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# PU 기반 솔리드 유한요소의 개발

# Development of the partition of unity based solid finite elements free from the linear dependence problem

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김 산(金 山 Kim, San)

한 국 과 학 기 술 원

Korea Advanced Institute of Science and Technology

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기계항공공학부/기계공학과

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## 김 산

# 위 논문은 한국과학기술원 박사학위논문으로 학위논문 심사위원회의 심사를 통과하였음

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# Development of the partition of unity based solid finite elements free from the linear dependence problem

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A dissertation/thesis submitted to the faculty of Korea Advanced Institute of Science and Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering

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The study was conducted in accordance with Code of Research Ethics<sup>1</sup>).

<sup>1)</sup> Declaration of Ethical Conduct in Research: I, as a graduate student of Korea Advanced Institute of Science and Technology, hereby declare that I have not committed any act that may damage the credibility of my research. This includes, but is not limited to, falsification, thesis written by someone else, distortion of research findings, and plagiarism. I confirm that my dissertation contains honest conclusions based on my own careful research under the guidance of my advisor.

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#### <u>초 록</u>

본 학위논문에서는 PU 기반 2차원 및 3차원 솔리드 유한요소(4절점 사각형 요소, 8절점 육면체 요소. 오면체 요소. 5절점 오면체 요소)를 제안한다. PU 기반 유한요소의 6절점 위해서 부분 선형종속문제(linear dependence problem)를 해결하기 각 요소별 선형 형상함수(piecewise linear shape function)를 제시하고, 이를 형상 및 변위 보간에 적용하였다. 다양한 격자에서 선형종속문제가 발견되지 않았으며, 좋은 수렴 성능을 나타내었다. 또한, 다양한 예제를 통해 요소의 성능 및 효율성을 입증하였다. 추가적으로 커버 함수(cover function)의 선택적 적용을 통해 유한요소 해석 결과의 정확도를 자동으로 개선하는 절차를 구현하고, 그 적용 가능성을 확인하였다. 이 해석 절차는 기존에 사용되었던 격자 재구성 또는 절점 추가가 필요 없으며, 일부 자유도의 증가만으로 유한요소 해석의 정확도를 크게 개선하였다.

<u>핵 심 낱 말</u> PU 기반 유한요소, 선형종속문제, 4절점 사각형 요소, 8절점 육면체 요소, 6절점 오면체 요소, 5절점 오면체 요소, 수렴 성능

#### Abstract

The partition of unity based 2D 4-node quadrilateral and 3D 8-node hexahedral, 6-node prismatic, 5-node pyramidal elements are presented. To resolve the linear dependence problem, sets of piecewise linear shape functions are proposed and adopted for the geometry and displacement interpolations. The rank deficiency was not observed with various mesh patterns and excellent convergence behaviors were observed, even when distorted meshes are used. The feasibility of the automatic procedure that improves the solution accuracy using the adaptive use of cover functions is also illustrated through several problems. Based on the error indicator proposed, the scheme automatically selects the order of cover functions in the procedure. The procedure provides good predictions for strain energy and stress with small increment of DOFs instead of any traditional local mesh refinement.

<u>Keywords</u> PU based finite element, Linear dependence problem, 4-node quadrilateral element, 8-node hexahedral element, 6-node prismatic element, 5-node pyramidal element, Convergence performance, Partition of unity

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## Chapter 1. Introduction

## 1.1. Research background

The finite element method can effectively model complex geometries using meshes and is considered robust and effective compared to other numerical methods. During the last decades, the finite element method has been widely used for numerical analysis of solids, fluids, and multi-physics problems [1-3]. However, the accuracy of solutions depends on the quality of the meshes used, and in engineering practice, it takes considerable effort to obtain a suitable mesh. Also, mesh refinements are often necessary to secure reliable solutions with required accuracy when non-smooth, near-singular, and high-gradient solutions are sought [1-4].

In order to obtain reliable solutions, the partition of unity methods have been developed in recent years. The partition of unity methods include the hp-cloud method [5-11], the numerical manifold method [7-18], the extended finite element method [9-27], the generalized finite element method [28-33], and enriched finite element method [34-37]. The partition of unity method forms the solution space by multiplying enrichment functions and the partition of unity functions. Babuska and Melenk showed a mathematical background for the partition of unity methods [38]. Applying enrichment functions suitable for a particular problem, a reliable solution can be obtained. To account for discontinuities and various singularities in solid mechanics problems, Belytschko and Black [19], Moes et al. [20], Duarte et al. [39], and Rabczuk et al. [40] incorporated enrichment functions. Harmonic functions were applied as enrichment function by Ham and Bathe [41] to solve wave propagation problem. For pipe analysis, special enrichment functions were embedded to represent pipe ovalization effect accurately [42,43]. Bathe and Chaudhary [44] and Yoon et al. [45] developed the beam finite element incorporated with warping functions.

There has also been studies of applying polynomials as enrichment functions that generally improve element performance rather than specific problems. Kim and Bathe [34,35] and Jeon et al. [36,37] studied and developed a finite element enriched by interpolation covers for solid and shell analysis. The enriched element showed improved convergence performance with low order finite element mesh, and the higher order enrichment is available. That is, enriched element can capture the high gradient of stress and reduce inter-element stress jump. In addition, adaptive use of the cover function can improve the solution accuracy without remeshing or adding nodes to the edge or center of the element.

However, linear dependence (LD) problem, in which global stiffness matrix becomes singular, can occur when functions in the enriched displacement interpolation become linearly dependent. The LD problem was first reported by Babuska and Melenk [38]. An at al. [46,47] proposed approach to predict rank deficiency of global stiffness matrix due to the LD problem in the partition of unity methods for 2D and 3D analyses, and various

attempts have been made to alleviate the LD problem.

Babuška and Melenk [38] designed partition of unity functions in order that the LD problem could be overcome in 1D analysis. Oden et al. [9] suggested the elimination of linear polynomial terms in the local approximation functions. Duarte et al. [39] and Stouboulis et al. [29] showed that such treatments are not enough to avoid the LD problem; then adopted special equation solvers in 2D and 3D analyses. Tian [48] suggested suppressing enriched degrees of freedoms (DOFs) corresponding to enriched functions at essential boundary and it is effective for 3-node 2D triangular and 4-node 3D tetrahedral elements [34,48].

