In the conventional weighted integral method, only the first order expansion is implemented resulting in the following first-order expanded displacement vector (Choi and Noh, 1996; Takada, 1990a,b)

\[
U \approx U_o^{(1)} - \sum_{e=1}^{n_e} \sum_{r=1}^{n_{pe}} (X^e_r - X^c_r)K^{-1} \left[ \frac{\partial K}{\partial X^e_r} \right] Eo U_o^{(1)}, \tag{2}
\]

where \( U_o^{(1)} \) denotes the mean displacement obtained by the first-order expansion based method and \( n_e \) the number of finite elements. \( n_e \) is the number of random variables \( X \) in each finite element. The random variable \( X \) is defined as a stochastic integration (Deodatis et al., 1991; Choi and Noh, 1996).

With Eq. (2), the mean displacement is obtained in precisely the same manner as in the deterministic analysis and the first-order covariance can also be found with relative low computational cost. In this regard, one motivation for choosing the first-order expansion is the high computational burden when a higher-order expansion is employed. The first-order methods, however, give reasonable results only if the coefficient of variation of the stochastic field is relatively low, i.e., about 0.1 (Choi and Noh, 1996, 2000; Papadopoulos et al., 2005). This means that as the degree of fluctuation of the random part is increased, the results are accordingly deteriorated (Graham and Deodatis, 1998; Noh, 2005a,b).

2.3. Higher-order expansion weighted integral formulation

2.3.1. Stochastic element stiffness

With the substitution of Eq. (1) into the constitutive matrix \( D \), the element stiffness matrix \( k^{(1)} = \int_{\Gamma_r} B^T D B d\Gamma \) can be divided into two parts, the mean and deviation

\[
k^{(1)} = k^{(e)} + \Delta k^{(1)}, \tag{3}
\]

in which the mean stiffness \( k^{(e)} \) is the same as the deterministic stiffness, and the deviatoric stiffness can be written as Eq. (4) if we apply the decomposition technique on the strain–displacement matrix \( B = B_0 p_i \) (Choi and Noh, 1996):

\[
\Delta k^{(1)} = \int_{\Gamma_r} f(x)B^T D B d\Gamma = B^T D B \mu X_{ij}^{(1)}, \tag{4}
\]

where the random variable is defined as \( X_{ij}^{(1)} = \int_{\Gamma_r} f(x)p_i p_j d\Gamma \); \( i, j = 1, 2, \ldots, n_p \) and \( n_p \) signifies the number of independent polynomials in the strain–displacement matrix \( B \). Considering the symmetry of the random variable, \( X_{ij}^{(1)} = X_{ji}^{(1)} \), the deviatoric stiffness can be written as follows:

\[
\Delta k^{(1)} = \sum_{r=1}^{n_{pe}} C_r^{(e)} Y_r^{(e)}. \tag{5}
\]

The index \( r \), in terms of two indexes \( i \) and \( j \) of the random variable \( X_{ij}^{(1)} \), takes a value as given in Table 1. Due to the symmetry of the random variable, \( C_2^{(1)} = B_1^1 D B_2 + B_2^1 D B_1 \); \( Y_2^{(1)} = X_{12}^{(1)} \), \( C_{np+2}^{(1)} = B_{np+2}^1 D B_3 + B_3^1 D B_2 \); \( Y_{np+2}^{(1)} = X_{23}^{(1)} \), and so on; and the vector of the random variable can be established as follows:
2.3.2. Higher-order expansion on the displacement vector

As a result, the element stiffness matrix is given as a function of the random variable \( Y_e^{(r)} \): \( k_e^{(r)} = k_e(Y_e^{(r)}) \), \( e = 1, \ldots, n_e, r = 1, \ldots, n_r \), where \( n_e \) signifies the number of finite elements in the domain under consideration.

### Table 1

<table>
<thead>
<tr>
<th>( i, j )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, \ldots, ( n_p )</td>
<td>( n_p + 1, \ldots, 2n_p - 1 )</td>
</tr>
<tr>
<td>2n_p, \ldots, 3n_p - 3</td>
<td>\ldots</td>
</tr>
<tr>
<td>( n_r = \frac{1}{2}n_p(n_p + 1) )</td>
<td></td>
</tr>
</tbody>
</table>

\[ Y = (Y^{(1)T}, Y^{(2)T}, \ldots, Y^{(n_e)T})^T, \]
\[ Y^{(i)} = (Y_1^{(i)}, Y_2^{(i)}, \ldots, Y_{n_e}^{(i)}). \]  

With respect to Eq. (8), it is of importance to note that the matrix \( \mathbf{K} \) should be evaluated as \( \mathbf{K} = \mathbf{K}|_{Y_e} = \mathbf{K} (\overline{Y}) \) and, therefore, is different from the deterministic stiffness.

Substituting Eq. (8) into Eq. (7), the displacement vector becomes

\[
U(Y) = \mathbf{U} (\overline{Y}) - \Delta Y_i^{(r)} \mathbf{K}^{-1} (K_{(ir)})|_{E} U + \Delta Y_j^{(r)} \Delta Y_k^{(r)} \mathbf{K}^{-1} (K_{(ir)} \mathbf{K}^{-1} K_{(jr)})|_{E} U \\
- \Delta Y_i^{(r)} \Delta Y_j^{(r)} \Delta Y_k^{(r)} \mathbf{K}^{-1} (K_{(ir)} \mathbf{K}^{-1} K_{(jr)} \mathbf{K}^{-1} K_{(kr)})|_{E} U + \cdots
\]  

In Eq. (9), it is noted that, with the aid of Eq. (4), \( K_{(ir)} = \frac{\partial k}{\partial Y_i} = C_i^{(r)} \), \( \Delta Y_j^{(r)} = Y_j^{(r)} \), and so on. Accordingly, it holds that \( \Delta Y_i^{(r)} (K_{(ir)})|_{E} = \Delta k^{(r)} \), the deviatoric stiffness as given in Eqs. (4) and (5). As a consequence, Eq. (9) can be re-written in the finite element-based form as follows:
\[ \mathbf{U}(\mathbf{Y}) = \mathbf{U}(\mathbf{Y}) + \mathbf{K}^{-1} \left( \sum_{k=1}^{n_y} \int_{V^{(k)}} f(\mathbf{x}_{e1}) \hat{\mathbf{k}}^e \hat{\mathbf{k}} \tips{dV^{(1)}} \right. \\
\left. + \sum_{e1=1}^{n_y} \sum_{e2=1}^{n_y} \int_{V^{(2)}} f(\mathbf{x}_{e1}) f(\mathbf{x}_{e2}) \hat{\mathbf{k}}^{e1} \hat{\mathbf{k}}^{e2} \hat{\mathbf{k}} \tips{dV^{(2)}} \right) \\
\left. - \sum_{e1=1}^{n_y} \sum_{e2=1}^{n_y} \sum_{e3=1}^{n_y} \int_{V^{(2)}} f(\mathbf{x}_{e1}) f(\mathbf{x}_{e2}) f(\mathbf{x}_{e3}) \hat{\mathbf{k}}^{e1} \hat{\mathbf{k}}^{e2} \hat{\mathbf{k}} \hat{\mathbf{k}} \hat{\mathbf{k}} \tips{dV^{(3)}} \right) \\
+ \cdots \right) \mathbf{U}, \tag{10} \]

