Finite Element Analysis of Structures

Mid-term Exam, Fall 2009 (30pt)

1. As shown in Fig. 1, we model a truss structure of uniform area (length = 2m, Area = Am^2) subjected to a uniform body force (\( \vec{f}_B = 2\vec{e}_x N/m \)) using a 3-node truss finite element. Assume that all DOFs for y- and z-directional translations are prescribed to be zero.

![Figure 1. A truss problem modeled by a 3-node truss element](image)

Find

(a) Interpolation of displacement, \( u(x) = \mathbf{H} \tilde{U} \). (3pt)

(b) Relation between strain and nodal displacements, \( \varepsilon_{xx} = \mathbf{B} \tilde{U} \). (2pt)

(c) Equilibrium equation, \( \mathbf{K} \tilde{U} = \tilde{\mathbf{R}} \). (5pt)

(d) Displacement and stress at \( x = -0.5m \). (5pt)

Note that \( u(x) \) is a quadratic polynomial, \( \tilde{U} = \{U_2, U_3\}^T \) is in (a), (b) and (c), and \( E \) denotes Young’s modulus.
2. We have a plane stress problem \((4m \times 4m \times t)\) subjected to a point load, \(\vec{p} = -4\vec{e}_y \, N\). The displacement BC is given as shown in Fig. 2. The 2-D problem is modeled by the uniform 2x2 mesh of plane stress finite elements.

The thickness is \(t\),

\[
\mathbf{C} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} \quad \text{N/m}^2
\]

![Figure 2. A plane stress problem subjected to a point load](image)

Calculate

(a) Diagonal component of the total stiffness matrix corresponding to \(U_7\), \(K_{\text{UU}_i}\). (5pt)

(b) Components of the load vector, \(R_{U_9}\) and \(R_{U_{10}}\). (5pt)

(c) Displacement \(U_{10}\) and strain energy stored in the structure. (2pt)

Assuming a dynamic analysis of the structure (mass density \(\rho\)), find

(d) Component of the total mass matrix, \(M_{\text{UU}_{10}}\). (3pt)
1. Using two 3-node triangular plane stress elements and two 2-node truss elements, we model a structure subjected to an X-directional tip loading \((P = 2000N)\), see Fig. 1. The properties are given:
- \(E = 2 \times 10^6 \text{N/}m^2\), \(\nu = 0.3\) and thickness = 0.1\(m\) for the plane stress elements,
- \(E = 2 \times 10^6 \text{N/}m^2\) and sectional area = 0.05\(m^2\) each for the truss elements.

Fig.1. A structural model in the XY and YZ planes

Considering "linear elastic analysis", calculate:

(a) the Jacobian matrix \(J\) and the determinant \(\text{det} J\) of the shaded triangular element (5 pt)
(b) the relation between strain and nodal displacements \(B(r,s)\) for the shaded triangular element (5 pt)
(c) the total stiffness \(K_{U_2U_2}\) corresponding to \(U_2\) (5 pt)
(d) the displacement \(U_2\) (5 pt)

Note that the 3-node triangular plane stress element has the interpolation functions:
\[
h_1 = 1 - r - s, \quad h_2 = r, \quad h_3 = s,
\]
and the material law is
\[
C = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu \\
\end{bmatrix} N/m^2.
\]
2. Consider the single degree of freedom system shown in Fig. 2.

![Nonlinear Spring Diagram](image)

Fig. 2. A nonlinear spring \( C = 0.1 \)

In the figure, \( U \) is the tip displacement, \( F \) is the internal force of the spring and \( R \) is the external loading. Recall that the stiffness of the system is \( K = \frac{\partial F}{\partial U} \).