Babuska et al. [49] suggested the stable generalized finite element method (SGFEM) that local approximation functions are modified by subtracting the standard finite element interpolation of local approximation functions to deal with the conditioning of the global stiffness matrix. In SGFEM, the LD problem is avoided by applying the flat top partition of unity functions for 1D and 2D analyses when polynomial is used as local approximation function [50]. A study using the flat-top partition of unity function was also conducted by Lee and Hong [51,52], expanded from earlier work of Babuška and Melenk [39], but construction of the flat-top partition of unity functions is not easy and requires artificial constant that effects matrix condition and solution accuracy. [51,52].

For 2D solid mechanics problem, 4-node element meshes are most widely used and 8-node hexahedral element meshes are widely used with 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral element meshes for analysis of 3D solid mechanics problems. Therefore, it is still necessary to effectively resolve the linear dependence problem of the enriched solid finite elements. In this thesis, we aim to develop, in a simple and effective way, solid finite element enriched by interpolation covers, which are free from the linear dependence problem. The developed enriched elements can be used for 2D and 3D mechanics problem and effectively improve the finite element solution accuracy.

#### 1.2. Research purpose & contents

The first objective of this thesis is to develop 2D and 3D solid finite elements (4-node quadrilateral, 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements) enriched by interpolation covers. The linear dependence problem of the enriched elements should be resolved in a simple and effective way, and the developed elements are require to pass the patch, zero energy mode, and isotropy tests for arbitrary enrichment. In addition, these elements should show good convergence behaviors and cover function should be applicable to local area to improve the solution accuracy.

The second objective in this thesis is to demonstrate the feasibility of the adaptive use of cover functions, an important advantage of the enriched elements. By applying cover function adaptively, the solution accuracy can be improved without any traditional local mesh refinement or adding nodes. If a procedure to automatically improve finite element solutions with the adaptive use of cover functions is developed, it will be useful in performing finite element analyses for various purposes. To implement this automatic procedure, an algorithm that determines appropriate orders of the cover functions for each node is required.

In the following chapter, the formulation of the enriched finite element is briefly reviewed, and the enriched 4node 2D solid finite element which is free from the linear dependence problem is presented. The linear dependence problem is tested using various mesh patterns and the effectiveness and performance of developed element is demonstrated through several problems: an ad hoc problem, a tool jig problem, a slender beam problem, automotive wheel problem, a cantilever beam with fillets problem, and an wrench problem.

In chapter 3, we present the enriched 3D solid finite elements (including 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements). Similar to the previous section, investigations of the linear dependence problem of the enriched 3D elements are presented. The performance and effectiveness of the developed elements are tested through several problems: an ad hoc problem, a tool jig problem, a straight beam problem, a curve beam problem, and a connecting rod problem.

In chapter 4, we show the feasibility of a procedure to automatically improve finite element solutions with the adaptive use of cover functions. For this automatic procedure, we present an error indicator and a scheme that select appropriate orders of cover functions based on the error indicator. We consider 3-node triangular and 4-node quadrilateral 2D solid finite elements. Through several 2D problems, the automatic procedure, that is based on the error indicator and the adaptive use of cover functions, is demonstrated.

In chapter 5, we propose the strain-smoothed 3-node triangular element enriched by interpolation covers. Strain of the enriched 3-node triangular element is divided into two parts. Then, the strain-smoothed element (SSE) method is applied for the strain part corresponding to the standard DOFs. The effectiveness and performance of developed element is tested through a cook beam problem and a tool jig problem.

In chapter 6, the conclusions and future works are presented.

#### Chapter 2. The enriched 4-node 2D solid finite element

In this chapter, formulation of enriched finite element is first reviewed. Then, we present the enriched 4-node 2D solid finite element for linear analysis. The investigations of the linear dependence problem and element performance are demonstrated through several numerical examples.

## 2.1. Formulation of the enriched 2D solid finite elements

#### 2.1.1. Enriching finite elements by interpolation covers

The enriched finite element solution procedure is theoretically clear but some researches has been proceeded to overcome difficulties such as the linear dependence problem. In this section, 2D analysis problem is briefly considered to introduce the basic procedure for the enrichment scheme used here.

Let's  $\mathbf{Q}^n = {\mathbf{x}_i\}_{i=1}^n}$  be a set of *N* nodal point position vectors  $\mathbf{x}_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T \in \Omega$ , and let  ${\lambda_h} = {\{\varphi^m\}_{m=1}^q}$  be a family of *q* quadrangles generated by  $\mathbf{Q}^n$ . The quadrangles corresponds to the domain  $\Omega$ , in which we seek the solution variable *u* 

$$\bigcup_{m=1}^{q} \varphi^m = \Omega.$$
(2.1)

The quadrangles do not overlap, that is,  $\varphi^j \cap \varphi^k = \emptyset$  for  $j \neq k$ . Fig. 2.1(a) shows the usual interpolation function  $h_i(x, y)$  which construct the partition of unity. Let  $C_i$  be the support domain of  $h_i$ , i.e.  $C_i = supp(h_i)$ ,  $\forall i = 1, ..., n$ , which we call the cover region. Hence the cover region  $C_i$  corresponds to the union of elements attached to the node *i*, see Fig. 2.1(b).



Figure 2.1 Description of sub-domain for enriched cover interpolations : (a) usual interpolation function, (b) cover region or elements affected by the interpolation cover, and, (c) an element.

For each  $\varphi^m$ , let  $i_c(m)$  be the set of cover indices defined by

$$i_c(m) = \{i : C_i \cap \varphi^m \neq \emptyset\}.$$
(2.2)

For the 4-node quadrilateral element, the overlapped region of the four cover regions  $C_i$ ,  $C_j$ ,  $C_k$  and  $C_l$  constitutes element *m* and hence  $i_c(m) = \{i, j, k, l\}$ , see Fig. 2.1(c). To enrich the standard finite element interpolation for the solution variable *u*, we use interpolation cover functions

$$\tilde{u} = \overline{u}_i + [\xi_i \quad \eta_i \quad \xi_i^2 \quad \xi_i \eta_i \quad \eta_i^2 \quad \cdots \quad \eta_i^d] \mathbf{a}_i$$
with  $\xi_i = \frac{(x - x_i)}{\chi_i}, \quad \eta_i = \frac{(y - y_i)}{\chi_i},$ 
(2.3)

where  $\bar{u}_i$  is the standard nodal point variable,  $\mathbf{a}_i = [a_{i1} \ a_{i2} \ \cdots]$  lists the additional degrees of freedom for the cover region, *d* is the order of the complete polynomial used, and  $\chi_i$  is the diameter of the largest finite element sharing the node *i*. The use of  $\chi_i$  can improve the conditioning of the coefficient matrix [34,37].