where the abbreviations \( F^{(n)} = f(\mathbf{x}_{e1}) f(\mathbf{x}_{e2}) \cdots f(\mathbf{x}_{e(n)}) \) and \( \hat{\mathbf{K}}^{(n)} = \hat{\mathbf{k}}^{e1} \hat{\mathbf{k}}^{e2} \cdots \hat{\mathbf{k}}^{en} \) are used, the dimension of \( \hat{\mathbf{k}}^{e} = \mathbf{B}^{T}_{e} \mathbf{D}_{e} \mathbf{B}_{e} \) is expanded to that of the global stiffness, and \( dV^{(n)} = dV^{(1)} dV^{(2)} \cdots dV^{(n)} \). Note that \( \mathbf{B}_{e} \) in \( \hat{\mathbf{k}}^{e} \) is not the constant matrix \( \mathbf{B}_{i} \) in Eq. (4) but the original strain–displacement matrix of the finite element \( q \).

As an extreme situation, we should consider the “random variable case” that corresponds to a state when the stochastic field assumes a constant value over the domain under consideration. In this special case, Eq. (10) becomes

\[ \mathbf{U}(\mathbf{Y}) = \mathbf{U}(\mathbf{Y}) + \mathbf{K}^{-1} \left( \sum_{k=1}^{n_y} \int_{V^{(k)}} F^{(k)} \hat{\mathbf{K}}^{(k)} \tips{dV^{(k)}} \right) \mathbf{U}, \tag{11} \]

where the stochastic field is assumed to be constant \( f_{W_1} \), i.e., \( f(\mathbf{x}_{e}) = f(\mathbf{x}_{e}) = f_{W_1} \).

### 2.3.2.1 Mean of displacement.

Applying the mean operation to Eq. (10), and noting that \( E[\mathbf{F}^{(k)}] = 0 \) when the power \( k \) is odd, the mean displacement vector is given by

\[ \mathbf{U}_{m} = E[\mathbf{U}(\mathbf{Y})] = \mathbf{U}(\mathbf{Y}) + \mathbf{K}^{-1} \left( \sum_{k=1}^{n_y} \int_{V^{(k)}} E[F^{(k)}] \hat{\mathbf{K}}^{(k)} \tips{dV^{(k)}} \right) \mathbf{U}, \tag{12} \]

where \( \sum_{e1=1}^{n_y} \sum_{e2=1}^{n_y} \cdots \sum_{e(k)=1}^{n_y} \int_{V^{(k)}} E[F^{(k)}] \hat{\mathbf{K}}^{(k)} \tips{dV^{(k)}} \mathbf{U} \) vanishes because the stochastic field function \( f(\mathbf{x}) \) is assumed to be a Gaussian distribution.

### 2.3.2.2 Covariance of displacement.

Since the covariance of the displacement is defined as \( \text{Cov}[\mathbf{U}] = E[(\mathbf{U} - \mathbf{U}_{m})(\mathbf{U} - \mathbf{U}_{m})^{T}] \), the mean centered deviation of the displacement vector needs be determined first. With Eqs. (10) and (12), the mean centered deviation of displacement can be written as

\[ \mathbf{U} - \mathbf{U}_{m} = \mathbf{K}^{-1} \left( \sum_{k=1}^{\infty} \left( \left( \sum_{e1=1}^{n_y} \sum_{e2=1}^{n_y} \cdots \sum_{e(k)=1}^{n_y} \int_{V^{(k)}} E[F^{(k)}] \hat{\mathbf{K}}^{(k)} \tips{dV^{(k)}} \right) \right) - \sum_{k=1}^{\infty} \left( \left( \sum_{e1=1}^{n_y} \sum_{e2=1}^{n_y} \cdots \sum_{e(k)=1}^{n_y} \int_{V^{(k)}} E[F^{(k)}] \hat{\mathbf{K}}^{(k)} \tips{dV^{(k)}} \right) \right) \right) \mathbf{U} \]

\[ = \sum_{(k)} - \sum_{(2k)}, \tag{13} \]
In the last line of Eq. (13), the subscripts \( (A) \) and \( (2k) \) are used to designate that each term corresponds to ‘all’ and \( 2k \)th terms. Substituting the mean centered deviation of the displacement vector in Eq. (13) into the definition of covariance, we obtain

\[
\text{Cov}[\mathbf{U}] = E\left[ (\mathbf{U} - \mathbf{U}_0)(\mathbf{U} - \mathbf{U}_0)^T \right]
\]

\[
= \left[ \sum_{(A)} \sum_{(2k)} \right] - \left[ \sum_{(2k)} \sum_{(2k)} \right]^T
\]

and

\[
E\left[ \sum_{(A)} \sum_{(A)} \right] = \mathbf{K}^{-1}(\{1,1\} + \{2,2\} + 2\{1,3\} + \{3,3\} + 2\{2,4\} + \cdots)\mathbf{K}^{-T},
\]

with

\[
(I,J) = E\left[ \left( \sum_{e \in I} \int_{Y(l)} F^{(l)} \tilde{\mathbf{R}}^{(l)} dV^{(l)} \right) \mathbf{U} \mathbf{U}^T \left( \sum_{e \in J} \int_{Y(l)} F^{(l)} \tilde{\mathbf{R}}^{(l)} dV^{(l)} \right)^T \right]
\]

\[
= \sum_{l=1}^{\infty} \sum_{h=1}^{\infty} \int_{Y(l)} \int_{Y(l)} E[F^{(l)}]F^{(l)} \tilde{\mathbf{R}}^{(l)} \mathbf{U} \mathbf{U}^T \tilde{\mathbf{R}}^{(l)} dV^{(l)} dV^{(l)}.
\]

The terms having an odd number for ‘\( I + J \)’ are omitted due to the consideration noted in Section 2.3.2.1.

