Finite Element Analysis of Structures

Mid-term Exam, Fall 2009 (30pt)

1. As shown in Fig. 1, we model a truss structure of uniform area (length $= 2m$, Area$= Am^2$) subjected to a uniform body force ($\vec{f}_B = 2\vec{e}_x \ N/m$) using ***a 3-node truss finite element***. Assume that all DOFs for y- and z-directional translations are prescribed to be zero.

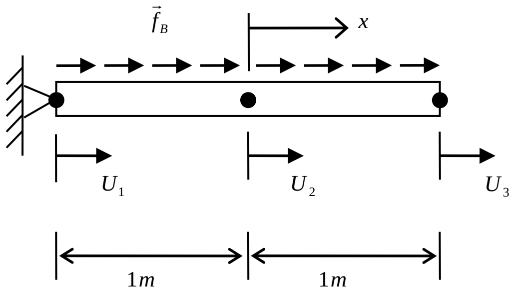


Figure 1. A truss problem modeled by a 3-node truss element

Find

(a) Interpolation of displacement, $u(x) = \mathbf{H}\vec{U}$. (3pt)
(b) Relation between strain and nodal displacements, $\varepsilon_{xx} = \mathbf{B}\vec{U}$. (2pt)
(c) Equilibrium equation, $\mathbf{K}\vec{U} = \vec{R}$. (5pt)
(d) Displacement and stress at $x = -0.5m$. (5pt)

Note that $u(x)$ is a quadratic polynomial, $\vec{U} = \{U_2 \ U_3\}^T$ is in (a), (b) and (c), and E denotes Young's modulus.

2. We have a plane stress problem ($4m \times 4m \times t$) subjected to a point load, $\vec{p} = -4\vec{e}_y\ N$. The displacement BC is given as shown in Fig. 2. The 2-D problem is modeled by the uniform 2x2 mesh of plane stress finite elements.

$$t = \text{thickness}, \quad \mathbf{C} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} N/m^2$$

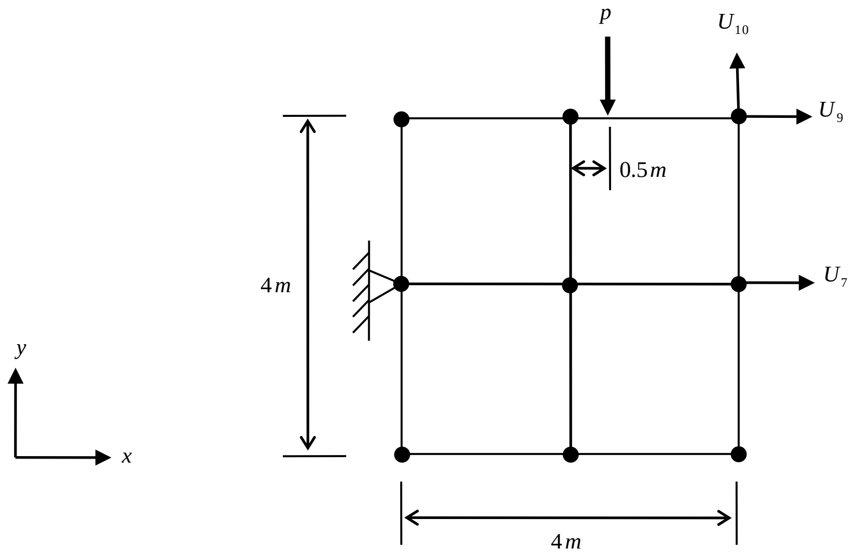


Figure 2. A plane stress problem subjected to a point load

Calculate

(a) Diagonal component of the total stiffness matrix corresponding to U_7, $K_{U_7U_7}$. (5pt)

(b) Components of the load vector, R_{U_9} and $R_{U_{10}}$. (5pt)

(c) Displacement U_{10} and strain energy stored in the structure. (2pt)

Assuming a dynamic analysis of the structure (mass density ρ), find

(d) Component of the total mass matrix, $M_{U_9U_{10}}$. (3pt)

Finite Element Analysis of Structures

Final Exam, Fall 2009 (40 pt)

1. Using *two 3-node triangular plane stress elements and two 2-node truss elements*, we model a structure subjected to an X-directional tip loading ($P = 2000N$), see Fig. 1. The properties are given
   - $E = 2\times10^6\ N/m^2$, $\nu = 0.3$ and thickness = $0.1m$ for the plane stress elements,
   - $E = 2\times10^6\ N/m^2$ and sectional area $= 0.05m^2$ each for the truss elements.

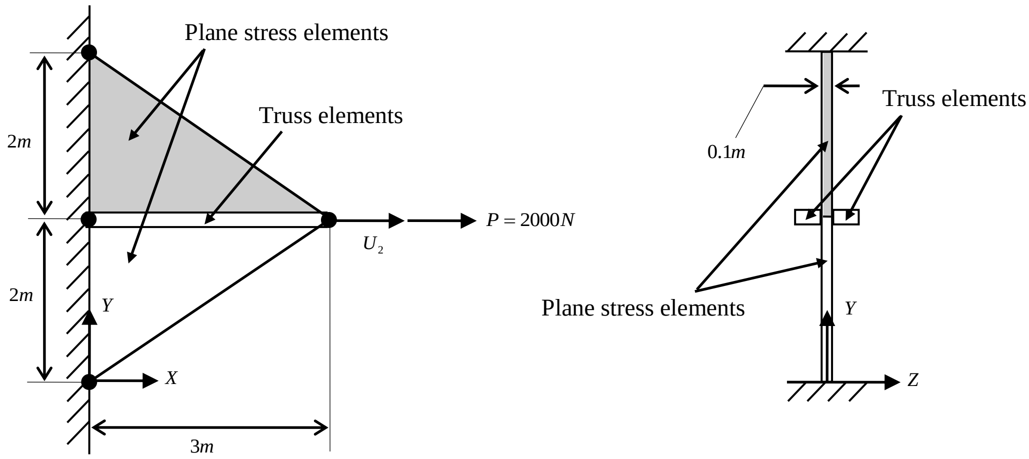


Fig.1. A structural model in the *XY* and *YZ* planes

Considering "linear elastic analysis", calculate

(a) the Jacobian matrix $\mathbf{J}$ and the determinant $\det\mathbf{J}$ of the shaded triangular element (5 pt)
(b) the relation between strain and nodal displacements $\mathbf{B}(r,s)$ for the shaded triangular element (5 pt)
(c) the total stiffness $K_{U_2U_2}$ corresponding to U_2 (5 pt)
(d) the displacement U_2 (5 pt)

Note that the 3-node triangular plane stress element has the interpolation functions:

$$h_1 = 1 - r - s, \quad h_2 = r, \quad h_3 = s,$$

and the material law is

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} N/m^2 .$$

2. Consider the single degree of freedom system shown in Fig. 2.

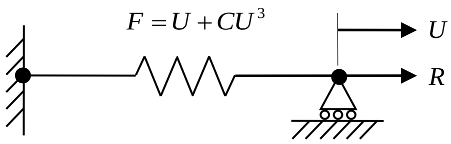


Fig. 2. A nonlinear spring ($C = 0.1$)

In the figure, U is the tip displacement, F is the internal force of the spring and R is the external loading. Recall that the stiffness of the system is $K = \dfrac{\partial F}{\partial U}$.