The enriched cover approximation of a field variable u is given by

$$u = \sum_{m=1}^{q} \sum_{i \in i_{c}(m)} h_{i} \tilde{u}_{i} = \sum_{m=1}^{q} \left( \sum_{i \in i_{c}(m)} h_{i} \overline{u}_{i} + \sum_{i \in i_{c}(m)} \mathbf{H}_{i} \mathbf{a}_{i} \right)$$
  
with  $\mathbf{H}_{i} = h_{i} [\xi_{i} \quad \eta_{i} \quad \xi_{i}^{2} \quad \xi_{i} \quad \eta_{i} \quad \eta_{i}^{2} \quad \cdots \quad \eta_{i}^{d} ].$  (2.4)

When  $\tilde{u}_i = \bar{u}_i$  is used for Eq. (2.4), then the enriched cover approximation of a field variable reduces to the standard linear finite element interpolation. The enriched interpolation consists of the standard finite element interpolation and enriched higher order interpolation. This procedure can be derived in a variety of different ways [5-8,12,15].

In order to obtain some insight into the enrichment scheme described above, let us consider the enriched cover approximation of a field variable u in element 1 shown in Fig. 2.2

$$u = \sum_{i \in \{1,2,4,5\}} h_i \tilde{u}_i$$

$$= \sum_{i \in \{1,2,4,5\}} h_i [\overline{u}_i + a_{i1} \xi_i + a_{i2} \eta_i + a_{i3} \xi_i \eta + \cdots].$$
(2.5)

The coefficients in Eq. (2.5) can be determined by differentiating interpolation cover functions in Eq. (2.3) at each node ( $\xi_i = 0, \eta_i = 0$ ) with respect to the nodal coordinate variables as follows:

$$\{a_{ij}\}_{j=0}^{j=(d+1)(d+2)/2-1} = \left\{\frac{1}{\alpha!\beta!} \frac{\partial^{(\alpha+\beta)}\tilde{u}_i}{\partial \zeta_i^{\alpha} \partial \eta_i^{\beta}}\Big|_{(\xi,\eta)=(0,0)}\right\}_{\alpha+\beta=0}^{\alpha+\beta=d},$$
(2.6)

in which  $a_{i0} = \overline{u}_i$ , the subscript j refers to the set of coefficients used,  $\alpha$  and  $\beta$  are integers such that

 $0 \le \alpha, \beta \le d$  and for  $\alpha = 0$  and  $\beta = 0$ , respectively, no derivative is taken. Substituting Eq. (2.6) into Eq. (2.5), the enriched cover approximation of a field variable *u* is given by

$$u = \sum_{i \in \{1,2,4,5\}} h_i \left[ a_{i0} + \frac{\partial \tilde{u}_i}{\partial \xi_i} \right|_{(\xi_i,\eta_i) = (0,0)} \xi_i + \frac{\partial \tilde{u}_i}{\partial \eta_i} \right|_{(\xi_i,\eta_i) = (0,0)} \eta_i + \frac{1}{2} \frac{\partial^2 \tilde{u}_i}{\partial \xi_i^2} \right]_{(\xi_i,\eta_i) = (0,0)} \xi_i^2 + \cdots \left].$$

$$(2.7)$$

This enriched approximation in the quadrilateral element can be interpreted as a bilinear interpolation of four cover functions that are defined by Taylor polynomials expanded along each cover coordinate variable. This interpolation spans higher spatial bases than the standard finite element interpolation [34].



Figure 2.2. Finite element model consisting of four quadrilateral elements.

#### 2.1.2. The enriched 2D solid finite elements

In this section, we present the geometry and displacement interpolations of 2D solid finite elements enriched by interpolation covers. The enriched displacement interpolation is based on the enriched cover approximation described in previous section. Then, the static equilibrium equation of the enriched finite elements in matrix form is presented.

The geometry interpolation of the enriched 2D solid finite elements is identical to that of the corresponding standard finite element

$$\mathbf{x}(r,s) = \sum_{i=1}^{n} h_i(r,s) \mathbf{x}_i \quad \text{with} \quad \mathbf{x}_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T,$$
(2.8)

where *n* is the number of nodes in each element,  $\mathbf{x}_i$  is the position vector of node *i* in the global Cartesian coordinate system shown in Fig. 2.3(a), and  $h_i(r,s)$  are the shape functions of standard isoparametric procedure

corresponding to node i defined in the natural coordinate systems in Fig. 2.3(b). The linear shape functions of standard 3-node triangular elements are

$$h_1(r,s) = (1-r-s), \ h_2(r,s) = r, \ h_3(r,s) = s.$$
 (2.9)

The bilinear shape functions of the standard 4-node quadrilateral elements are

$$h_1(r,s) = (1+r)(1+s)/4, \quad h_2(r,s) = (1-r)(1+s)/4,$$
  
$$h_3(r,s) = (1-r)(1-s)/4, \quad h_4(r,s) = (1+r)(1-s)/4.$$
 (2.10)

The 2D shape functions,  $h_i$  satisfy the partition of unity requirement,  $\sum_{i=1}^{n} h_i = 1$ . Therefore, the displacement interpolation of the enriched 2D solid finite element is given by multiplying the shape functions by cover functions defined in the cover region  $C_i$  as follows [28,34,36,38]:

$$\mathbf{u}(r,s) = \sum_{i=1}^{n} h_i(r,s) \tilde{\mathbf{u}}_i \quad \text{with} \quad \tilde{\mathbf{u}}_i = \begin{bmatrix} \tilde{u}_i & \tilde{v}_i \end{bmatrix}^T,$$
(2.11)

in which  $\tilde{u}_i$  and  $\tilde{v}_i$  are cover functions corresponding to the displacements in the *x*- and *y*-directions, respectively, and the cover  $C_i$  is the union of elements attached to node *i* (see Fig. 2.1).