(a) Write “incremental equilibrium equation” to calculate the response of the system. (5 pt)

(b) We want to find the tip displacement \( U \) corresponding to \( R = 1.0 \). Using the full Newton-Raphson method, perform iterations until the solution is converged \( |R - F^{(i)}| < 10^{-2} \). (5 pt)

3. Consider the 4-node plane stress element shown, where \( ^0 \tau_{11} = ^0 \tau_{22} = ^0 \tau_{12} = 0 \).

Using the total Lagrangian formulation, calculate the nonlinear strain incremental stiffness matrix \( ^0 K_{NL} \). (10 pt)

![4-Node Plane Stress Element Diagram](image)

Fig. 3. A 4-node plane stress element under rotation

** If you think that the calculation is too heavy, calculate any component of \( ^0 K_{NL} \).
Finite Element Analysis of Structures

Mid-term Exam, Fall 2010 (30pt)

We have a plane stress problem subjected to point loads, $P_1$ and $P_2$, and the displacement BC as shown in Fig. The 2D problem is modeled by a 4-node plane stress finite element,

$$\text{thickness} = 1.0 \text{m}, \quad C = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} \text{ N/m}^2.$$

![Figure. A plane stress problem subjected to point loads](image)

1. Considering the isoparametric procedure, evaluate
   (a) Global coordinate $(X, Y)$ at $r=s=0$ (2pt)
   (b) Jacobian matrix at $r=s=0$ (3pt)
   (c) $B$-matrix at $r=s=0$ (5pt)
   (d) $B^T CB$ at $r=s=0$ corresponding to $K_{U_1 U_2}$ (5pt)
   and calculate
   (e) Components of the load vector, $R_{U_1}$ and $R_{U_2}$ (5pt).

2. Let's assume that we obtain the displacement $U_1 = U_2 = 0.001 m$, calculate
   (f) Displacements at $r=s=0$ (3pt)
   (g) Strains and stresses at $r=s=0$ (5pt)
   (h) Strain energy stored in the structure (2pt)
Finite Element Analysis of Structures

Final exam, 2010-12-20 (40pt.)

1. Let us consider a 2-node iso-beam (Timoshenko beam) element as shown in Fig. 1.

The geometry and displacement interpolation functions and the nodal displacement vector are given:

\[
\begin{align*}
(x, y) &= \left( \sum \frac{h_i x_i}{a} \right), \\
(u, v) &= \left( \sum \frac{h_i u_i - \frac{s}{2} \sum a h_i \theta_i}{\sum h_i v_i} \right) \\
\bar{U} &= [u_1, u_2, v_1, v_2, \theta_1, \theta_2]^T.
\end{align*}
\]

The material law used is

\[
\begin{pmatrix}
\tau_{xx} \\
\tau_{xy}
\end{pmatrix} = \mathbf{C} \begin{pmatrix}
\varepsilon_{xx} \\
\gamma_{xy}
\end{pmatrix} \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}.
\]

Considering the standard isoparametric procedure for linear elastic problems, calculate

(a) Jacobian matrix (5pt.)

(b) \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) in terms of the nodal displacement vector (5pt.)

(c) \( \mathbf{B} \)-matrix (5pt.)

(d) Components of the \( \mathbf{K} \)-matrix, \( K_{u_1 u_1} \) and \( K_{\theta_1 v_1} \) (5pt.)
2. Considering the total Lagrangian formulation, we analyze an elastic bar problem by using one 2-node truss element. Assume that the cross section does not change during deformation and the material law is

\[ \sigma_{0x} = E_0 \varepsilon_{0x}. \]

![Diagram of an elastic bar problem](image)

**Fig. 2. An elastic bar problem**

For the configuration at time 0, calculate

(a) Tangential stiffness, \( \mathbf{K}_{u_1 u_1} \) (5pt.)

For the configuration at time \( t \), calculate

(b) Linear part of the tangential stiffness, \( \mathbf{K}_L u_1 u_1 \) (5pt.)

(c) Nonlinear part of the tangential stiffness, \( \mathbf{K}_{NL} u_1 u_1 \) (5pt.)

(d) Component of the force vector \( \mathbf{F} \) corresponding to \( u_1 \) (5pt.)
1. (25pt.) Let us consider a 2D 5-node plane stress element as shown in Figure.