(a) Write "incremental equilibrium equation" to calculate the response of the system. (5 pt)
(b) We want to find the tip displacement U corresponding to $R = 1.0$. Using the *full Newton-Raphson method*, perform iterations until the solution is converged $\left| R - F^{(i)} \right| < 10^{-2}$. (5 pt)

3. Consider the 4-node plane stress element shown, where ${}^{0}\tau_{11} = {}^{0}\tau_{22} = {}^{0}\tau_{12} = 0$. Using the *total Lagrangian formulation*, calculate the nonlinear strain incremental stiffness matrix ${}^{t}_{0}\mathbf{K}_{NL}$. (10 pt)

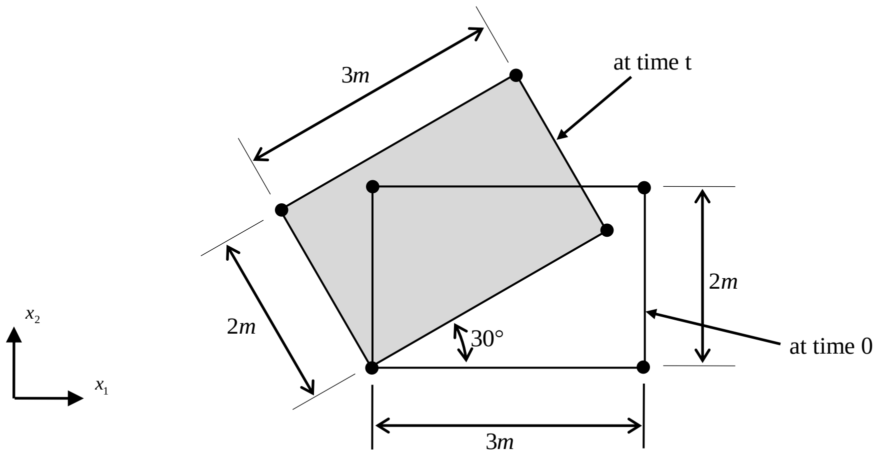


Fig. 3. A 4-node plane stress element under rotation

** If you think that the calculation is too heavy, calculate any component of ${}^{t}_{0}\mathbf{K}_{NL}$.

Finite Element Analysis of Structures

Mid-term Exam, Fall 2010 (30pt)

We have a plane stress problem subjected to point loads, P_1 and P_2, and the displacement BC as shown in Fig. The 2D problem is modeled by a 4-node plane stress finite element,

$$\text{thickness}=1.0m,\quad \mathbf{C}=\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} N/m^2 .$$

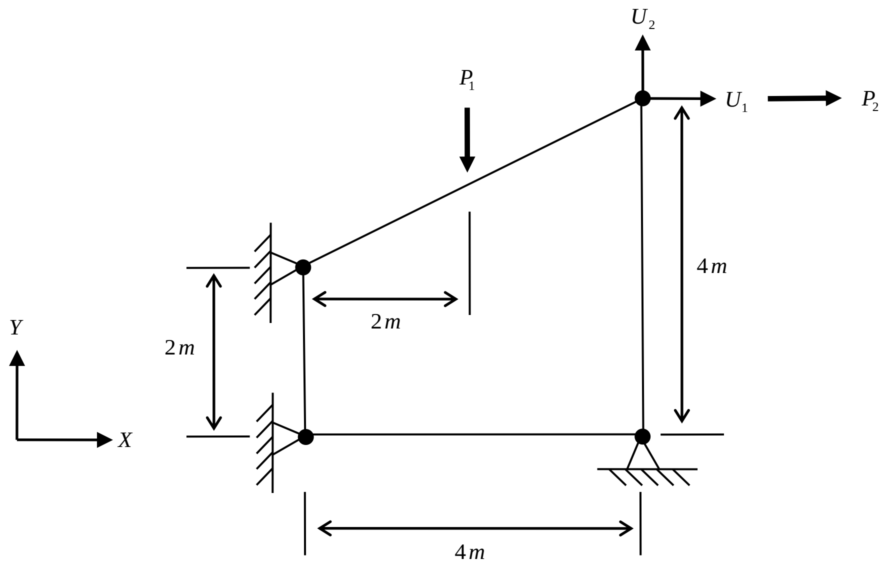


Figure. A plane stress problem subjected to point loads

1. Considering the isoparametric procedure, evaluate
   - (a) Global coordinate (X,Y) at r=s=0 (2pt)
   - (b) Jacobian matrix at r=s=0 (3pt)
   - (c) **B**-matrix at r=s=0 (5pt)
   - (d) $\mathbf{B}^{\mathbf{T}}\mathbf{CB}$ at r=s=0 corresponding to $K_{U_1U_2}$ (5pt)

   and calculate
   - (e) Components of the load vector, R_{U_1} and R_{U_2} (5pt).

2. Let's assume that we obtain the displacement $U_1 = U_2 = 0.001m$, calculate
   - (f) Displacements at r=s=0 (3pt)
   - (g) Strains and stresses at r=s=0 (5pt)
   - (h) Strain energy stored in the structure (2pt)

Finite Element Analysis of Structures

Final exam, 2010-12-20 (40pt.)

1. Let us consider a 2-node iso-beam (Timoshenko beam) element as shown in Fig. 1.

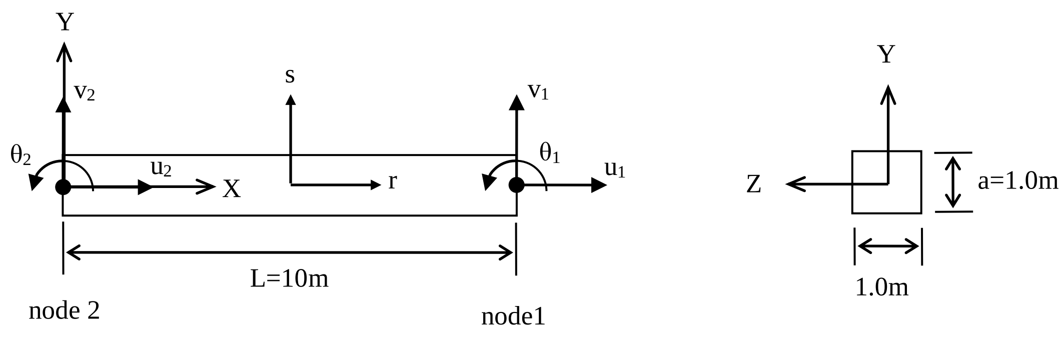


Fig. 1. A 2-node iso-beam element in 2D ($-1.0 \le r \le 1.0$ and $-1.0 \le s \le 1.0$)

The geometry and displacement interpolation functions and the nodal displacement vector are given:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum h_i x_i \\ \frac{s}{2} a \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sum h_i u_i - \frac{s}{2} \sum a\, h_i \theta_i \\ \sum h_i v_i \end{pmatrix} \quad \text{and} \quad \vec{U} = [u_1 \quad u_2 \quad v_1 \quad v_2 \quad \theta_1 \quad \theta_2]^T.$$

The material law used is

$$\begin{Bmatrix} \tau_{xx} \\ \tau_{xy} \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} \epsilon_{xx} \\ \gamma_{xy} \end{Bmatrix} \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}.$$

Considering the standard isoparametric procedure for linear elastic problems, calculate

(a) Jacobian matrix (5pt.)

(b) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in terms of the nodal displacement vector (5pt.)

(c) **B**-matrix (5pt.)

(d) Components of the **K**-matrix, $\mathbf{K}_{u_1 u_1}$ and $\mathbf{K}_{\theta_1 v_1}$ (5pt.)