The evaluation of the second term of covariance, \( \sum_{(2k)} \sum_{(2k)} \), is straightforward. With Eq. (13), it can be written as follows:

\[
\sum_{(2k)} \sum_{(2k)}^T = \left( \mathbf{K}^{-1} \sum_{k=1}^{\infty} \left\{ \sum_{e \in e^{(2k)}} \int_{Y^{(2k)}} E[F^{(2k)}]F^{(2k)} \tilde{\mathbf{R}}^{(2k)} dV^{(2k)} \right\} \right) \left( \mathbf{K}^{-1} \sum_{k=1}^{\infty} \left\{ \sum_{e \in e^{(2k)}} \int_{Y^{(2k)}} E[F^{(2k)}]F^{(2k)} \tilde{\mathbf{R}}^{(2k)} dV^{(2k)} \right\} \right)^T.
\]

In order to reduce the computational burden in practical applications, the infinite summation with respect to index \( k \) in Eqs. (15) and (17) needs to be truncated with tolerable error.

3. Monte Carlo analysis and establishment of a higher order stochastic field function

The displacement vector in the Neumann expansion Monte Carlo simulation (Yamazaki et al., 1988) is obtained as

\[
\mathbf{U}_{\text{MCS}} = (\mathbf{I} - \mathbf{J} + \mathbf{J}^2 - \mathbf{J}^3 + \cdots)\mathbf{U},
\]

where \( \mathbf{J} = \mathbf{K}^{-1}\Delta\mathbf{K} \) and \( \mathbf{U} = \mathbf{K}^{-1}\mathbf{P}\). \( \Delta\mathbf{K} \) and \( \Delta\mathbf{K} \) signify the mean and deviatoric stiffness matrices. The convergence of the Neumann expansion in Eq. (19) is guaranteed only if the absolute values of all eigenvalues of \( \mathbf{J} \) are less than one (Matthies et al., 1997). This implies that the stochastic field should be in the range of \( |f(x)| < 1.0 \). In other words, the intensity of randomness of the random field needs to be in a moderate range.

3.1. Random variable case

As a special concern, the case when the stochastic field becomes a random variable, i.e., when the correlation length of the stochastic field is infinite, needs to be examined, as done for the higher-order weighted integral formulation in Section 2.3.2 (Eq. 11). In this case, the matrix \( \mathbf{J} \) becomes an identity matrix multiplied by a random variable, because the random field assumes a constant value, e.g. \( f(x) = f_M \). Accordingly, the Neumann expansion in Eq. (18) becomes
\( \mathbf{U}_{\text{MCS}} = (1 - f_M + f_M^2 - f_M^3 + \cdots) \mathbf{U} \).

It is of particular interest that two equations, (11) and (19), are identical. This analogousness shows the equivalence between the MCS and the higher-order weighted integral scheme.

### 3.2. Random field case

For the "random variable case" it is clear that the two displacement vectors of the higher order weighted integral scheme (HWI) and of the MCS are equivalent. However, in the random field case, adoption of Eq. (19) with the replacement of \( f_M \) with \( f(x) \) is not clear. Before a mathematical investigation on the characteristics of the random field case we examine some simple illustrative systems. This provides insight into what happens in the system matrix for the random field case. In the following, \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) denote the realization of a random field corresponding to a specified spring element, \( \alpha_i = f(x_i) \).

We first examine a spring system connected in a linear manner as follows:

(a) 2 element and 2 DOF system

\[
\mathbf{K} = \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \Delta \mathbf{K} = \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 & -\alpha_2 k_2 \\ -\alpha_2 k_2 & \alpha_2 k_2 \end{bmatrix}
\]

\[
\mathbf{J} = \mathbf{K}^{-1} \Delta \mathbf{K} = \frac{1}{\det \mathbf{K}} \begin{bmatrix} k_2 & k_2 \\ k_2 & k_2 + k_2 \end{bmatrix} \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 & -\alpha_2 k_2 \\ -\alpha_2 k_2 & \alpha_2 k_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_1 - \alpha_2 & \alpha_2 \end{bmatrix}; \quad \det \mathbf{K} = k_1 k_2
\]

(b) 3 element and 3 DOF system

\[
\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}, \quad \Delta \mathbf{K} = \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 & -\alpha_2 k_2 & 0 \\ -\alpha_2 k_2 & \alpha_2 k_2 + \alpha_3 k_3 & -\alpha_3 k_3 \\ 0 & -\alpha_3 k_3 & \alpha_3 k_3 \end{bmatrix}
\]

\[
\mathbf{J} = \mathbf{K}^{-1} \Delta \mathbf{K} = \frac{1}{\det \mathbf{K}} \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 & -\alpha_2 k_2 & 0 \\ -\alpha_2 k_2 & \alpha_2 k_2 + \alpha_3 k_3 & -\alpha_3 k_3 \\ 0 & -\alpha_3 k_3 & \alpha_3 k_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_1 - \alpha_2 & \alpha_2 & 0 \\ \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \alpha_3 \end{bmatrix}; \quad \det \mathbf{K} = k_1 k_2 k_3
\]

As a general case, for an \( n \) spring elements system, the matrix \( \mathbf{J} \) is

\[
\mathbf{J} = \begin{bmatrix} \alpha_1 \\ \Delta \alpha_{12} & \alpha_2 \\ \vdots & \vdots & \ddots \\ \Delta \alpha_{12} & \Delta \alpha_{23} & \cdots & \alpha_{n-2} \\ \Delta \alpha_{12} & \Delta \alpha_{23} & \cdots & \Delta \alpha_{(n-2)(n-1)} & \alpha_{n-1} \\ \Delta \alpha_{12} & \Delta \alpha_{23} & \cdots & \Delta \alpha_{(n-2)(n-1)} & \Delta \alpha_{(n-1)n} & \alpha_n \end{bmatrix}
\]

where \( \det \mathbf{K} = k_1 k_2 \cdots k_n \) and \( \Delta \alpha_{ij} = \alpha_i - \alpha_j \).

In cases where the degrees of freedom are connected in a linear form, as shown in Figs. 1 and 2, the coefficients of the matrix \( \mathbf{J} \) consist only of constants, which are realizations of a random field. In particular, note

![Fig. 1. Two spring system.](image-url)
that the off-diagonal coefficients in the lower part of the matrix have zero values from the perspective of statistical quality.

Next, we look into more complex systems where two or more springs are connected to a nodal point.