Figure 2.3 Coordinate systems for a 3-node triangular and a 4-node quadrilateral elements: (a) Global Cartesian coordinate system (x, y) and nodal local coordinate systems  $(\xi_i, \eta_i)$ , and (b) Natural coordinate systems.

The cover functions are given by

$$\tilde{u}_i = \mathbf{p}_i(\mathbf{x})\mathbf{u}_i^u, \quad \tilde{v}_i = \mathbf{p}_i(\mathbf{x})\mathbf{u}_i^v \quad \text{in } C_i$$
(2.12)
with

$$\mathbf{p}_{i}(\mathbf{x}) = \begin{bmatrix} 1 & \zeta_{i} & \eta_{i} & \zeta_{i}^{2} & \cdots & \eta_{i}^{d} \end{bmatrix}, \quad \zeta_{i} = \frac{(x - x_{i})}{\chi_{i}}, \quad \eta_{i} = \frac{(y - y_{i})}{\chi_{i}}, \\ \mathbf{u}_{i}^{u} = \begin{bmatrix} u_{i}^{1} & u_{i}^{\xi} & u_{i}^{\eta} & u_{i}^{\xi^{2}} & \cdots & u_{i}^{\eta^{d}} \end{bmatrix}^{T}, \quad \mathbf{u}_{i}^{v} = \begin{bmatrix} v_{i}^{1} & v_{i}^{\xi} & v_{i}^{\eta} & v_{i}^{\xi^{2}} & \cdots & v_{i}^{\eta^{d}} \end{bmatrix}^{T}, \quad (2.13)$$

in which  $\mathbf{p}(\mathbf{x})$  is a polynomial basis vector for node *i*, *d* is the degree of polynomial bases,  $\chi_i$  is the largest edge length of elements attached to node *i*, and  $\mathbf{u}_i^u$  and  $\mathbf{u}_i^v$  are the degrees of freedom (DOFs) vectors corresponding to polynomial bases for the displacements *u* and *v*, respectively.

Substituting Eq. (2.12) into Eq. (2.11), the displacement interpolation of the enriched 2D solid finite element is obtained

$$\mathbf{u}(r,s) = \overline{\mathbf{u}}(r,s) + \hat{\mathbf{u}}(r,s) = \sum_{i=1}^{n} h_i(r,s)\overline{\mathbf{u}}_i + \sum_{i=1}^{n} \hat{\mathbf{H}}_i(r,s)\hat{\mathbf{u}}_i$$
(2.14)

with

$$\overline{\mathbf{u}}_{i} = \begin{bmatrix} \overline{u}_{i} \\ \overline{v}_{i} \end{bmatrix}, \quad \hat{\mathbf{u}}_{i} = \begin{bmatrix} \hat{\mathbf{u}}_{i}^{u} \\ \hat{\mathbf{u}}_{i}^{v} \end{bmatrix}, \quad \hat{\mathbf{H}}_{i}(r,s) = \begin{bmatrix} \hat{\mathbf{h}}_{i}(r,s) & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{h}}_{i}(r,s) \end{bmatrix}, \quad (2.15)$$

in which  $\mathbf{\bar{u}}_i$  is the standard nodal displacement vector at node *i* in the global Cartesian coordinate system, and  $\hat{\mathbf{u}}_i$ and  $\hat{\mathbf{H}}_i(r,s)$  are the enriched DOFs vector and the corresponding interpolation matrix, respectively.

For the linear cover functions used (i.e., d = 1), the components of the interpolation matrix and the enriched DOFs vector become

$$\hat{\mathbf{h}}_{i}(r,s) = h_{i}(r,s)[\boldsymbol{\xi}_{i} \quad \boldsymbol{\eta}_{i}], \ \hat{\mathbf{u}}_{i}^{\boldsymbol{\mu}} = [\hat{\boldsymbol{\mu}}_{i}^{\boldsymbol{\xi}} \quad \hat{\boldsymbol{\mu}}_{i}^{\boldsymbol{\eta}}]^{T}, \ \hat{\mathbf{u}}_{i}^{\boldsymbol{\nu}} = [\hat{\boldsymbol{\nu}}_{i}^{\boldsymbol{\xi}} \quad \hat{\boldsymbol{\nu}}_{i}^{\boldsymbol{\eta}}]^{T}.$$

$$(2.16)$$

When the quadratic cover functions are used (d = 2), the following components and vector are employed

$$\hat{\mathbf{h}}_{i}(r,s) = h_{i}(r,s)[\xi_{i} \quad \eta_{i} \quad \xi_{i}^{2} \quad \xi_{i}\eta_{i} \quad \eta_{i}^{2}],$$

$$\hat{\mathbf{u}}_{i}^{u} = [\hat{u}_{i}^{\xi} \quad \hat{u}_{i}^{\eta} \quad \hat{u}_{i}^{\xi^{2}} \quad \hat{u}_{i}^{\xi\eta} \quad \hat{u}_{i}^{\eta^{2}}]^{T}, \quad \hat{\mathbf{u}}_{i}^{v} = [\hat{v}_{i}^{\xi} \quad \hat{v}_{i}^{\eta} \quad \hat{v}_{i}^{\xi^{2}} \quad \hat{v}_{i}^{\xi\eta} \quad \hat{v}_{i}^{\eta^{2}}]^{T}.$$
(2.17)