2. Considering the total Lagrangian formulation, we analyze an elastic bar problem by using one 2-node truss element. Assume that the cross section does not change during deformation and the material law is

$$ {}_0^tS_{xx} = E\,{}_0^t\epsilon_{xx}. $$

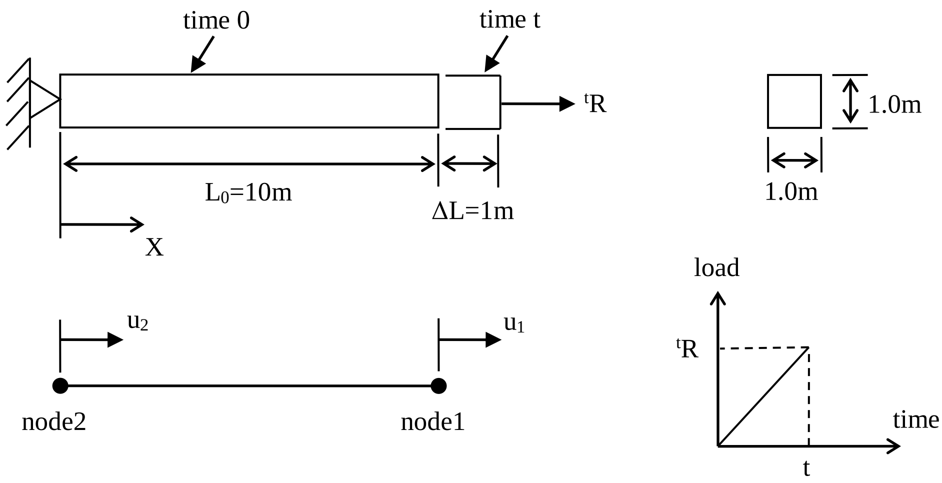


Fig. 2. An elastic bar problem

For the configuration at time 0, calculate

(a) Tangential stiffness, ${}_0^0\mathbf{K}_{u_1u_1}$ (5pt.)

For the configuration at time t, calculate

(b) Linear part of the tangential stiffness, $({}_0^t\mathbf{K}_L)_{u_1u_1}$ (5pt.)

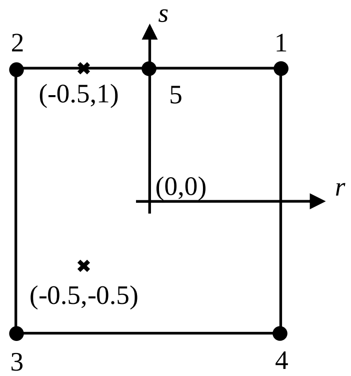
(c) Nonlinear part of the tangential stiffness, $({}_0^t\mathbf{K}_{NL})_{u_1u_1}$ (5pt.)

(d) Component of the force vector $({}_0^t\mathbf{F})$ corresponding to u_1 (5pt.)

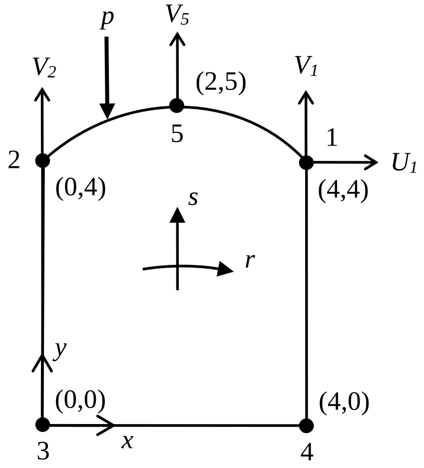
Finite Element Analysis of Structures

Mid-term Exam, Fall 2011 (40pt.)

1. (25pt.) Let us consider a 2D 5-node plane stress element as shown in Figure.



Natural coordinate



Global coordinate

(a) Find the shape function h_5 when $h_1 = \frac{1}{4}r(1+r)(1+s)$, $h_2 = -\frac{1}{4}r(1-r)(1+s)$, $h_3 = \frac{1}{4}(1-r)(1-s)$,$h_4 = \frac{1}{4}(1+r)(1-s)$.

(b) Find (x_p, y_p) corresponding to $r = -0.5$ and $s = 1$.

(c) Calculate the Jacobian matrix $\boldsymbol{J}$ and $det\boldsymbol{J}$ at $r = s = 0$.

(d) Calculate the column vector of the strain-displacement matrix $\boldsymbol{B}$ at $r = s = 0$ corresponding to U_1. Note that $\vec{\varepsilon} = \{\varepsilon_{xx} \ \ \varepsilon_{yy} \ \ \gamma_{xy}\}^T$.

(e) Calculate the components R_{V_1}, R_{V_2} and R_{V_5} of the load vector for the point load $\vec{P} = -p\vec{e}_y$ applied at (x_p, y_p).

2. (15pt.) We study the convergence of finite element solutions in potential energy. The potential energy is defined by

$$\pi(\vec{v}) = \frac{1}{2}a(\vec{v}, \vec{v}) - (\vec{f}, \vec{v}),$$

in which $\vec{v}$ is the displacement vector. Prove $\pi(\vec{u}) \le \pi(\vec{u}_h)$, where $\vec{u}$ is the exact solution and $\vec{u}_h$ is the finite element solution, and explain the meaning of the inequality.

Finite Element Analysis of Structures

Final Exam, Fall, 2011-12-20 (50pt.)

1. Let us consider a truss structure shown in Fig. 1. The structure consists of two bar members connected by a pin and the pin connection is supported by a spring. Each bar member of length L is modeled by a 2-node bar element.

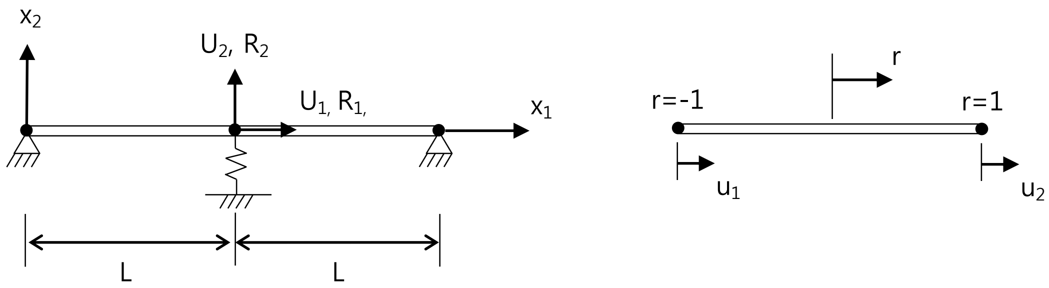


Figure 1. Left: A <u>linear elastic</u> truss problem (L = 5, EA = 1, spring constant = k). Right: 2-node bar element in the natural coordinate system.

(a) (5pt.) Find the **B**-matrix of the bar element.

(b) (10pt.) Find the stiffness matrix of the truss problem, $\vec{U} = [U_1 \quad U_2]^T$.

(c) (5pt.) Find the tension forces of the bar members when k = 1 and $U_1 = U_2 = 0.1$.

(d) (10pt.) When k = 0 and $R_2 = 1$, what happens in the linear analysis? Plot the nonlinear load-displacement curve that you expect (between R_2 and U_2) as the load increases.

2. Let us consider the 4-node plane stress element shown in Fig. 2.

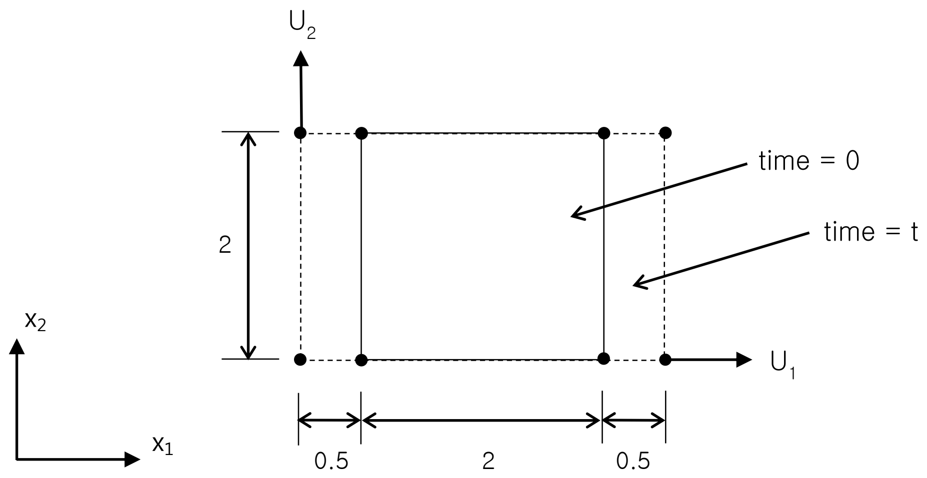


Figure 2. A 2-D plane stress problem.