(c) 3 element and 2 DOF system

\[
\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 - k_3 \\ -k_2 - k_3 & k_2 + k_3 \end{bmatrix}, \quad \Delta \mathbf{K} = \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 & -\alpha_2 k_2 - \alpha_3 k_3 \\ -\alpha_2 k_2 - \alpha_3 k_3 & \alpha_2 k_2 + \alpha_3 k_3 \end{bmatrix}
\]

\[
\mathbf{J} = \mathbf{K}^{-1} \Delta \mathbf{K} = \frac{1}{\det \mathbf{K}} \begin{bmatrix} k_2 + k_3 & k_3 + k_3 \\ k_2 + k_3 & k_1 + k_2 + k_3 \end{bmatrix} \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 & -\alpha_2 k_2 - \alpha_3 k_3 \\ -\alpha_2 k_2 - \alpha_3 k_3 & \alpha_2 k_2 + \alpha_3 k_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_1 - c_{\beta_1} & c_{\beta_1} \end{bmatrix};
\]

\[
\det \mathbf{K} = k_1(k_2 + k_3); \quad c_{\beta_1} = \frac{\alpha_2 k_2 + \alpha_3 k_3}{k_2 + k_3}
\]

(d) 4 element and 3 DOF system

\[
\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 - k_3 & 0 \\ -k_2 - k_3 & k_2 + k_3 + k_4 & -k_4 \\ 0 & -k_4 & k_4 \end{bmatrix}, \quad \Delta \mathbf{K} = \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 & -\alpha_2 k_2 - \alpha_3 k_3 & -\alpha_2 k_2 - \alpha_3 k_3 & 0 \\ -\alpha_2 k_2 - \alpha_3 k_3 & \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 & -\alpha_2 k_2 - \alpha_3 k_3 & 0 \\ 0 & -\alpha_4 k_4 & \alpha_4 k_4 \end{bmatrix}
\]

\[
\mathbf{J} = \mathbf{K}^{-1} \Delta \mathbf{K} = \frac{1}{\det \mathbf{K}} \begin{bmatrix} k_{23}k_4 & k_{23}k_4 & k_{23}k_4 \\ k_{23}k_4 & k_{123}k_4 & k_{123}k_4 \\ k_{23}k_4 & k_{123}k_4 & k_{123}k_4 - k_{23}^2 \end{bmatrix} \begin{bmatrix} \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 & -\alpha_2 k_2 - \alpha_3 k_3 & -\alpha_2 k_2 - \alpha_3 k_3 & 0 \\ -\alpha_2 k_2 - \alpha_3 k_3 & \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 & -\alpha_2 k_2 - \alpha_3 k_3 & 0 \\ 0 & -\alpha_4 k_4 & \alpha_4 k_4 \end{bmatrix}
\]

\[
= \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_1 - c_{\beta_1} & c_{\beta_1} & 0 \\ \alpha_1 - c_{\beta_1} & c_{\beta_1} - \alpha_4 & \alpha_4 \end{bmatrix}; \quad \det \mathbf{K} = k_1(k_2 + k_3)k_4; \quad k_{ijk} = k_ikjk_k; \quad c_{\beta_1} = \frac{\alpha_2 k_2 + \alpha_3 k_3}{k_2 + k_3}
\]

As shown in Figs. 3 and 4, when more than 2 springs are connected at one node, the diagonal coefficient corresponding to that node is given as a weighted coefficient in terms of the pertinent stiffness, and it appears in the consecutive coefficients of the matrix J. In this case, however, the off-diagonal coefficients are also given as zero-valued coefficients similar to the systems in Figs. 1 and 2.

Here, it is worth noting that, from a stochastic analysis point of view, the response statistics are to be found in the direction normal to the plane of the system stochastic field. In other words, the response statistics are
given in an ensemble concept. Therefore, in the statistical operation, with respect to all stochastic fields that satisfy the statistical characteristics of the random field under consideration, the effect of off-diagonal coefficients in the matrix \( J \) vanishes having almost no influence on the resulting response statistics. This can also be applied to the case of \( J^n, n = 1, 2, \ldots \).

Importantly, from Eqs. (20)–(24), we find that the matrix \( J \) consists of random numbers that come from the original random field. Some of the coefficients of the matrix \( J \), however, appear to be weighted by pertinent stiffness related to a specific degree of freedom, and therefore constitute a distinct random field different from the original random field. This discrepancy is caused by the pre-multiplication of the inverse of the deterministic stiffness matrix to the deviatoric stiffness. Regarding this discrepancy, a short numerical examination is given in Section 5.1.3. In the following, the equivalence of the random fields between the MCS and higher order Weighted Integral methodology is addressed.

For this purpose, we need to revisit Eqs. (10), the higher order Weighted Integral equation, and Eq. (18), the Neumann expansion MCS. With substitution of \( J = \mathbf{K}^{-1} \Delta \mathbf{K} \), Eq. (18) can be written as follows:

\[
U_{\text{MCS}} = (I - J + J^2 - J^3 + \cdots) \tilde{U} = \tilde{U} + \mathbf{K}^{-1}(-\Delta \mathbf{K} + \Delta \mathbf{K} \mathbf{K}^{-1} \Delta \mathbf{K} - \Delta \mathbf{K} \mathbf{K}^{-1} \Delta \mathbf{K} \mathbf{K}^{-1} \Delta \mathbf{K} + \cdots) \tilde{U}. \tag{25}
\]

For comparison, Eq. (10) is repeated here in a different form as follows:

\[
U(Y) = \tilde{U}(Y) + \mathbf{K}^{-1} \left( - \sum_{e=1}^{n_e} \int_{V_e} f(x_e) \tilde{e}^1 dV^1 \right) + \sum_{e=1}^{n_e} \int_{V_e} f(x_e) \tilde{e}^1 dV^1 \mathbf{K}^{-1} \sum_{e=2}^{n_e} \int_{V_e} f(x_e) \tilde{e}^2 dV^2 \left( \Delta \mathbf{K}^{-1} \right) \bigg|_{\tilde{e}^2} + \sum_{e=1}^{n_e} \int_{V_e} f(x_e) \tilde{e}^3 dV^3 \left( \Delta \mathbf{K}^{-1} \right) \bigg|_{\tilde{e}^3} + \cdots \bigg) \tilde{U}. \tag{26}
\]

As quickly noted, each term in the parenthesis in the second line of Eq. (25) and the corresponding term in the parenthesis of Eq. (26) constitute well-matched pairs. Therefore, the proposed higher order expanded displacement vector in the weighted integral formulation is equivalent to that of the MCS. The only difference is that while Eq. (25) deals with the stochastic field as a discrete field, Eq. (26) treats it as a semi-continuous field. Even though the summation is performed with different indexes, it must be noted that \( \Delta \mathbf{e}^j \) is the same as \( \Delta \mathbf{K}^j \) (refer also to the equivalent expression of the MCS, Eq. (25)). Therefore, we can simplify Eq. (26) as follows:

\[
U(Y) = \tilde{U}(Y) - \mathbf{K}^{-1} \sum_{e=1}^{n_e} \int_{V_e} \left( f(x_e) - f^2(x_e) + f^3(x_e) - \cdots \right) \tilde{e}^j dV \tilde{U} \tag{27}
\]

Consequently, the higher order stochastic field is obtained as follows from Eq. (27):

\[
f_\infty(x_e) = f(x_e) - f^2(x_e) + f^3(x_e) + \cdots = \sum_{k} (-1)^{k+1} f^k(x_e). \tag{28}
\]

It is of particular interest that the displacement vector in Eq. (27) has the same form as that in the first-order expansion weighted integral scheme except that the higher order stochastic field function \( f_\infty(x_e) \). Eq. (28), is employed. This means that we can develop a first-order weighted integral scheme with the higher order
stochastic field function $f_\infty(x_e)$, which is compatible with the stochastic field in the MCS; in addition, this also means that the derived formulation is expected to show MCS compatible response statistics. The validity of Eqs. (27) and (28) is supported via numerical demonstrations in Section 5 for numerical verifications.