(a) Find the shape function \( h_5 \) when \( h_1 = \frac{1}{4} r(1 + r)(1 + s) \), \( h_2 = -\frac{3}{4} r(1 - r)(1 + s) \), \( h_3 = \frac{1}{4} (1 - r)(1 - s) \), \( h_4 = \frac{1}{4} (1 + r)(1 - s) \).

(b) Find \((x_p, y_p)\) corresponding to \( r = -0.5 \) and \( s = 1 \).

(c) Calculate the Jacobian matrix \( J \) and \( det J \) at \( r = s = 0 \).

(d) Calculate the column vector of the strain-displacement matrix \( B \) at \( r = s = 0 \) corresponding to \( U_1 \). Note that \( \vec{\varepsilon} = \{ \varepsilon_{xx} \ \varepsilon_{yy} \ \gamma_{xy} \}^T \).

(e) Calculate the components \( R_{V_1}, R_{V_2} \) and \( R_{V_5} \) of the load vector for the point load \( \vec{P} = -p \vec{e}_y \) applied at \((x_p, y_p)\).

2. (15pt.) We study the convergence of finite element solutions in potential energy. The potential energy is defined by

\[
\pi(\vec{\vartheta}) = \frac{1}{2} a(\vec{\vartheta}, \vec{\vartheta}) - (\vec{f}, \vec{\vartheta}),
\]

in which \( \vec{\vartheta} \) is the displacement vector. Prove \( \pi(\vec{u}) \leq \pi(\vec{u}_h) \), where \( \vec{u} \) is the exact solution and \( \vec{u}_h \) is the finite element solution, and explain the meaning of the inequality.
Finite Element Analysis of Structures

Final Exam, Fall, 2011-12-20 (50pt.)

1. Let us consider a truss structure shown in Fig. 1. The structure consists of two bar members connected by a pin and the pin connection is supported by a spring. Each bar member of length $L$ is modeled by a 2-node bar element.

(a) (5pt.) Find the $B$-matrix of the bar element.

(b) (10pt.) Find the stiffness matrix of the truss problem, $\bar{U} = [U_1 \ U_2]^T$.

(c) (5pt.) Find the tension forces of the bar members when $k = 1$ and $U_1 = U_2 = 0.1$.

(d) (10pt.) When $k = 0$ and $R_2 = 1$, what happens in the linear analysis? Plot the nonlinear load-displacement curve that you expect (between $R_2$ and $U_2$) as the load increases.
2. Let us consider the 4-node plane stress element shown in Fig. 2.

![Figure 2. A 2-D plane stress problem.](image)

At time \( t \), \( \dot{S}_{22} = 10 \) and all other stresses = 0 are given in the element. Assume that the material law with Young’s modulus \( E \) and Poisson’s ratio \( (\nu = 0) \) relates the incremental 2nd Piola-Kirchhoff stresses to the incremental Green-Lagrange strains and assume thickness = 1.0 at time = 0.

Using the total Lagrangian formulation, calculate the following.

(a) (5pt.) The components of the linear stiffness matrices \( \dot{0}K_L \) and \( \dot{\delta}K_L \) corresponding to \( \delta U_1 \) and \( U_1 \).

(b) (10pt.) The component of the nonlinear stiffness matrix \( \dot{0}K_{NL} \) corresponding to \( \delta U_1 \) and \( U_1 \).

(c) (5pt.) The component of the force vector \( \dot{0}F \) corresponding to \( U_2 \).
Finite Element Analysis of Structures

Mid-term Exam, Fall 2012 (40pt.)

1. (15pt.) As shown in Fig. 1, a 8-node 3D solid element is subjected to a uniformly distributed normal pressure $q$ (force per area). Calculate the nodal point consistent loads at nodes 1, 2, 3 and 4.