At time = t, ${}_0^tS_{22}$ =10 and all other stresses = 0 are given in the element. Assume that the material law with Young's modulus E and Poisson's ratio ($\upsilon = 0$) relates the incremental 2nd Piola-Kirchhoff stresses to the incremental Green-Lagrange strains and assume thickness = 1.0 at time = 0.

Using the total Lagrangian formulation, calculate the following.

(a) (5pt.) The components of the linear stiffness matrices ${}_0^0\mathbf{K}_\mathrm{L}$ and ${}_0^t\mathbf{K}_\mathrm{L}$ corresponding to $\delta \mathrm{U}_1$ and U_1.

(b) (10pt.) The component of the nonlinear stiffness matrix ${}_0^t\mathbf{K}_\mathrm{NL}$ corresponding to $\delta \mathrm{U}_1$ and U_1.

(c) (5pt.) The component of the force vector ${}_0^t\mathbf{F}$ corresponding to U_2.

Finite Element Analysis of Structures

Mid-term Exam, Fall 2012 (40pt.)

1. (15pt.) As shown in Fig. 1, a 8-node 3D solid element is subjected to a uniformly distributed normal pressure q (force per area). Calculate the nodal point consistent loads at nodes 1, 2, 3 and 4.

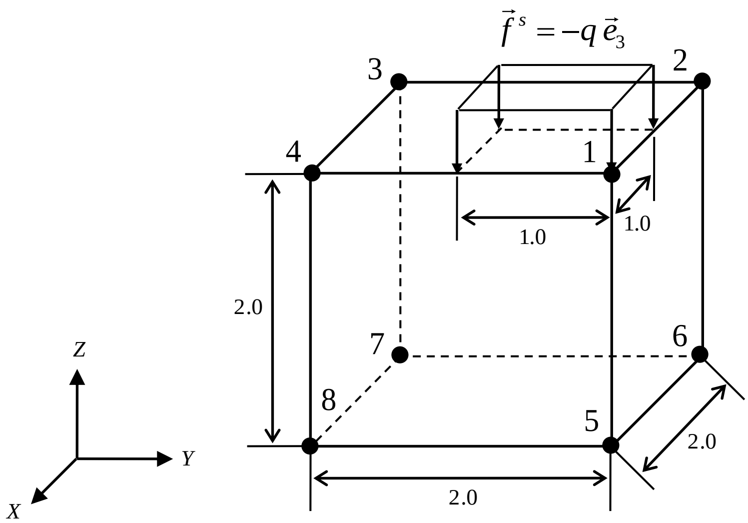


Fig. 1. A 8-node 3D solid element with a pressure loading.

2. (25pt.) Let us consider a triangular cantilever problem modeled by a 3-node plane stress element as shown Fig. 2. The force and displacement BCs are presented in Fig. 2, thickness is 1.0,

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{XX} \\ \varepsilon_{YY} \\ \gamma_{XY} \end{Bmatrix} \quad \text{and} \quad \mathbf{C} = \frac{E}{(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \text{ with } \upsilon = 0 .$$

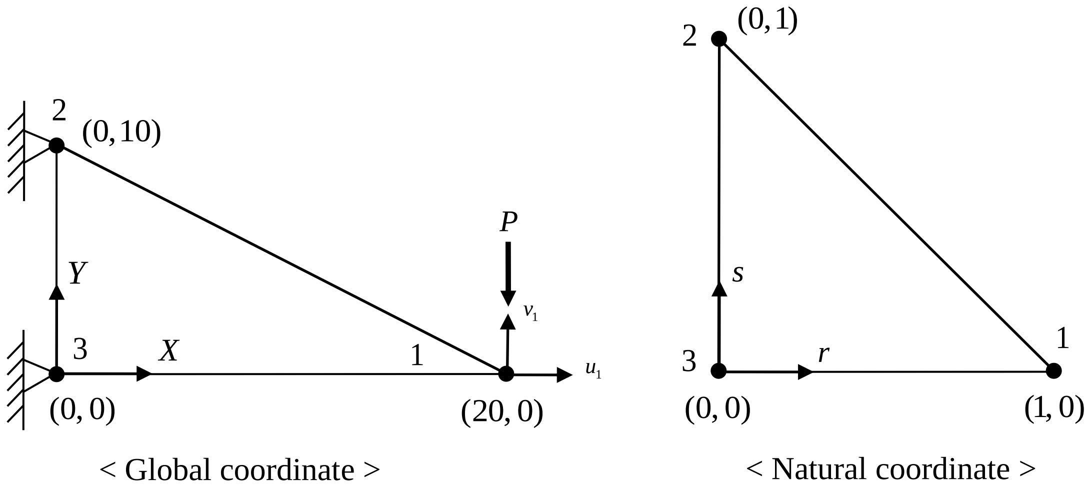


Fig. 2. A triangular cantilever subjected to a point load P

The shape functions of the 3-node element are given by

$$h_1 = r\,,\ h_2 = s \text{ and } h_3 = 1 - r - s\,.$$

Calculate the followings

(a) Global coordinates (X,Y) corresponding to r=s=0.5

(b) Jacobian matrix $\mathbf{J}$

(c) $\mathbf{B}$-matrix [3 by 2] corresponding to u_1 and v_1

(d) Stiffness matrix $\mathbf{K}$ [2 by 2] corresponding to u_1 and v_1

(e) Tip displacements u_1 and v_1, and stress at X=Y=0

Finite Element Analysis of Structures

Final Exam, Fall 2012 (45pt.)

1. We here want to extend our experience on finite element analysis of structures into heat transfer problems. For 2D heat transfer problems, the governing equations are given

$$k\frac{\partial^2\theta}{\partial x_i^2} + q^B = 0 \qquad \text{in } V, \qquad \text{(differential heat flow equilibrium equation)}$$

$$\theta = \theta^S \qquad \text{on } S_\theta, \qquad \text{(essential boundary condition)}$$

$$k\frac{\partial\theta}{\partial x_i}n_i = q^S \qquad \text{on } S_q, \qquad \text{(natural boundary condition)}$$

where k and θ are the thermal conductivity and the temperature of the body, q^B is the rate of heat generated per unit volume, θ^S is the surface temperature on S_θ, and q^S is the heat flux input on the surface S_q, see Fig. 1.

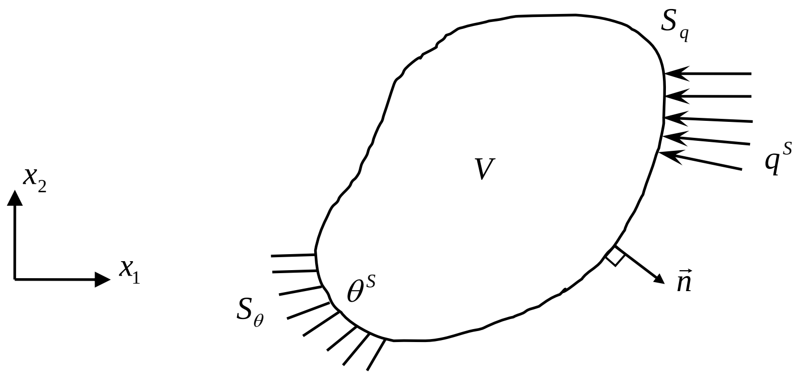


Fig. 1. A 2D body subjected to heat transfer ($S = S_\theta \bigcup S_q$, $S_\theta \bigcap S_q = 0$)

(a) (10pt.) Derive the principle of virtual temperatures given as follows

$$\int_V \frac{\partial\delta\theta}{\partial x_i}k\frac{\partial\theta}{\partial x_i}dV = \int_V \delta\theta\, q^B dV + \int_{S_q}\delta\theta\, q^S dS,$$

in which $\delta\theta$ is the virtual temperature distribution ($\delta\theta = 0$ on S_θ).