4. First-order approximation using a higher order stochastic field

As demonstrated in Sections 2.3.2.1 and 2.3.2.2, the mean and covariance of displacement can be evaluated in the form of a higher order expansion. The use of these equations for evaluation of response statistics, however, is virtually impossible due to extensive computation time and an excessive memory requirement. In this regard, the use of Eq. (27) or Eq. (28) in the first order expansion weighted integral formulation is not only an alternative but also of considerable advantage. As a special case, if we take $f_\infty = f$, the formulation for the weighted integral method is derived to be the same as the original first-order weighted integral formulation.

4.1. Definition of random variable

If we revisit Eq. (27) with $\hat{k}^v = B_q^T D_q B_q$, as defined previously, the equation can be written as

$$U(Y) = \overline{U(Y)} - K^{-1} \sum_{e=1}^{n_e} \int_Y f_\infty(x_e) B_e^T D_e B_e \, dV \overline{U}.$$  \hspace{1cm} (29)

Following the same procedure as in Section 2.3.1 and with decomposition of $B = B_q p$, the deviatoric stiffness of an element $(e)$ in Eq. (29) can then be derived as

$$\Delta k^{(e)} = \int_Y f_\infty(x_e) B_e^T D_e B_e \, dV = \sum_{r=1}^{n_r} C_r^{(e)} \hat{Y}_r^{(e)}.$$  \hspace{1cm} (30)

In this case, the random variable corresponding to $\hat{Y}_r^{(e)}$ is defined with the higher order stochastic field $f_\infty(x)$ as $\hat{X}_ij^{(e)} = \int_Y f_\infty(x)p_j \, dV^e$. The definition of the matrix $C_r^{(e)}$ and the index $r$ of the random variable $\hat{Y}_r^{(e)}$ are defined as in Section 2.3.1.

4.2. Mean displacement

As noted in the mathematical form of the random variables $\hat{X}_ij^{(e)}$ or $\hat{Y}_r^{(e)}$, some of them have non-zero mean values. Therefore, if we perform a mean operation on the displacement vector in Eq. (29), the mean is obtained,

$$U_o = E[U(Y)] = \overline{U(Y)} - K^{-1} \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C_r^{(e)} E[\hat{Y}_r^{(e)}] \overline{U}.$$  \hspace{1cm} (31)

Since we assume that the stochastic field $f(x)$ is zero-mean Gaussian, $E[\hat{Y}_r^{(e)}]$ is active only for a stochastic field in an even power. Therefore, if we write the random variable as

$$\hat{Y}_r^{(e)} = \hat{Y}_r^{(e)} + \tilde{Y}_r^{(e)} = \sum_{k=1}^{\infty} \int_{Y^e} f^{2k-1}(x) p_r \, dV^e - \sum_{k=1}^{\infty} \int_{Y^e} f^{2k}(x) p_r \, dV^e,$$  \hspace{1cm} (32)

the mean displacement is given by

$$U_o = E[U(Y)] = \overline{U(Y)} + K^{-1} \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C_r^{(e)} E[\hat{Y}_r^{(e)}] \overline{U},$$  \hspace{1cm} (33)

because $E[\tilde{Y}_r^{(e)}] = 0$ for a zero-mean Gaussian random variable. Here, it is worth noting that the mean in Eq. (33) is equivalent to Eq. (12).
4.3. Covariance of displacement

Following the definition of the covariance operation, we determine the mean-centered displacement in advance, which is given by Eqs. (29), (30) and (33),

\[
\Delta \mathbf{U}(Y) = \mathbf{U}(Y) - E[\mathbf{U}(Y)]
\]

\[
= -\mathbf{K}^{-1} \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C^{(e)}_r \tilde{Y}^{(e)}_r \mathbf{U} - \mathbf{K}^{-1} \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C^{(e)}_r E[\tilde{Y}^{(e)}_r] \mathbf{U}
\]

\[
= \sigma_A + \bar{\sigma}.
\]

Accordingly, the covariance is

\[
\text{Cov}[\mathbf{U}(Y)] = E[\Delta \mathbf{U}(Y) \Delta \mathbf{U}(Y)^T]
\]

\[
= E[(\sigma_A + \bar{\sigma})(\sigma_A + \bar{\sigma})^T]
\]

\[
= E[\sigma_A \sigma_A^T] - E[\bar{\sigma} \bar{\sigma}^T].
\]

The first term of the covariance can be given by

\[
E[\sigma_A \sigma_A^T] = \mathbf{K}^{-1} \left( \sum_{e=1}^{n_e} \sum_{f=1}^{n_f} \sum_{r=1}^{n_r} \sum_{s=1}^{n_s} C^{(e)}_r \tilde{Y}^{(e)}_r \mathbf{U} \mathbf{U}^T C^{(f)}_s \tilde{Y}^{(f)}_s \right) \mathbf{K}^{-T},
\]

which is equivalent to Eq. (15). The second term is evaluated as

\[
E[\bar{\sigma} \bar{\sigma}^T] = \mathbf{K}^{-1} \left( \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C^{(e)}_r E[\tilde{Y}^{(e)}_r] \mathbf{U} \mathbf{U}^T \left( \sum_{e=1}^{n_e} \sum_{r=1}^{n_r} C^{(e)}_r E[\tilde{Y}^{(e)}_r] \right) \right) \mathbf{K}^{-T},
\]

which is equivalent to Eq. (17). In Eq. (36), the summation in parentheses can be rearranged as

\[
E[\Delta k^{(e)} \mathbf{U} \mathbf{U}^T \Delta k^{(f)/T}] = \int_{\gamma_\text{ex}} \int_{\gamma_\text{in}} E[f_{\text{ex}}(x_i) f_{\text{ex}}(x_j)] \tilde{k}^{(e)} \mathbf{U} \mathbf{U}^T \tilde{k}^{(f)} \mathbf{d}V^{(e)} \mathbf{d}V^{(f)},
\]

where the expectation operation in the integrand can be replaced by modified auto-correlation functions defined as

\[
E[f_{\text{ex}}(x_i) f_{\text{ex}}(x_j)] = R_{f_{\text{ex}},f_{\text{ex}}} (\xi = x_j - x_i),
\]

and the stiffness notation in Eq. (38) is an abbreviation of \( \tilde{k}^{(e)} = B^{(e)^T} \mathbf{D} k^{(e)} \).