Fig. 1. A 8-node 3D solid element with a pressure loading.
2. (25pt.) Let us consider a triangular cantilever problem modeled by a 3-node plane stress element as shown Fig. 2. The force and displacement BCs are presented in Fig. 2, thickness is 1.0,

\[
\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad \text{and} \quad C = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad \text{with} \ \nu = 0.
\]

![Fig. 2. A triangular cantilever subjected to a point load \( P \)](image)

The shape functions of the 3-node element are given by

\[ h_1 = r, \quad h_2 = s \quad \text{and} \quad h_3 = 1 - r - s. \]

Calculate the followings

(a) Global coordinates \((X,Y)\) corresponding to \(r=s=0.5\)

(b) Jacobian matrix \(J\)

(c) \(B\)-matrix \([3 \times 2]\) corresponding to \(u_i\) and \(v_i\)

(d) Stiffness matrix \(K\) \([2 \times 2]\) corresponding to \(u_i\) and \(v_i\)

(e) Tip displacements \(u_i\) and \(v_i\), and stress at \(X=Y=0\)
1. We here want to extend our experience on finite element analysis of structures into heat transfer problems. For 2D heat transfer problems, the governing equations are given

\[ k \frac{\partial^2 \theta}{\partial x_i^2} + q^B = 0 \quad \text{in } V, \quad \text{(differential heat flow equilibrium equation)} \]

\[ \theta = \theta^s \quad \text{on } S_\theta, \quad \text{(essential boundary condition)} \]

\[ k \frac{\partial \theta}{\partial x_i} n_i = q^s \quad \text{on } S_q, \quad \text{(natural boundary condition)} \]

where \( k \) and \( \theta \) are the thermal conductivity and the temperature of the body, \( q^B \) is the rate of heat generated per unit volume, \( \theta^s \) is the surface temperature on \( S_\theta \), and \( q^s \) is the heat flux input on the surface \( S_q \), see Fig. 1.

![2D body subjected to heat transfer](image)

**Fig. 1.** A 2D body subjected to heat transfer \((S = S_\theta \cup S_q, \quad S_\theta \cap S_q = 0)\)

(a) (10pt.) Derive the principle of virtual temperatures given as follows

\[ \int_V \frac{\partial \theta}{\partial x_i} k \frac{\partial \theta}{\partial x_i} dV = \int_V \partial \theta \ q^B dV + \int_{S_q} \partial \theta \ q^s dS, \]

in which \( \partial \theta \) is the virtual temperature distribution \((\partial \theta = 0 \text{ on } S_\theta)\).

(Hint) Divergence theorem: \[ \int_V \frac{\partial f}{\partial x_i} dV = \int_{S_q} f \ n_i dS \] with a scalar function \( f \).
(b) (5pt.) Assume that the interpolation of temperature is $\theta = H \tilde{\theta}$, and the relation between $\tilde{\alpha} = \begin{bmatrix} \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{bmatrix}^T$ and the nodal temperature vector $\tilde{\theta}$ is $\tilde{\alpha} = B \tilde{\theta}$. Derive the finite element formulation for 2D heat transfer problems, $K \tilde{\theta} = \tilde{R}$.

(c) (10pt.) Considering the 4-node finite element shown in Fig. 2, find the component of stiffness matrix ($K_{11}$) corresponding to $\partial \theta_1$ and $\theta_1$.

(Hint) $\int_{-1}^{1} \int_{-1}^{1} (1 \pm x)^2 dxdy = \frac{16}{3}$, $\int_{-1}^{1} \int_{-1}^{1} (1 \pm x)(1 \mp y) dxdy = \int_{-1}^{1} \int_{-1}^{1} (1 \pm x)(1 \pm y) dxdy = 4$

Fig. 2. Four-node finite element for 2D heat transfer problems

2. The configurations of a body at time $0$, $t$ and $t + \Delta t$ and the second Piola-Kirchhoff stresses $\mathbf{S}$ for the plane strain four-node element are shown in Fig. 3.