(Hint) Divergence theorem: $\int_V \frac{\partial f}{\partial x_i}dV = \int_S f\, n_i dS$ with a scalar function f.

(b) (5pt.) Assume that the interpolation of temperature is $\theta = \mathbf{H}\vec{\theta}$, and the relation between $\vec{\alpha} = \begin{bmatrix} \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{bmatrix}^T$ and the nodal temperature vector $\vec{\theta}$ is $\vec{\alpha} = \mathbf{B}\vec{\theta}$. Derive the finite element formulation for 2D heat transfer problems, $\mathbf{K}\vec{\theta} = \vec{R}$.

(c) (10pt.) Considering the 4-noede finite element shown in Fig. 2, find the component of stiffness matrix (K_{11}) corresponding to $\delta\theta_1$ and θ_1.

(Hint) $\int_{-1}^{1}\int_{-1}^{1}(1 \pm x)^2 dxdy = \frac{16}{3}$, $\int_{-1}^{1}\int_{-1}^{1}(1 \pm x)(1 \mp y)dxdy = \int_{-1}^{1}\int_{-1}^{1}(1 \pm x)(1 \pm y)dxdy = 4$

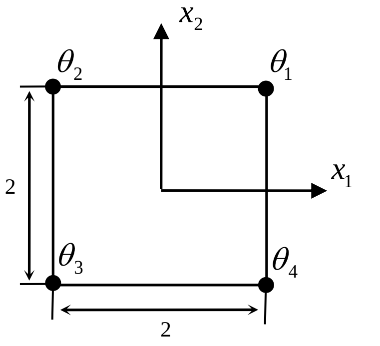


Fig. 2. Four-node finite element for 2D heat transfer problems

2. The configurations of a body at time 0, t and $t + \Delta t$ and the second Piola-Kirchhoff stresses ${}_0^t\mathbf{S}$ for the plane strain four-node element are shown in Fig. 3.

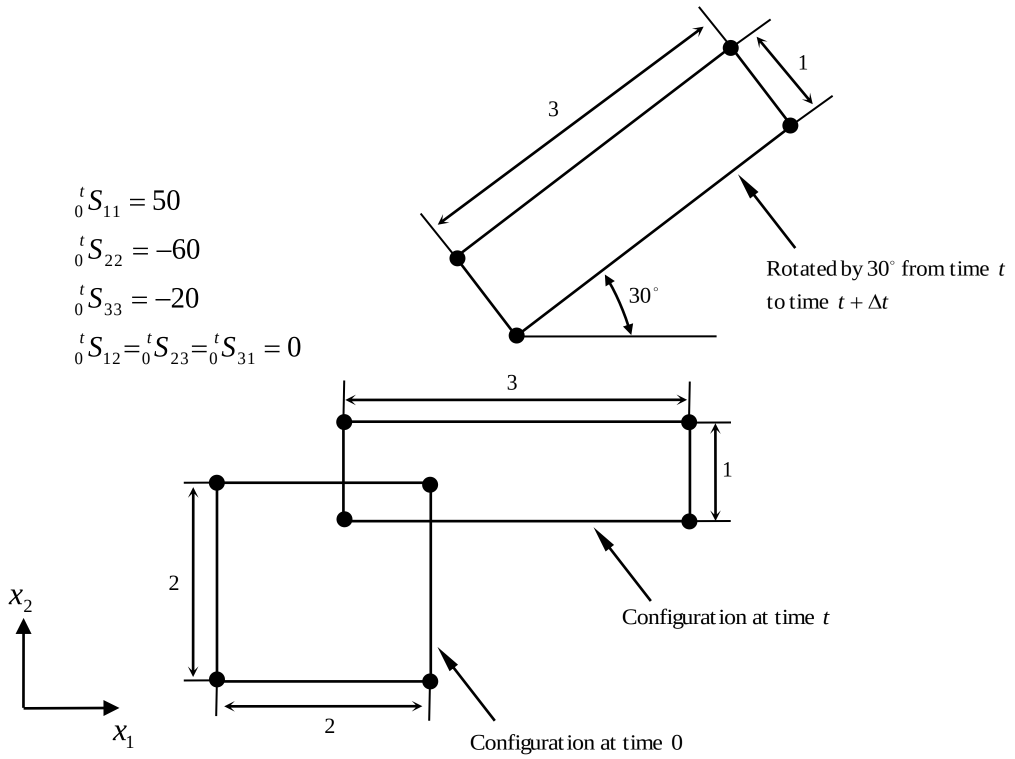


Fig. 3. Four-node finite element subjected to stretching and rotation

(Unit thickness at all time steps)

Calculate the followings

(a) (5pt.) Deformation gradients ${}_{0}^{t}\mathbf{X}$ and ${}_{0}^{t+\Delta t}\mathbf{X}$

(b) (5pt.) Cauchy stresses at time t , ${}^{t}\boldsymbol{\tau}$

(c) (5pt.) Second Piola-Kirchoff stresses at time $t+\Delta t$, ${}_{0}^{t+\Delta t}\mathbf{S}$

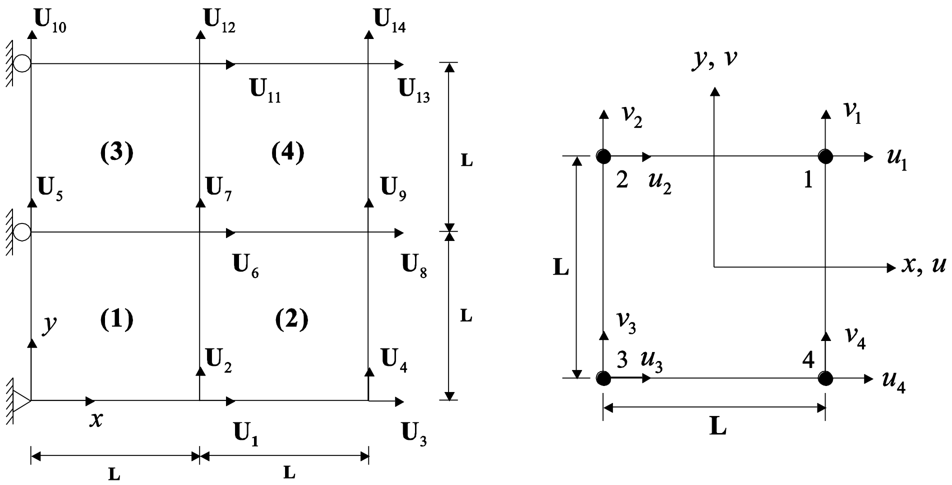
(d) (5pt.) Cauchy stresses at time $t+\Delta t$, ${}^{t+\Delta t}\boldsymbol{\tau}$

(Hint) ${}^{t}\boldsymbol{\tau} = \dfrac{{}^{t}\rho}{{}^{0}\rho}\, {}_{0}^{t}\mathbf{X}\ {}_{0}^{t}\mathbf{S} \left({}_{0}^{t}\mathbf{X}\right)^{T}$

Finite Element Analysis of Structures

Mid-term Exam, Fall 2013 (40pt.)