The principle of virtual work for linear elastic solid mechanics [1] is

$$\int_{V} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\tau} \, dV = \int_{V} \delta \boldsymbol{u}^{T} \boldsymbol{f}^{B} \, dV + \int_{S_{f}} \delta \boldsymbol{u}^{S_{f}T} \boldsymbol{f}^{S} , \qquad (2.18)$$

where  $\varepsilon$  is the strain vector,  $\tau$  is the stress vector,  $\mathbf{f}^{B}$  and  $\mathbf{f}^{S_{f}}$  are the body force vector and surface traction vector, respectively, and the  $\delta$  denotes a virtual quantity. The displacement vector in Eq. (2.14) can be written in matrix form for an element *m* as follows:

$$\mathbf{u}^{(m)} = \mathbf{H}^{(m)} \mathbf{u}^{(m)} , \qquad (2.19)$$

where  $\mathbf{H}^{(m)}$  is the interpolation matrix corresponding to the nodal DOFs vector  $\mathbf{u}^{(m)}$ , and the nodal DOFs vector  $\mathbf{u}^{(m)}$  includes  $\overline{\mathbf{u}}_i$  and  $\hat{\mathbf{u}}_i$ . Using the displacement-strain relation, the strain vector and the stress vector for an element *m* is obtained by [34]

$$\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)} \mathbf{u}^{(m)}, \qquad (2.20)$$

$$\boldsymbol{\tau}^{(m)} = \mathbf{C}\boldsymbol{\varepsilon}^{(m)} = \mathbf{C}\mathbf{B}^{(m)}\mathbf{u}^{(m)}, \qquad (2.21)$$

in which  $\mathbf{B}^{(m)}$  is the displacement-strain relation matrix and  $\mathbf{C}$  is the material law matrix.

Substituting Eqs. (2.19), (2.20) and (2.21) into the principle of virtual work in Eq. (2.18), the following matrix equation (static equilibrium equation) is obtained

$$\mathbf{K}\mathbf{U} = \mathbf{R} = \mathbf{R}_B + \mathbf{R}_S, \qquad (2.22)$$

with

$$\mathbf{K} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \mathbf{K}^{(m)} \mathbf{L}^{(m)} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C} \mathbf{B}^{(m)} dV \mathbf{L}^{(m)}$$
  

$$\mathbf{R}_{B} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \mathbf{R}_{B}^{(m)} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B} dV ,$$
  

$$\mathbf{R}_{S} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \mathbf{R}_{S}^{(m)} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \int_{S_{f}^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{S} dV ,$$
(2.23)

where **K** and **R** are the global stiffness matrix and load vector, **U** is the vector of the nodal DOFs of the entire finite element model,  $\mathbf{L}^{(m)}$  is the assemblage Boolean matrix for element *m*, *e* is the number of elements used, and summation signs denote the assembly procedure [1-3].

Note that the enriched displacement interpolation consists of the standard finite element interpolation  $\bar{\mathbf{u}}$  and the additional enriched higher order interpolation  $\hat{\mathbf{u}}$ . This polynomial enrichment scheme produces convergence behavior similar to the *p*-version of the finite element method. In addition, cover functions of the different polynomial degrees can be applied to each node.

When both the partition of unity functions and cover functions are composed of polynomial bases, the linear dependence (LD) problem occurs in some mesh topologies and leads to singular global stiffness matrices. To alleviate the LD problem, we enforce not only  $\overline{\mathbf{u}}_i = \mathbf{0}$ , but also  $\hat{\mathbf{u}}_i = \mathbf{0}$  when imposing the essential boundary condition at nodes [34,36,48]. When a finite element model consists of 3-node triangular elements, such treatment derives a well-conditioned stiffness matrix irrespective of mesh topology. However, if a finite element model consists only of 4-node quadrilateral elements or contains 4-node quadrilateral elements, the rank deficiency could appear in the global stiffness matrix depending on the mesh topology.

## 2.2. Piecewise linear shape functions for 4-node quadrilateral element

In order to resolve the LD problem of the enriched 4-node element formulated in the previous section, we here derive new shape functions  $\hat{h}_i$  satisfying the following requirements:

- Kronecker delta property:  $\hat{h}_i(r,s) = \delta_{ij}$  at node j with i, j = 1, 2, 3, 4,
  - $(\delta_{ij} = 1 \text{ if } i = j \text{ and } 0, \text{ otherwise}), \text{ see Fig. 2.2.}$
- Partition of unity:  $\sum_{i=1}^{4} \hat{h}_i(r,s) = 1$
- · Compatibility: linear variation along edges of the element
- · Completeness: displacement interpolations able to represent rigid body modes and constant strain states.

Satisfying the Kronecker delta property and the partition of unity, boundary conditions can easily be imposed as mentioned in Section 2.1 [1,34,36,48], and the shape functions can be used as the partition of unity functions [38,39]. Compatibility and completeness are required for monotonic convergence [1-3].

The key idea to avoiding the LD problem is to employ piecewise linear shape functions. This idea comes from the fact that enriched 3-node triangular elements with linear shape functions do not suffer from the LD problem.

The bilinear shape function of the standard 4-node elements for node *i* in Eq. (2.7) has the following form  $h_i(r,s) = a_i + b_i r + c_i s + d_i rs$ (2.24)

with four coefficients  $a_i$ ,  $b_i$ ,  $c_i$ , and  $d_i$  determined to satisfy the Kronecker delta property.



Figure 2.4. Triangular subdivision of the 4-node quadrilateral element.

Let us define the center point corresponding to r = s = 0. Then, the quadrilateral domain can be divided into four

triangular sub-domains as shown in Fig. 2.4. Note that at the center point, the bilinear shape function gives the value of 0.25.

In each triangular sub-domain, the linear interpolation for node *i* can be given as

$$\hat{h}_i(r,s) = \hat{a}_i + \hat{b}_i r + \hat{c}_i s$$
, (2.25)

where three coefficients  $\hat{a}_i$ ,  $\hat{b}_i$ , and  $\hat{c}_i$  are determined to satisfy the Kronecker delta property. Note that, unlike the bilinear shape functions in Eq. (2.10), the bilinear term (rs) is not included.