In constructing the modified auto-correlation function in Eq. (39), we employed the general expression for the \( n \)th moment of random variables as follows:

\[
E[X_1 X_2 \cdots X_n] = \sum_{k \neq j} E[X_k X_j] E[X_k X_j]
\]

As expected, implementation of Eq. (40) is complex for order \( n \) higher than 4. Therefore, an independent computer program is developed in order to generate a source code for the modified auto-correlation functions.

For general statically indeterminate finite element systems, even if the loads are assumed to be deterministic, the statically indeterminate reactions in general are obtained to be functions of the stochastic field that describes the random material properties. In such a case, we can use the first or higher order approximation of the response displacements, e.g. Eqs. (2), (10) and (29), in the evaluation of statistical terms of the reaction vector. The pseudo-algorithmic chart in Fig. 5 shows the flow of implementation of the proposed scheme.

4.4. Extra remarks

Even if the structure behaves in a linear manner, not to mention the case of non-linear behavior, the response statistics subjected to the Gaussian random variable are to be non-Gaussian. This is attributable to the inversion of the system matrix involved in the solution process, which renders the probability of
response non-Gaussianity. This means that the response variability will be different from that of the input statistics. This can also be deduced from Eqs. (19) and (27), which show that the response statistics are dependent on \( f_1(x) \), which is non-Gaussian, but not on \( f(x) \), the original Gaussian stochastic field. In the case of the first-order expansion methods, however, the response variability is obtained to be the same as the input statistics due to the linear approximation of the non-Gaussian stochastic field \( f_1(x) \), i.e. \( f_1(x) \approx f(x) \). Thus, the difference in the resulting statistics between the MCS and the first-order expansion method is increased as the coefficient of variation of the stochastic field increases, as has been demonstrated in various previous works (Chakraborty and Bhattacharyya, 2002; Kaminski, 2001; Noh, 2005a,b; Yamazaki et al., 1988). In the formulation proposed in this paper, it should be noted that Eq. (39) implies the non-Gaussianity of the response, thus relieving the degree of discrepancy with MCS.

Strictly speaking, it is not possible to deal with Gaussian random variables because the effect of the negative tail of the Gaussian probability density function is tremendous, making the response tend to infinity and thus deteriorating the response statistics. The effect of the negative tail increases remarkably when the standard deviation of the stochastic field is relatively high. In this case the probability of the negative system parameter is extremely high, as can be noticed in Eq. (1) when \( f(x) < -1.0 \), such that the response variability becomes immeasurable (see also the conceptual drawing in Fig. 6 for a large \( \sigma \)). In practice, however, we frequently assume the random variable as Gaussian because the uncertain characteristics of random media are well described by this probabilistic distribution when the standard deviation \( \sigma \) is in an acceptable range (see...
In this study, we also assume that the probabilistic distribution of the random Young's modulus is within the practical limit.

In the Monte Carlo analysis for the stochastic field having a coefficient of variation of 0.25, it is highly probable that negative material properties might be encountered. If the MCS analysis is performed excluding only the samples that contain negative material properties, the statistical moments are, for certain, contaminated. Therefore, we discarded total samples generated if any of random number is smaller than $-1.0$. The results given in this paper are obtained with a set of samples that do not have any irregular, i.e., less than one, random numbers. For a COV of 0.25, the largest value taken in this study, the probability $p(x < -1)$ is about 0.0032%.

5. Numerical verifications

In order to validate the proposed formulation, we choose two example structures: in-plane and plate bending problems. The response variability is evaluated only for a stochastic field of standard deviation up to 0.25, because the probability of a negative elastic modulus is relatively high for a Gaussian random variable with higher standard deviation. It is evident that the response becomes very sensitive to the negative tail of the probability distribution, since the elastic modulus near zero makes the response tend to infinity. In order to show the adequacy of the outcomes, the numerical results obtained by the proposed scheme are compared with those yielded by a MCS. In compliance with the requirement of the random field generation scheme employed (Yamazaki and Shinozuka, 1990), $4 \times n_f \times n_{elem}$ number of samples are generated and used in the direct MCS. Specifically, we take $n_f = 5$, $n_{elem} = 6 \times 6$, and use 9 data for each finite element for a local averaging scheme; therefore, 6480 samples are used in the direct MCS.

For the original auto-correlation function, we choose a function

$$R_f(\xi) = \sigma_f^2 e^{-|\xi_j|/b_j - |\xi_i|/b_i}, \quad \text{(41)}$$

where $\xi$ is a separation vector defined by the distance between two distinct points under consideration, $\xi_1$, the components of $\xi$, $b_i$ the correlation distance in the $i$-direction, and $\sigma_f^2$ the variance of the stochastic field. We consider the condition $b_1 = b_2 = b$ in the numerical demonstrations. In the following, the COV (coefficient of variation) denotes

$$\text{COV} = \sqrt{\frac{\sigma_R^2}{\mu_R^2}}, \quad \text{(42)}$$

where $\sigma_R^2$ signifies the variance of response $R$ and $\mu_R$ the mean response.

5.1. In-plane problem

Let us consider a $10 \times 10$ in-plane structure subject to a uniform load on the upper edge as shown in Fig. 7. The mean elastic modulus is assumed as $2.1 \times 10^6$, and the mean Poisson’s ratio as 0.20. Any unit can be
employed if it is used in a consistent manner. The example structure is modeled with a displacement-based 4-
node finite element. In this analysis, the structure in the plane-strain condition only is considered, because the
results in the plane-stress condition can be obtained with ease and of a similar quality, as given in the follow-
ing. The mean thickness is taken as 1.0. The response variability is found at point A in Fig. 7.

5.1.1. Convergence check

In Fig. 8, the convergence characteristic of the response variability in terms of the highest order \( m \) in Eq.
(44) is illustrated. The “relative improvement” in the response COV is evaluated as

\[
\text{Relative improvement}_{(k)} = \frac{\delta \text{COV}_{(k)}}{\text{COV}_{(k)}} \times 100; \quad \delta \text{COV}_{(k)} = \text{COV}_{(k+1)} - \text{COV}_{(k)}.
\]

In Eq. (43), the subscript \( (k) \) denotes the highest order \( m-1 \) in Eq. (44), as noted in the title of the abscissa of
Fig. 8. For example, the percentile of relative improvement for \( k = 3 \) depends on the COVs of \( m = 3 \) and 4,
and is evaluated as the ratio of deviation of the COV between \( m = 3 \) and 4 (\( \delta \text{COV}_{(3)} = \text{COV}_{(4)} - \text{COV}_{(3)} \)) to
the COV for \( m = 3 \) (COV(3)).