$$\begin{align*}
^0_0 S_{11} &= 50 \\
^0_0 S_{22} &= -60 \\
^0_0 S_{33} &= -20 \\
^0 S_{12} = 0, S_{23} = 0, S_{31} &= 0
\end{align*}$$

Fig. 3. Four-node finite element subjected to stretching and rotation
(Unit thickness at all time steps)

Calculate the followings

(a) (5pt.) Deformation gradients $\dot{\mathbf{X}}$ and $\dot{\mathbf{X}}_{0}^{+\Delta t}$

(b) (5pt.) Cauchy stresses at time $t$, $\tau$

(c) (5pt.) Second Piola-Kirchoff stresses at time $t + \Delta t$, $\mathbf{S}_{0}^{+\Delta t}$

(d) (5pt.) Cauchy stresses at time $t + \Delta t$, $\mathbf{S}_{t}^{+\Delta t}$

(Hint) $\tau = \dot{\mathbf{X}}^{T} \mathbf{S} \dot{\mathbf{X}}^{T}$
1. (10 pt.) Let us consider a 2D finite element model shown in Fig. 1.

Fig. 1. A 2D cantilever plate problem. (a) Finite element model (2x2 mesh), (b) A 4-node plane stress element, (c) Stiffness matrix of the 4-node plane stress element.

Find the components of the global stiffness matrix, $K_{u_i u_j}$, $K_{u_i v_j}$ and $K_{u_j v_{i+1}}$, in terms of the components of $K_{e}^{(m)}$ in Fig. 1 (c).
2. (15 pt.) Let us consider a tapered bar problem and its finite element model in Fig. 2.

![Tapered Bar Problem Diagram]

Fig 2. A tapered bar problem \((E=\text{Young's modulus})\). (a) Problem description, (b) Finite element model \((R = qa)\).

The exact solution of this problem is given by

\[
    u_e(x) = \frac{RL}{Ea} \ln \left( \frac{2}{2 - x/L} \right).
\]

Using a 2-node bar element, the following FE solution is obtained

\[
    u_h(x) = h_1(x)u_1 + h_2(x)u_2 \quad \text{with} \quad h_1(x) = 1 - x/L, \ h_2(x) = x/L, \ u_1 = 0 \ \text{and} \ u_2 = \frac{2LR}{3aE}.
\]

The principle of virtual work specialized to this bar problem is given by

\[
    \int_0^L \frac{d\delta u}{dx} \frac{EA}{dx} \frac{du}{dx} dx = R\delta u_{1=L} \quad \text{with} \quad \left. u \right|_{x=0} = 0, \ \left. \delta u \right|_{x=0} = 0
\]

(a) For the following 4 cases, check whether the principle of virtual work is satisfied or not.

- \(u = u_e(x), \ \delta u = a_0h_2(x), \ \text{and} \ u = u_e(x), \ \delta u = a_0x^2\)
- \(u = u_h(x), \ \delta u = a_0h_2(x), \ \text{and} \ u = u_h(x), \ \delta u = a_0x^2\)

(Hint) \[
    \frac{du_e}{dx} = \frac{R}{Ea \left( 2 - x/L \right)}
\]

(b) Discuss the results.
3. (15 pt.) A 3-node bar finite element is shown in Fig. 3.

![3-node bar finite element diagram](image)

Fig 3. A 3-node bar finite element ($E$=Young’s modulus, $A$=area, $\rho$=density).

Considering the isoparametric procedure, the shape functions for the 3-node bar finite element are given by

$$
\begin{align*}
    h_1 &= \frac{1}{2} r(r-1), \\
    h_2 &= \frac{1}{2} r(r+1), \\
    h_3 &= (1 + r)(1 - r).
\end{align*}
$$

Calculate the followings

(a) Jacobian

(b) B-matrix, $B(r)$ when $\varepsilon_{x_3}(r) = B(r)u$ with $u = [u_1 \ u_2 \ u_3]^T$.