For example, the coefficients for the linear shape function  $\hat{h}_1$  for node 1 on the sub-domain T1 is determined with the following three conditions

$$\hat{h}_{1}(1,1)=1, \ \hat{h}_{1}(-1,1)=0, \ \hat{h}_{1}(0,0)=0.25,$$
(2.26)

and the resulting shape function is

$$\hat{h}_1 = (1+2r+s)/4$$
 on T1. (2.27)

The linear shape function  $\hat{h}_2$  for node 2 on the sub-domain T1 should satisfy the following three conditions

$$\hat{h}_2(1,1) = 0, \ \hat{h}_1(-1,1) = 1, \ \hat{h}_1(0,0) = 0.25,$$
(2.28)

and the resulting shape function is

$$\hat{h}_2 = (1 - 2r + s)/4$$
 on T1. (2.29)

Similarly, the linear shape functions are obtained in all the triangular sub-domains

$$\hat{h}_{1} = (1+2r+s)/4, \quad \hat{h}_{2} = (1-2r+s)/4, \quad \hat{h}_{3} = (1-s)/4, \quad \hat{h}_{4} = (1-s)/4 \quad \text{on T1},$$

$$\hat{h}_{1} = (1+r)/4, \quad \hat{h}_{2} = (1-r+2s)/4, \quad \hat{h}_{3} = (1-r-2s)/4, \quad \hat{h}_{4} = (1+r)/4 \quad \text{on T2},$$

$$\hat{h}_{1} = (1+s)/4, \quad \hat{h}_{2} = (1+s)/4, \quad \hat{h}_{3} = (1-2r-s)/4, \quad \hat{h}_{4} = (1+2r-s)/4 \quad \text{on T3},$$

$$\hat{h}_{1} = (1+r+2s)/4, \quad \hat{h}_{2} = (1-r)/4, \quad \hat{h}_{3} = (1-r)/4, \quad \hat{h}_{4} = (1+r-2s)/4 \quad \text{on T4}.$$
(2.30)

These piecewise linear shape functions satisfy all the requirements previously mentioned and also the element isotropy, that is, the element behavior does not depend on the sequence of node numbering [53]. The piecewise linear shape functions provide linear variation along edges of the element, shown in Fig. 2.5(c), like the linear and bilinear shape functions that are used for standard 3- and 4-node finite elements, respectively. Therefore, interelemental compatibility is satisfied with the standard elements.

Substituting the bilinear shape functions in Eq. (2.10) with the piecewise linear shape functions in Eq. (2.30), a new enriched 4-node finite element is simply constructed. When using the piecewise linear shape functions for geometry and displacement interpolations, the linear dependence (LD) problem is resolved. Analytical and numerical studies on this issue will be presented in Section 2.3.



Figure 2.5. Interpolation functions of 4-node quadrilateral elements on the cover region  $C_i$ : (a) Cover region (shaded) corresponding to node *i*, (b) and (c) Interpolation functions constructed by the bilinear and piecewise linear shape functions, respectively, on the cover region.

To evaluate the element stiffness matrix and load vector, the Gauss integration is separately adopted for each triangular sub-domain in an element, because  $C^1$  continuity is only satisfied in each triangular sub-domain, not between sub-domains. The triangular Gaussian integrations of degree 2 (3-point integration) and 4 (6-point integration) are adopted in each triangular sub-domain as shown in Fig. 2.6 to evaluate element stiffness matrices with linear and quadratic cover functions, respectively [54].

The piecewise linear shape functions are similar to the combination of shape functions of four triangular elements in a quadrilateral domain. However, when one quadrilateral element is replaced by four triangular elements, an additional node is necessary at the center of the quadrilateral element. The total degrees of freedom in a whole finite element model increases by almost twice. Of course, one quadrilateral element may be replaced by two triangular elements. In this case, the solution depends on the direction of the element and thus element isotropy is not satisfied. On the other hand, the new enriched element does not require any additional node and maintains the isotropy.



Figure 2.6. Gauss integration points used for the new enriched element with (a) Linear and (b) Quadratic covers.

## 2.3. Investigation of the linear dependence problem

We investigate the linear dependence (LD) problem of the new enriched 4-node element proposed in this thesis. We analytically identify whether the new enriched interpolation functions of a square element are linearly independent. Then the rank deficiency (RD) of the global stiffness matrix due to the LD problem is numerically evaluated by counting the number of zero eigenvalues in various finite element mesh topologies.

Two enriched 4-node quadrilateral elements are considered: the previous enriched element using the bilinear shape functions and the new enriched element using the piecewise linear shape functions. For comparison, the enriched 3-node triangular element is also considered.

For a single square element shown in Fig. 2.7, it is easy to find the analytical expressions for the enriched displacement interpolation. The essential boundary conditions are enforced for both  $\bar{\mathbf{u}}_i$  (standard DOFs) and  $\hat{\mathbf{u}}_i$  (enriched DOFs) at nodes. When the previous element is enriched by the linear cover function for the single square element, the enriched displacement interpolations *u* and *v* are given by

$$u = \alpha(1+r)(1+s)\overline{u_1} - \underline{\beta(1+r)(1-r)(1+s)}\hat{u}_1^{\xi} - \beta(1+r)(1+s)(1-s)\hat{u}_1^{\eta} + \alpha(1-r)(1+s)\overline{u_2} + \underline{\beta(1+r)(1-r)(1+s)}\hat{u}_2^{\xi} - \beta(1-r)(1+s)(1-s)\hat{u}_2^{\eta} ,$$

$$+ \alpha(1+r)(1-s)\overline{u_4}$$
(2.31)

$$v = \alpha(1+r)(1+s)\overline{v_1} - \frac{\beta(1+r)(1-r)(1+s)}{\beta_1^{\xi}} \hat{v}_1^{\xi} - \beta(1+r)(1-s)(1-s)\hat{v}_1^{\eta} + \alpha(1-r)(1+s)\overline{v_2} + \frac{\beta(1+r)(1-r)(1+s)}{\beta_2^{\xi}} \hat{v}_2^{\xi} - \beta(1-r)(1+s)(1-s)\hat{v}_2^{\eta},$$
(2.32)

where  $\alpha = 0.25$ ,  $\beta = 0.125$  and the underlined functions corresponding to DOFs  $\hat{u}_1^{\xi}$ ,  $\hat{u}_2^{\xi}$  and  $\hat{v}_1^{\xi}$ ,  $\hat{v}_2^{\xi}$  are identical each other; that is, the functions are linearly dependent [34].



Figure 2.7. A single square element for the investigation of the linear dependence (LD) problem.