1. (10 pt.) Let us consider a 2D finite element model shown in Fig. 1.



(a) (b)

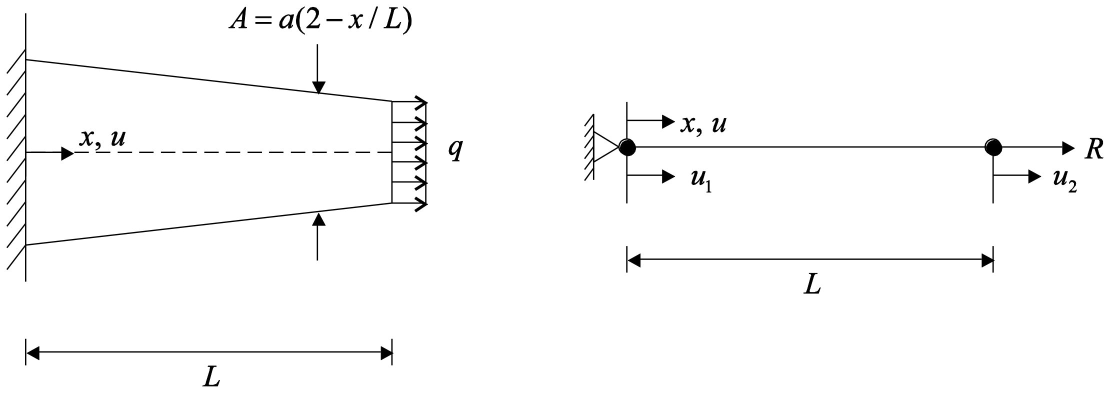
$$
\mathbf{K}_e^{(m)} = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\
k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\
k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88}
\end{bmatrix}
\qquad
\mathbf{u}^{(m)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}
$$

(c)

Fig. 1. A 2D cantilever plate problem. (a) Finite element model (2x2 mesh), (b) A 4-node plane stress element, (c) Stiffness matrix of the 4-node plane stress element.

Find the components of the global stiffness matrix, $K_{U_7U_7}$, $K_{U_6U_7}$ and $K_{U_2U_{12}}$, in terms of the components of $\mathbf{K}_e^{(m)}$ in Fig. 1 (c).

2. (15 pt.) Let us consider a tapered bar problem and its finite element model in Fig. 2.



(a) **(b)**

Fig 2. A tapered bar problem (E=Young's modulus). (a) Problem description, (b) Finite element model ($R = qa$).

The exact solution of this problem is given by

$$u_e(x) = \frac{RL}{Ea} \ln\left(\frac{2}{2 - x/L}\right).$$

Using a 2-node bar element, the following FE solution is obtained

$$u_h(x) = h_1(x)u_1 + h_2(x)u_2 \quad \text{with} \quad h_1(x) = 1 - x/L,\ h_2(x) = x/L,\ u_1 = 0 \ \text{and} \ u_2 = \frac{2LR}{3aE}.$$

The principle of virtual work specialized to this bar problem is given by

$$\int_0^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx = R\delta u\big|_{x=L}, \quad \text{with } u\big|_{x=0} = 0, \quad \delta u\big|_{x=0} = 0$$

(a) For the following 4 cases, check whether the principle of virtual work is satisfied or not.

- $u = u_e(x)$, $\delta u = a_0 h_2(x)$, and $u = u_e(x)$, $\delta u = a_0 x^2$
- $u = u_h(x)$, $\delta u = a_0 h_2(x)$, and $u = u_h(x)$, $\delta u = a_0 x^2$

(Hint) $\dfrac{du_e}{dx} = \dfrac{R}{Ea\,(2 - x/L)}$

(b) Discuss the results.

3. (15 pt.) A 3-node bar finite element is shown in Fig. 3.

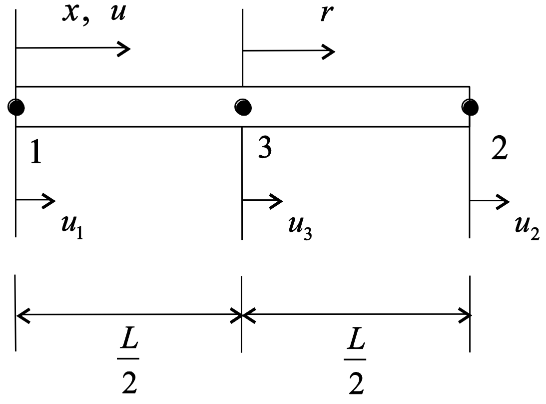


Fig 3. A 3-node bar finite element (E=Young's modulus, A=area, ρ =density).

Considering the **isoparametric procedure**, the shape functions for the 3-node bar finite element are given by

$$h_1 = \frac{1}{2}r(r-1), \quad h_2 = \frac{1}{2}r(r+1), \quad h_3 = (1+r)(1-r).$$

Calculate the followings

(a) Jacobian

(b) B-matrix, $\boldsymbol{B}(r)$ when $\varepsilon_{xx}(r) = \boldsymbol{B}(r)\mathbf{u}$ with $\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$.

(c) Component of the stiffness matrix, $K_{u_2u_2}$.

(d) Component of the mass matrix, $M_{u_1u_2}$.

Finite Element Analysis of Structures

Final Exam, Fall 2013 (40pt.)

1. (15 pt.) A 3-node triangular plane stress element shown in Fig. 1 is subjected to the prescribed displacement Δ.

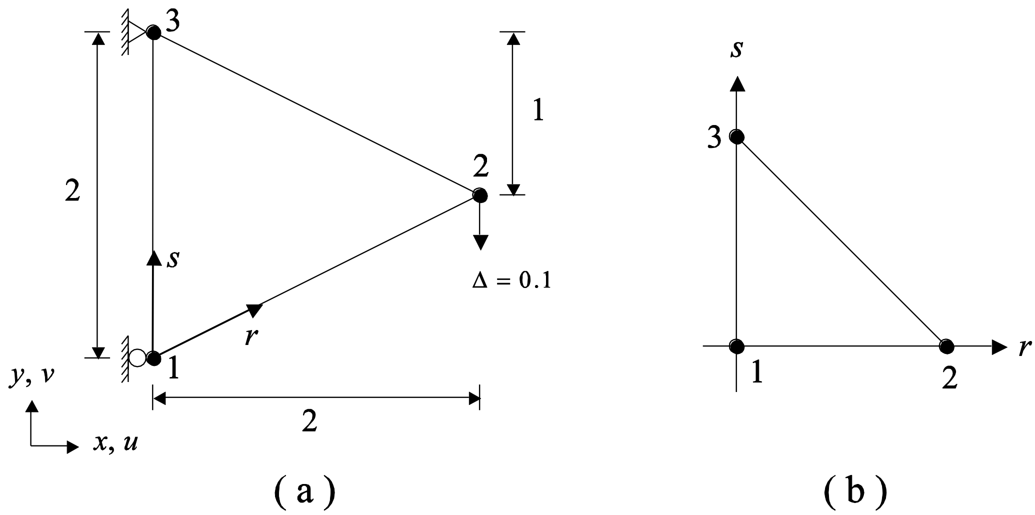


Fig. 1. A 3-node triangular plane stress element

(a) in the Cartesian coordinate system, (b) in the natural coordinate system.

Thickness = 0.1, $\mathbf{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$, E = 1 and ν = 0.

Shape functions: $h_1 = 1 - r - s, \quad h_2 = r, \quad h_3 = s$

Calculate the followings

(a) Strain-displacement matrix, **B**

(b) 3 by 3 stiffness matrix where the boundary condition is imposed, **K**

(c) Displacements at nodes 1 and 2

(d) Reaction force corresponding to the prescribed displacement

2. (10 pt.) Consider a 4-node plane strain element in the configurations at time 0 and t shown in Fig. 2.