(c) Component of the stiffness matrix, $K_{u_1u_2}$.

(d) Component of the mass matrix, $M_{u_1u_2}$.
1. (15 pt.) A 3-node triangular plane stress element shown in Fig. 1 is subjected to the prescribed displacement $\Delta$.

![Diagram](image)

**Fig. 1.** A 3-node triangular plane stress element
(a) in the Cartesian coordinate system, (b) in the natural coordinate system.

Thickness = 0.1, \[ C = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - 2\nu) / 2 \end{bmatrix}, \quad E = 1 \text{ and } \nu = 0. \]

Shape functions: \[ h_1 = 1 - r - s, \quad h_2 = r, \quad h_3 = s \]

Calculate the followings

(a) Strain-displacement matrix, $B$
(b) 3 by 3 stiffness matrix where the boundary condition is imposed, $K$
(c) Displacements at nodes 1 and 2
(d) Reaction force corresponding to the prescribed displacement
2. (10 pt.) Consider a 4-node plane strain element in the configurations at time 0 and \( t \) shown in Fig. 2.

![Diagram of a 4-node plane strain element at time 0 and \( t \)](image)

Fig. 2. A 4-node plane strain element at time 0 and \( t \)

Calculate the followings at \( r_1 = r_2 = 0 \)

(a) Jacobian matrices, \( \mathbf{J}^0 \) and \( \mathbf{J}^t \)

(b) Deformation tensor, \( \mathbf{X}^t \)

(c) Green-Lagrange strain tensor at time \( t \), \( \mathbf{\varepsilon}^t \)
3. (15 pt.) Using the total Lagrangian formulation, we consider a truss structure modeled by two 2-node truss elements in Fig. 3. Assume that the cross section \( (A_0) \) is not changed during the deformation and the material law is given by

\[
_{\varepsilon_{11}}^tS_{11} = E_0^t \varepsilon_{11}
\]

When \( ^t u_2^2 = -3 \) for the configuration at time \( t \), evaluate the followings

(a) Component of the linear part of the tangent stiffness matrix \( (_{\varepsilon}^tK_L) \) corresponding to \( \Delta u_2 \) and \( u_2^2 \)

(b) Component of the nonlinear part of the tangent stiffness matrix \( (_{\varepsilon}^tK_{NL}) \) corresponding to \( \Delta u_2 \) and \( u_2^2 \)

(c) Component of the internal force vector \( (_tF) \) corresponding to \( u_2^2 \)

(Hint) Due to symmetry, you may consider a half-symmetric model.
Let us consider a thin membrane structure of thickness \( t \) subjected to a uniform temperature variation \( \Delta \theta = 1 \, ^\circ C \) as shown in Figure. The structure is clamped along left, right and bottom edges. We model the structure using a uniform 2x2 mesh of four 4-node isoparametric plane stress elements.

For this thermal expansion problem, we use the stress-strain law given by

\[
\tau = C(\varepsilon - \varepsilon^t) \quad \text{with} \quad \tau = \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix}, \quad C = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \end{bmatrix}, \quad \varepsilon^t = \alpha \Delta \theta, \quad \Delta \theta,
\]

in which \( \tau \) is the stress vector, \( \varepsilon \) is the total strain vector, \( \varepsilon^t \) is the thermal strain vector, \( C \) is the material law matrix with Young’s modulus \( E \), and \( \alpha \) is the thermal coefficient of expansion.

(a) (8pt.) Specialize the principle of virtual work to this problem considering the given stress-strain law. Write down the finite element formulation for the nodal load vector due to the thermal strain.

(b) (8pt.) Calculate the stiffness components \( k_{uv} \) of the element (1), see Figure (b).

Note that \( \int_{-1}^{1} \int_{-1}^{1} (1 + r)^2 drds = 16/3 \) and \( \int_{-1}^{1} \int_{-1}^{1} (1 + s)(1 - s) dsdr = 8/3 \).