The convergence is observed to be satisfactory even though the rate of convergence decreases as the COV of
the stochastic field increases. The response pattern for the “relative improvement” is attributable to the fact
that the stochastic field function in the odd power does contributes only to the variance of response without
making any contribution to the mean response. This means that the COV of the response is evaluated with the same mean value for \( m = 2 \) and \( 3(k = 1 \text{ and } 2) \), and \( m = 4 \) and \( 5(k = 3 \text{ and } 4) \), and so on.

Based on observations from Fig. 8, the higher order stochastic field function is truncated as

\[
f_{\infty}(x) \approx \sum_{k=1}^{m} (-1)^{k+1} f^k(x),
\]

where \( m \) is taken as 6, and employed in the numerical analyses. Even though there is not any theoretical base in the predetermination of the highest order \( m \) in Eq. (44), \( m = 6 \) or 7 might be appropriate as demonstrated in the following numerical examples. Furthermore, it is probable that the highest order \( m \) is dependent on a given problem. Due to the computational difficulties, the use of the proposed scheme is not feasible if the order \( m \) of the stochastic field in Eq. (44) is greater than 7.

5.1.2. Comparison with conventional methods

In order to validate the proposed scheme, we compare the proposed scheme and the conventional linear and higher order methods introduced by Shinozuka and Yamazaki (1995). For this purpose, the in-plane problem shown in Fig. 7 is analyzed.

As shown in Fig. 9, the response variability obtained by the proposed scheme agrees well with those given in the literature. Here, the discrepancy between the first- and second-order methods is very small, as also noted by Shinozuka and Yamazaki (1995). In order to demonstrate the validity of the proposed scheme in comparison with the Monte Carlo analysis, the results for the case when \( m = 6 \) (see Eq. (44)) are also compared with the results of a MCS given by Shinozuka and Yamazaki (1995).

5.1.3. Numerical investigation for the case of random field

In order to investigate the discrepancy between the original random field and that obtained in the matrix \( J \), we examine the random process at specific degree of freedoms (d.o.f.s): two points, A and B, in Fig. 7. The locations in the matrix \( J \) of the corresponding d.o.f.s are illustrated in Fig. 10. The standard deviation (SD) of the stochastic field is assumed to be 0.2, and the correlation distance \( b = 10.0 \). Note that this investigation corresponds to the MCS.

The random processes for specific d.o.f.s in the matrix \( J \) are shown in Figs. 11–13. As expected, the random process at \((N, N)\) shows exactly the same statistics as the pre-assigned statistics having the mean and SD as 0.0 and 0.2, respectively. In the case of \((m, m)\), the random process is obtained to have a zero mean but with some discrepancies in the second moment. The SD for this d.o.f. (\(y\)-displacement at point B) is obtained to be 0.1849, slightly less than 0.2. This is attributable to a weighting effect of the related stiffness at point B, as shown in Eqs. (23b) and (24b). As noted in Section 3.2, the off-diagonal term shows a random process almost equal to zero, as depicted in Fig. 13.

![Fig. 9. Comparison with results in the literature.](image-url)
Fig. 14 shows the variation of the SD of the random process for the d.o.f. of y-displacement at point B as a function of the correlation distance \( b \). The SD is also affected by refinement of the finite element mesh, and converges to the SD of the stochastic field (0.2) as the correlation length \( b \) increases, which corresponds to the case of a random variable (see Section 3.1).
5.1.4. Results for in-plane problem

The effect of correlation distance $b$ of the stochastic field on the response variability is shown in Fig. 15, where the results of the proposed scheme are compared with those of a MCS for a standard deviation (SD) of the stochastic field 0.1, 0.2, and 0.25. In Fig. 15, $X$ and $Y$ signify displacement components in the $x$- and $y$-directions, respectively. The results designated by a dotted line denote the corresponding results of the MCS. As shown in Fig. 15, the agreement between the proposed scheme and MCS is observed to be outstanding even for a stochastic field with a high intensity of uncertainty.
The additional response variability obtained by the proposed scheme is shown in Fig. 16 for the displacement in the Y-direction and for various values of correlation distance $b$. It is observed that the increasing rate of response variability is accelerated together with the increase in the COV of the stochastic field with noticeable agreement between the proposed scheme and MCS. It is not possible to obtain any additional response variability using the original first-order weighted integral method; which is attributable to the linear relationship between the COV of the stochastic field and the response variability. The additional response variability when $\log(b) = 3$ reaches roughly 28% and 27% for the MCS and the proposed scheme, respectively, when the COV of the stochastic field is 0.25.

Fig. 17 shows the fields of Eq. (38). It is noteworthy that Eq. (38) is a covariance of load-equivalent terms, $\Delta k^\epsilon U$, and the plots are observed to be highly correlated with the corresponding loading condition of the example structure.

Even though the plot in Fig. 17 fluctuates relatively depending on the loading condition, the field of the response COV over the structural domain is obtained to be constant, as shown in Fig. 18a, for the MCS. As expected, similar constant fields are obtained in the proposed weighted integral scheme, and the difference between the MCS and the proposed scheme is depicted in Fig. 18b. The deviation of the proposed scheme from the MCS is less than 1% (refer also to Fig. 15 for $\log(b) = 3.0$).

5.2. Plate bending problem

As a second example, we consider the plate bending problem shown in Fig. 19. A square plate of dimensions $20 \times 20$ with uniform thickness is subjected to uniform pressure normal to the flat surface ($-z$-direction) and all edges are simply supported. Due to symmetry, we consider only a one-quarter model employing symmetric boundary conditions. The mean elastic modulus is assumed as 10920.0, and the mean Poisson's ratio as 0.30. The thickness of the plate is taken as 1.0. In modeling the plate problem, we adopted the 9-node Heterosis plate element (Hughes and Cohen, 1978) that uses different shape functions for transverse displacement and nodal rotations.

Fig. 17. Fields of Eq. (38) in the proposed formulation. (a) $X$-component. (b) $Y$-component.
In the plate bending problem, we also obtain a convergence rate that is similar to the case of the in-plane problem in Section 5.1.4.

The effect of the correlation distance $b$ of the stochastic field on the variability of the transverse displacement is shown in Fig. 20, where the results of the proposed formulation are compared with those of the MCS.
Fig. 20. Comparison of COV variation as a function of correlation distance $b$. 