On the contrary, with the new enriched element, we obtain the linearly independent displacement interpolation u on each sub-domain as follows:

$$u = \alpha(1+2r+s)\overline{u_{1}} - \beta(1+2r+s)(1-r)\hat{u}_{1}^{\xi} - \beta(1+2r+s)(1-s)\hat{u}_{1}^{\eta} \quad \text{on T1},$$
  

$$+ \alpha(1-2r+s)\overline{u_{2}} + \beta(1-2r+s)(1+r)\hat{u}_{2}^{\xi} - \beta(1-2r+s)(1-s)\hat{u}_{2}^{\eta} + \alpha(1-s)\overline{u_{4}} \quad \text{on T2},$$
  

$$u = \alpha(1+r)\overline{u_{1}} - \beta(1+r)(1-r)\hat{u}_{1}^{\xi} - \beta(1+r)(1-s)\hat{u}_{1}^{\eta} \quad \text{on T2},$$
  

$$+ \alpha(1-r+2s)\overline{u_{2}} + \beta(1-r+2s)(1+r)\hat{u}_{2}^{\xi} - \beta(1-r+2s)(1-s)\hat{u}_{2}^{\eta} + \alpha(1+r)\overline{u_{4}} \quad \text{on T3},$$
  

$$u = \alpha(1+s)\overline{u_{1}} - \beta(1+s)(1-r)\hat{u}_{1}^{\xi} - \beta(1+s)(1-s)\hat{u}_{1}^{\eta} \quad \text{on T3},$$
  

$$+ \alpha(1+s)\overline{u_{2}} + \beta(1+s)(1+r)\hat{u}_{2}^{\xi} - \beta(1+r)(1-s)\hat{u}_{1}^{\eta} + \alpha(1+2r-s)\overline{u_{4}} \quad \text{on T4}.$$
  

$$u = \alpha(1+r+2s)\overline{u_{1}} - \beta(1+r+2s)(1-r)\hat{u}_{1}^{\xi} - \beta(1+r+2s)(1-s)\hat{u}_{1}^{\eta} \quad \text{on T4}.$$
  

$$(2.33)$$

The enriched displacement interpolation v of the new enriched element is also linear independent as follows:

$$v = \alpha(1+2r+s)\overline{v_1} - \beta(1+2r+s)(1-r)\hat{v}_1^{\xi} - \beta(1+2r+s)(1-s)\hat{v}_1^{\eta} \quad \text{on T1}, + \alpha(1-2r+s)\overline{v_2} + \beta(1-2r+s)(1+r)\hat{v}_2^{\xi} - \beta(1-2r+s)(1-s)\hat{v}_2^{\eta} \quad \text{on T2}, v = \alpha(1+r)\overline{v_1} - \beta(1+r)(1-r)\hat{v}_1^{\xi} - \beta(1+r)(1-s)\hat{v}_1^{\eta} \quad \text{on T2}, + \alpha(1-r+2s)\overline{v_2} + \beta(1-r+2s)(1+r)\hat{v}_2^{\xi} - \beta(1-r+2s)(1-s)\hat{v}_2^{\eta} \quad \text{on T3}, + \alpha(1+s)\overline{v_1} - \beta(1+s)(1-r)\hat{v}_2^{\xi} - \beta(1+s)(1-s)\hat{v}_1^{\eta} \quad \text{on T3}, + \alpha(1+s)\overline{v_2} + \beta(1+s)(1+r)\hat{v}_2^{\xi} - \beta(1+r)(1-s)\hat{v}_2^{\eta} \quad \text{on T4}.$$
 (2.34)

We then consider plane stress problems modeled using various quadrilateral and triangular mesh patterns in Figs. 2.8 and 2.9. Two boundary condition cases described in Table. 2.1 are considered. In the boundary condition case (i), regardless of the mesh topology and the shape functions used, the enriched 2D solid finite elements yield the RD including the three rigid body modes and numerical results are given in the Tables 2.2 and 2.4. Except for zero eigenvalues corresponding to the three rigid body modes, the RD is calculated due to the LD of the additional DOFs.

Table 2.1. The boundary condition cases for the meshes shown in the Figs. 2.8, 2.9, and 2.10.

	Boundary conditions			
Case	Node <i>m</i>	Node <i>n</i>		
(i)	-	-		
(ii)	$\overline{u}_m = 0$ , $\overline{v}_m = 0$ , $\hat{\mathbf{u}}_m = 0$	$\overline{v}_n = 0$ , $\hat{\mathbf{u}}_m = 0$		

The analytical prediction of the RD for square and regular (right-angled triangle) meshes due to the LD was given by An [46] as follows:

RD(d) = d(d+2) for right-angled triangle meshes, (2.35)

 $RD(d) = (N-E)((d-1)(4-d)/2+1) + (d-1)^2 + 1$  for square meshes, (2.36)

where N, E are the numbers of nodes and elements, respectively, d is the degree of cover functions. Note that analytical prediction in Eqs. (2.35) and (2.36) are for one variable. Considering that the stiffness matrices used to calculated RD in Tables 2. 2 and 2.4 take into account two variables (u,v) and include three rigid body modes, the numerical results are in good agreement with the analytical predictions.

Table 2.3 shows the calculated rank deficiencies (RD) for various quadrilateral meshes shown in Fig. 8 when the boundary condition case (ii) is applied. When the previous enriched 4-node element is used in square and trapezoidal meshes (i.e., meshes with parallel edges), the RD increases as the number of element layers, or the degree of the cover functions, increases [46,48]. On the other hand, when the new enriched element is used with the boundary condition case (ii), the RD is not observed regardless of mesh patterns, the degree of the cover functions, or the number of element layers. Table 2.5 shows that the enriched 3-node triangular element does not exhibit the RD [34,36,48].

The new enriched 4-node element is compatibly usable with the enriched 3-node element to construct a finite element model. Quadrilateral and triangular mixed mesh patterns in Fig. 2.10 are also considered. The LD problem does not occur when the new enriched 4-node element is used with the enriched 3-node elements in the mixed mesh patterns, see Table 2.6. As in the meshes in Figs. 2.8 and 2.10, no rank deficiency is observed in the meshes used in the following sections, when using the new enriched element with the boundary condition case (ii).



Figure 2.8. Quadrilateral meshes for the investigation of the LD problem: (a) Square meshes, (b) Trapezoidal meshes, and (c) Distorted meshes.



Figure 2.9. Triangular meshes for the investigation of the LD problem: (a) Regular and (b) Distorted meshes.