(c) (8pt.) Construct the 2x2 total stiffness matrix \( K \) using \( k_{uv} \) of the corresponding nodal displacement vector is \( U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T \). Note that the problem is symmetric.

(d) (8pt.) Calculate the nodal load vector, \( R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}^T \).

(e) (8pt.) Assuming \( U_1 = 32/100 \) and \( U_2 = 8/100 \), calculate the stress jump \( \Delta \tau_{yy} \) at point A between the elements (1) and (3), see Figure (a).
Finite Element Analysis of Structures

Final Exam, Fall 2014 (40pt.)

1. (10pt.) Consider a single degree of freedom (DOF) system subjected to the force \( p(t) \) as shown in Figure 1.

For this SDOF system, the linear equation of motion is given by

\[
m\ddot{u}(t) + c\dot{u}(t) + ku(t) = p(t)
\]
with \( u(0) = 0 \) and \( \dot{u}(0) = 0 \), in which \( m \), \( c \), \( k \), and \( u \) are the mass, damping coefficient, spring constant and displacement, respectively.

Using the Central Difference Method (CDM) with time step size \( \Delta t = \Delta t_v \), calculate the response of the system from time 0 to 1 sec. (Fill in the blanks (a), (b) and (c) in Table 1.)

Table 1. Response of the SDOF system.

<table>
<thead>
<tr>
<th>Time [sec]</th>
<th>0</th>
<th>( \Delta t )</th>
<th>( 2\Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(t) )</td>
<td>(a)</td>
<td>(b)</td>
<td>(c)</td>
</tr>
</tbody>
</table>

Note that the critical time step size \( \Delta t_v = T_n / \pi \) and the free-vibration period of the system \( T_n \) is 0.5\( \pi \). In CDM, the following approximations are used for the discretization of time

\[
\dot{u}(t) = \frac{1}{2\Delta t}[u(t + \Delta t) - u(t - \Delta t)] \quad \text{and} \quad \ddot{u}(t) = \frac{1}{(\Delta t)^2}[u(t + 2\Delta t) - 2u(t) + u(t - \Delta t)].
\]
2. (15pt.) Let us consider a 4-node axisymmetric finite element as shown in Figure 2. The finite element is clamped along the bottom edge.

Figure 2. A 4-node axisymmetric finite element. (a) Global DOFs, (b) Local DOFs

For linear elastic analysis, the material law matrix $C$ and the strain vector $\varepsilon$ are given by

\[
C = \begin{bmatrix}
E & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
0 & 0 & 0.5E & 0 \\
0 & 0 & 0 & E
\end{bmatrix}
\quad \text{and} \quad
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy} \\
\varepsilon_{zz}
\end{bmatrix}
\]

with Young’s modulus $E$. $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, and $\varepsilon_{zz} = \frac{u}{x}$.

Using the isoparametric procedure, calculate the stiffness component $K_{U_iU_j}$.

Note that
\[
\int_{-1}^{1} \int_{-1}^{1} (4 + r)(1 + s)^2 dr ds = 64/3, \quad \int_{-1}^{1} \int_{-1}^{1} (4 + r)(1 + r)^2 dr ds = 24 \quad \text{and} \quad \int_{-1}^{1} \int_{-1}^{1} \frac{(1 + r)^2(1 + s)^2}{4 + r} ds dr = 8/5.
\]
3. (15pt.) Let us consider a plane strain element as shown in Figure 3.

Figure 3. A 2D plane strain element.

The Cauchy stress at time $t$, not including $\tau_{zz}$, is given by

$$
\tau = \begin{bmatrix}
0 & 2.0 \times 10^7 \\
2.0 \times 10^7 & 0
\end{bmatrix} \text{ Pa.}
$$

Using the Total Lagrangian Formulation, compute the component of the nonlinear stiffness $\mathbf{K}_{NL}$ corresponding to $U_1$ and $\delta U_1$. Note that $\mathbf{K} = \mathbf{K}_L + \mathbf{K}_{NL}$. 