Fig. 21. Additional response variability: plate bending.

Fig. 22. Fields for Eq. (38) in the proposed formulation. (a) $X$-rotation component. (b) $Y$-rotation component. (c) Transverse displacement component.
for the same cases analyzed in the previous example. In these analyses, the agreement between the proposed weighted integral scheme and the MCS is also observed to be outstanding.

The additional response variability obtained by the proposed scheme is shown in Fig. 21 for various values of correlation distance $b$. We observe that the increasing rate of response variability is accelerated with an increase in the coefficient of variation of the stochastic field and also there appears good agreement between the proposed scheme and the MCS. The quantity of additional variability of the plate structure is slightly larger than that of the in-plane plate. In the plate structure under consideration, the additional response variability when $\log(b) = 3$ reaches about 33% and 32% for the MCS and the proposed scheme, respectively, when the COV of the stochastic field is 0.25. As expected, the additional response variability is negligible if the COV of the stochastic field is 0.1, which shows the first-order approximate methods gives acceptable results if the intensity of uncertainty is relatively small.

In Fig. 22, the fields of Eq. (38) are displayed for the plate example structure. As analogous to the previous in-plane example, the plots are observed to be highly correlated with the loading condition of the example.

![Fig. 23. Comparison between MCS and proposed scheme. (a) COV field of MCS in X- and Y-rotation and transverse displacement (COV of the stochastic field = 0.20). (b) Percent difference between MCS and proposed scheme in COV.](image-url)
plate bending structure. The peaks in Figs. 22a and b, as well as the dimples in Fig. 22c, are attributable to the characteristic of the employed Heterosis element in which different shape functions are used to interpolate transverse displacement and rotations.

Even though the plot in Fig. 22 shows a characteristic view of the distribution of the force-equivalent terms that depend on the loading condition, the field of the response COV over the structural domain is obtained to be constant as shown in Fig. 23a. Similar constant fields are obtained in the proposed weighted integral scheme, and the difference between the MCS and the proposed scheme is shown in Fig. 23b. The deviation of the proposed scheme from the MCS is observed to be less than 1% (refer to Fig. 20 for log(b) = 3.0).

5.3. Mean and variance in MCS and the weighted integral scheme proposed

The variation of the mean and standard deviation of displacement is illustrated in Fig. 24 as a function of the correlation distance $b$. The mean is overestimated to a degree in the proposed weighted integral scheme when compared with the MCS. However, the increasing trend found for larger values of correlation distance $b$ is similar in both analyses schemes.

The origin of the increase of the mean response in the proposed scheme can be deduced from Eq. (31) or Eq. (33), where the expectation of the random variable $E[Y^e_r]$ or $E[Y^p_r]$ becomes larger as the correlation distance $b$ increases. The explanation for the increase of the mean response in the MCS, however, is somewhat different: it cannot be deduced from any mathematical equation used in the MCS but can only be deduced from a statistical point of view. Specifically, when $b$ tends to infinity, it is well known that the stochastic field tends to the random variable case (Deodatis et al., 2003). In this case, the probabilistic distribution of the response assumes a form analogous to the log-normal distribution, especially when the uncertain elastic mod-

![Fig. 24. Mean and standard deviation of displacement as a function of correlation distance $b$: in-plane structure. (a) X-displacement at point A in Fig. 1. (b) Y-displacement at point A in Fig. 1.](image)

![Fig. 25. Probability density functions (pdfs) of random variable and corresponding response.](image)
ulus is taken into account. In this type of distribution, the distribution has its mass eccentric to the left side and the mean is to slightly greater than that of a normal distribution (Fig. 25). Accordingly, the mean increases as $b$ increases.

The variance is also observed to be similar in both analyses except for the fact that the variance in the MCS is slightly larger than that in the proposed weighted integral scheme.

In the case of plate bending, as shown in Fig. 26, the same tendency in the mean and standard deviation is observed as in the case of the in-plane problem.

5.4. Comparison of computational efficiency

Considering that the stochastic analyses are considerably more involved when compared to deterministic methods, the aspect of computational efficiency cannot be ignored. In this study, the computation time of the proposed scheme is compared with that of the MCS in terms of the highest order $m$ in Eq. (44). We also examined the effect of the number of cosine terms, NF, involved in the random field generation technique (Yamazaki and Shinozuka, 1990). As noted in Yamazaki and Shinozuka (1990), as more cosine terms are included in the generation, the generated random fields become closer to Gaussian. This means that we need...
to use as many random samples as possible in order to satisfy the Gaussian assumption on the random field. In Fig. 27, we investigate four cases: NF=5, 10, 15 and 20.

In the case of the proposed scheme, the computation time increases rapidly, as expected, as the highest order $m$ increases. The increase, however, remains within an acceptable range even for $m = 6$ when compared to the MCS, and is found to be moderate for lesser orders. Notably, the computation time of the proposed scheme for $m = 5$ is much less than that of the MCS adopting NF=5, which suggests that the proposed scheme is highly efficient in terms of computational effort. In the sense that the computation time increases tremendously as the order $m$ increases, however, the proposed method and virtually all the non-statistical numerical approaches with higher order expansion are inferior to the MCS as MCS includes implicitly all infinite terms that appear in the infinite series expansion.

6. Conclusions

Based on a theoretical formulation of the weighted integral scheme that employs an infinite series expansion, and by comparison of this formulation with the Monte Carlo analysis, we suggested a higher order stochastic field function. Employing the suggested higher order stochastic field, we proposed a new weighted integral formulation that provides response variability comparable to the MCS for stochastic systems with a relatively high degree of uncertainty. The proposed formulation is found to be equivalent to the theoretical infinite series expansion scheme.

In order to demonstrate the adequacy of the proposed formulation, two exemplificative problems (an in-plane problem and a plate bending problem) were analyzed. Outstanding agreement of response variability between the proposed scheme and the MCS was observed even for cases where the coefficient of variation of the stochastic field was as high as 0.25.

As expected, it was also observed that the increase in the additional response variability is accelerated as the coefficient of variation of the stochastic field becomes larger. This phenomenon is well described by the proposed scheme showing noticeable agreement with the MCS. The additional response variability is estimated to be up to 28% and 33% of the coefficient of variation of the stochastic field for in-plane and plate structures, respectively, for the MCS, and 27% and 32% for the proposed scheme. Regarding the mean and standard deviation of the response, it is shown that these two statistical terms in the proposed scheme vary in an analogous manner to the MCS. In particular, we observed a monotonous increase in the mean response as the correlation distance of the stochastic field becomes larger. A short explanation on this phenomenon was presented for the MCS and the proposed scheme, respectively.

References