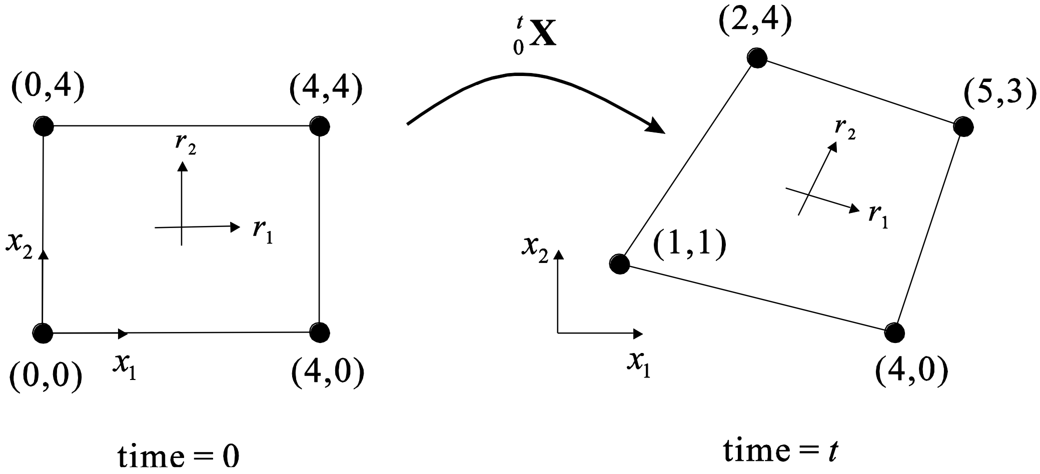


Fig. 2. A 4-node plane strain element at time 0 and t

Calculate the followings at $r_1 = r_2 = 0$

(a) Jacobian matrices, ${}^0\mathbf{J}$ and ${}^t\mathbf{J}$

(b) Deformation tensor, ${}_0^t\mathbf{X}$

(c) Green-Lagrange strain tensor at time t, ${}_0^t\boldsymbol{\varepsilon}$

3. (15 pt.) Using the total Lagrangian formulation, we consider a truss structure modeled by two 2-node truss elements in Fig. 3. Assume that the cross section (A_0) is not changed during the deformation and the material law is given by

$$ {}_0^tS_{11} = E\,{}_0^t\varepsilon_{11} $$

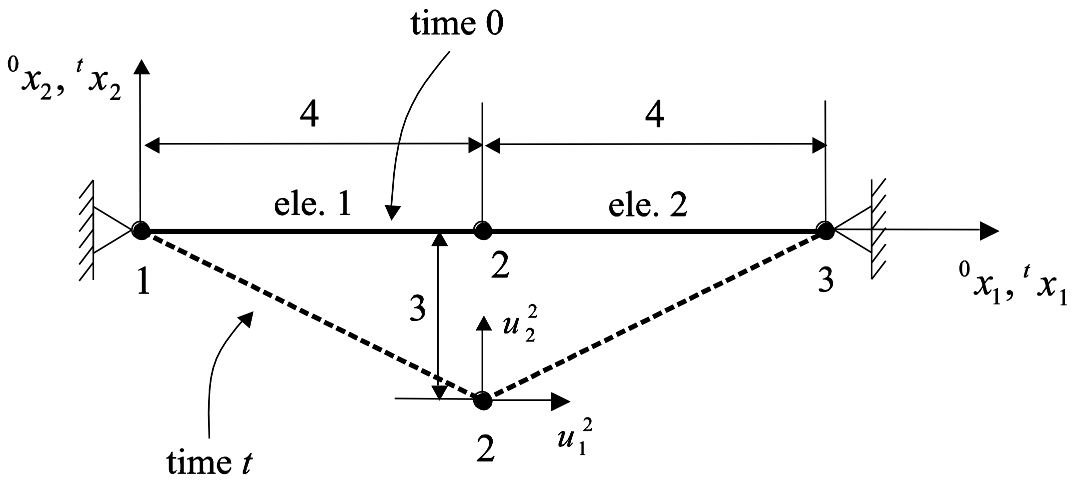


Fig. 3. A truss structure modeled by two 2-node truss elements.

When ${}^tu_2^2 = -3$ for the configuration at time t, evaluate the followings

(a) Component of the linear part of the tangent stiffness matrix (${}_0^t\boldsymbol{K}_L$) corresponding to δu_2^2 and u_2^2

(b) Component of the nonlinear part of the tangent stiffness matrix (${}_0^t\boldsymbol{K}_{NL}$) corresponding to δu_2^2 and u_2^2

(c) Component of the internal force vector (${}_0^t\boldsymbol{F}$) corresponding to u_2^2

(Hint) Due to symmetry, you may consider a half-symmetric model.

Finite Element Analysis of Structures

Mid-term Exam, Fall 2014 (40pt.)

Let us consider a thin membrane structure of thickness t subjected to a uniform temperature variation $\Delta\theta = 1^{\circ}$C as shown in Figure. The structure is clamped along left, right and bottom edges. We model the structure using a uniform 2x2 mesh of four 4-node *isoparametric* plane stress elements.

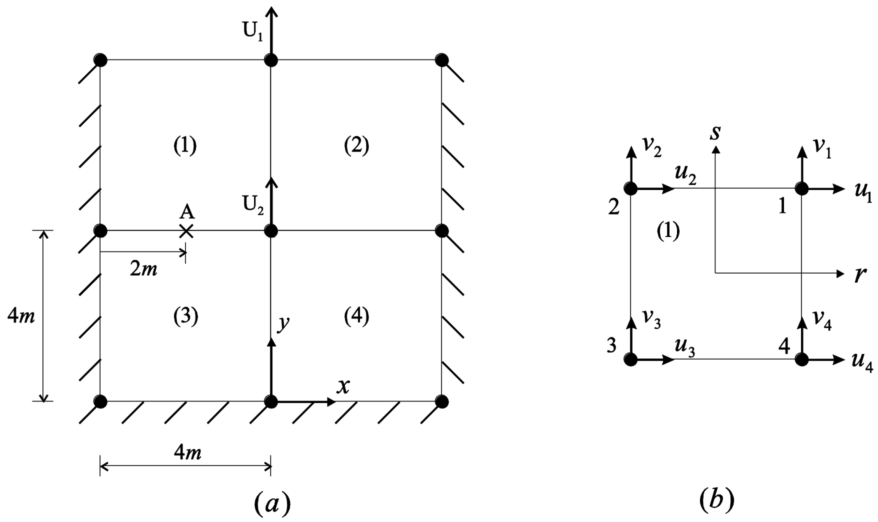


Figure. Thermal expansion of a thin membrane structure.
(a) Finite element model, (b) A 4-node plane stress element.

For this thermal expansion problem, we use the stress-strain law given by

$$\boldsymbol{\tau} = \mathbf{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{th}) \quad \text{with} \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{th} = \alpha \begin{bmatrix} \Delta\theta \\ \Delta\theta \\ 0 \end{bmatrix},$$

in which $\boldsymbol{\tau}$ is the stress vector, $\boldsymbol{\varepsilon}$ is the total strain vector, $\boldsymbol{\varepsilon}^{th}$ is the thermal strain vector, $\mathbf{C}$ is the material law matrix with Young's modulus E, and α is the thermal coefficient of expansion.

(a) (8pt.) Specialize the principle of virtual work to this problem considering the given stress-strain law. Write down the finite element formulation for the nodal load vector due to the thermal strain.

(b) (8pt.) Calculate the stiffness components $k_{v_1 v_1}$, $k_{v_4 v_4}$ and $k_{v_1 v_4}$ of the element (1), see Figure (b).
Note that $\int_{-1}^{1}\int_{-1}^{1}(1+r)^2\,drds = 16/3$ and $\int_{-1}^{1}\int_{-1}^{1}(1+s)(1-s)\,dsdr = 8/3$.

(c) (8pt.) Construct the 2x2 total stiffness matrix $\mathbf{K}$ using $k_{v_1 v_1}$, $k_{v_4 v_4}$ and $k_{v_1 v_4}$. The corresponding nodal displacement vector is $\mathbf{U} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$. Note that the problem is symmetric.

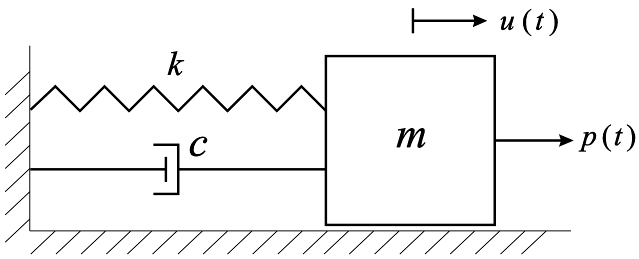
(d) (8pt.) Calculate the nodal load vector, $\mathbf{R} = \begin{bmatrix} R_1 & R_2 \end{bmatrix}^T$.

(e) (8pt.) Assuming $U_1 = 32/100$ and $U_2 = 8/100$, calculate the stress jump $\Delta\tau_{yy}$ at point A between the elements (1) and (3), see Figure (a).

Finite Element Analysis of Structures

Final Exam, Fall 2014 (40pt.)

1. (10pt.) Consider a single degree of freedom (DOF) system subjected to the force $p(t)$ as shown in Figure 1.



$$m = 2,\ c = 0,\ k = 6\ and\ \ p(t) = 10$$

Figure 1. A single degree of freedom system.

For this SDOF system, the linear equation of motion is given by

$$m\ddot{u}(t) + c\dot{u}(t) + k\,u(t) = p(t) \quad \text{with } u(0) = 0 \text{ and } \dot{u}(0) = 0\,,$$

in which m, c, k, and u are the mass, damping coefficient, spring constant and displacement, respectively.

Using the *Central Difference Method* (CDM) with time step size $\Delta t = \Delta t_{cr}$, calculate the response of the system from time 0 to 1 sec. (Fill in the blanks (a), (b) and (c) in Table 1.)

Table 1. Response of the SDOF system.

Time [sec]	0	Δt	$2\Delta t$
$u(t)$	(a)	(b)	(c)

Note that the critical time step size $\Delta t_{cr} = T_n / \pi$ and the free-vibration period of the system T_n is 0.5π. In CDM, the following approximations are used for the discretization of time

$$\dot{u}(t) = \frac{1}{2\Delta t}\left[u(t+\Delta t) - u(t-\Delta t)\right] \text{ and } \ddot{u}(t) = \frac{1}{(\Delta t)^2}\left[u(t+\Delta t) - 2u(t) + u(t-\Delta t)\right].$$

2. (15pt.) Let us consider a 4-node axisymmetric finite element as shown in Figure 2. The finite element is clamped along the bottom edge.

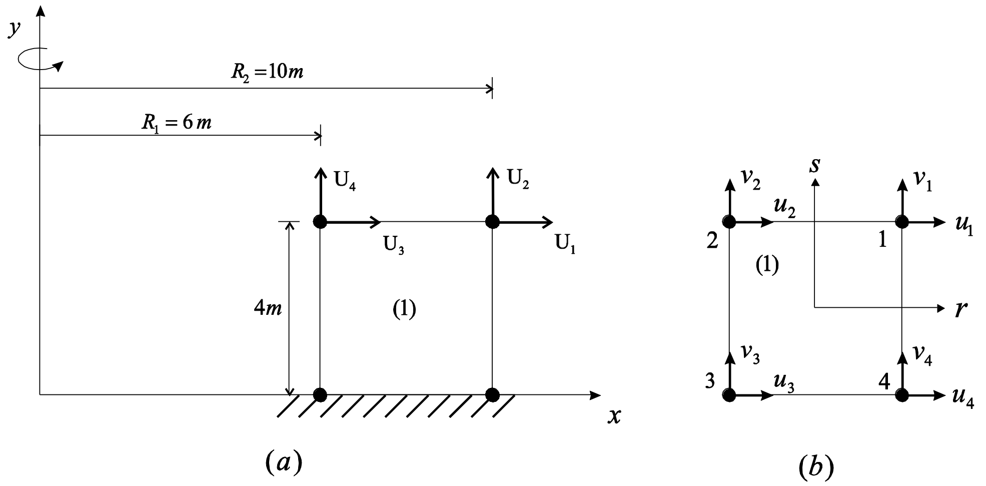


Figure 2. A 4-node axisymmetric finite element. (a) Global DOFs, (b) Local DOFs

For linear elastic analysis, the material law matrix $\mathbf{C}$ and the strain vector $\boldsymbol{\varepsilon}$ are given by

$$\mathbf{C} = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 0.5E & 0 \\ 0 & 0 & 0 & E \end{bmatrix} \text{ and } \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{zz} \end{bmatrix}$$

with Young's modulus E, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, and $\varepsilon_{zz} = \frac{u}{x}$.

Using the isoparametric procedure, calculate the stiffness component $K_{U_1U_1}$.

Note that $\int_{-1}^{1}\int_{-1}^{1}(4+r)(1+s)^2\,drds = 64/3$, $\int_{-1}^{1}\int_{-1}^{1}(4+r)(1+r)^2\,drds = 24$ and $\int_{-1}^{1}\int_{-1}^{1}\frac{(1+r)^2(1+s)^2}{4+r}\,dsdr = 8/5$.

3. (15pt.) Let us consider a plane strain element as shown in Figure 3.

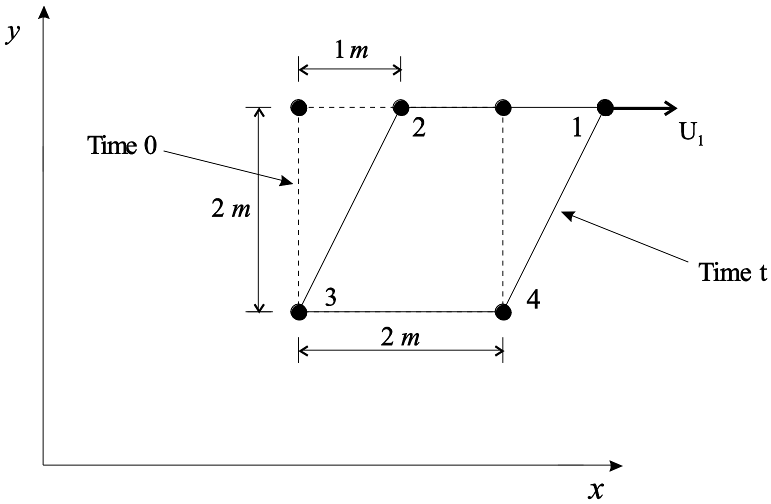


Figure 3. A 2D plane strain element.

The Cauchy stress at time t, not including ${}^t\tau_{zz}$, is given by

$$ {}^t\boldsymbol{\tau} = \begin{bmatrix} 0 & 2.0\times10^7 \\ 2.0\times10^7 & 0 \end{bmatrix} \text{ Pa.} $$

Using the <u>*Total Lagrangian Formulation*</u>, compute the component of the nonlinear stiffness ${}_0^t\mathbf{K}_{NL}$ corresponding to U_1 and δU_1. Note that ${}_0^t\mathbf{K} = {}_0^t\mathbf{K}_L + {}_0^t\mathbf{K}_{NL}$.