Element	Number of element layers	RD / Total DOFs					
		Square mesh		Trapezoidal mesh		Distorted mesh	
		<i>d</i> = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2
Previous	1	11/24	25/48	11/24	25/48	9/24	19/48
	2	15/54	37/108	15/54	37/108	9/54	19/108
	4	23/150	61/300	23/150	61/300	9/150	19/300
	8	39/486	109/972	39/486	109/972	9/486	19/972
New	1	9/24	19/48	9/24	19/48	9/24	19/48
	2	9/54	19/108	9/54	19/108	9/54	19/108
	4	9/150	19/300	9/150	19/300	9/150	19/300
	8	9/486	19/972	9/486	19/972	9/486	19/972

Table 2.2. Rank deficiency (RD) of the global stiffness matrices when the enriched 4-node finite elements are used with the meshes shown in Fig. 2.8 (d: degree of cover functions). The boundary condition case (i) is applied.

Table 2.3. Rank deficiency (RD) of the global stiffness matrices when the enriched 4-node finite elements are used with the meshes shown in Fig. 2.8 (d: degree of cover functions). The boundary condition case (ii) is applied.

Element	Number of element layers	RD / Total DOFs					
		Square mesh		Trapezoidal mesh		Distorted mesh	
		<i>d</i> = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	<i>d</i> = 1	<i>d</i> = 2
Previous	1	2/13	6/25	2/13	6/25	0/13	0/25
	2	6/43	18/85	6/43	18/85	0/43	0/85
	4	14/139	42/277	14/139	42/277	0/139	0/277
	8	30/475	90/949	30/475	90/949	0/475	0/949
New	1	0/13	0/25	0/13	0/25	0/13	0/25
	2	0/43	0/85	0/43	0/85	0/43	0/85
	4	0/139	0/277	0/139	0/277	0/139	0/277
	8	0/475	0/949	0/475	0/949	0/475	0/949

Number of	RD / Total 1	RD / Total DOFs				
layers	Regular me	sh	Distorted m	Distorted mesh		
	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2		
1	9/18	19/36	9/18	19/36		
2	9/24	19/48	9/24	19/48		
4	9/54	19/108	9/54	19/108		
8	9/150	19/300	9/150	19/300		

Table 2.4. Rank deficiency (RD) of the global stiffness matrices when the enriched 3-node elements are used with the meshes shown in Fig. 2.9 (*d*: degree of cover functions). The boundary condition case (i) is applied.

Table 2.5. Rank deficiency (RD) of the global stiffness matrices when the enriched 3-node elements are used with the meshes shown in Fig. 2.9 (*d*: degree of cover functions). The boundary condition case (ii) is applied.

Number of element layers	RD / Total DOFs				
	Regular mesh		Distorted mesh		
	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	
1	0/13	0/25	0/13	0/25	
2	0/43	0/85	0/43	0/85	
4	0/139	0/277	0/139	0/277	
8	0/475	0/949	0/475	0/949	


Figure 2.10. Mixed mesh patterns for the investigation of the LD problem.

Table 2.6. Rank deficiency (RD) of the global stiffness matrices when the new enriched 4-node and the enriched 3-node elements are used together with the meshes shown in Fig. 2.10 (d: degree of cover functions). The boundary condition case (ii) is applied.

Number of	RD / Total DOFs						
element layers	Mesh (a)		Mesh (b)		Mesh (c)		
	<i>d</i> = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	
2	0/43	0/85	0/43	0/85	0/43	0/85	
4	0/139	0/277	0/139	0/277	0/139	0/277	

## 2.4. Numerical examples

The new enriched element passes the isotropy, zero energy mode, and patch tests considering three constant stress states ( $\tau_{xx}$ ,  $\tau_{xy}$ ,  $\tau_{yy}$ ) for arbitrary enrichment, see Fig. 2.11 [1,36]. In all the tests, the enriched DOFs  $\hat{\mathbf{u}}_i$  are suppressed at two boundary nodes (nodes 3 and 4) in Fig. 2.11 [48].



Figure 2.11. Finite element models for isotropy, zero energy mode and patch tests: (a) Single element for isotropy and zero energy mode tests and (b) Mesh for patch tests.

In the following sections, we investigate the performance and effectiveness of the new enriched 4-node element. Convergence is explored using the ad hoc and tool jig problems. The slender beam problem is considered with three different meshes. We illustrate the adaptive use of cover functions in the automotive wheel problem. The essential boundary conditions are imposed as mentioned in Sections 2.2 and 2.3, and the nodal loads corresponding not only to the standard DOFs but also to the enriched DOFs are considered. In addition, through free vibration analysis of a cantilever beam problem, we show that the new enriched element is suitable not only for static analysis but also for dynamic analysis.

In the ad hoc problem, the convergence study is performed using the s-norm defined as follows [32]

$$\left\|\mathbf{u}-\mathbf{u}_{h}\right\|_{s}^{2}=\int_{\Omega}\Delta\boldsymbol{\varepsilon}^{T}\Delta\boldsymbol{\tau}\,dV \quad \text{with} \quad \Delta\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_{h}, \quad \Delta\boldsymbol{\tau}=\boldsymbol{\tau}-\boldsymbol{\tau}_{h}, \quad (2.37)$$

where  $\mathbf{u}$  is the exact solution, and  $\mathbf{u}_h$  is the solution obtained using the finite element discretization. The snorm is suitable for identifying whether the finite element formulation satisfies the consistency and inf-sup conditions [55-58]. When the exact solution is not available, a fine mesh reference solution can be used.

For the relative error  $E_h$ , the theoretical convergence behavior can be estimated to be

$$E_{h} = \frac{\left\|\mathbf{u} - \mathbf{u}_{h}\right\|_{s}^{2}}{\left\|\mathbf{u}\right\|_{s}^{2}} \cong ch^{k} , \qquad (2.38)$$

in which h indicates the element size and c is a constant. If an element is uniformly optimal, the k represents the optimal order of convergence: k = 2, 4, and 6 for linear, quadratic and cubic elements.

#### 2.4.1. Ad hoc problem

Considering the ad hoc plane stress problem shown in Fig. 2.12 [1,36], we investigate the convergence behavior with regular and distorted meshes.



Figure 2.12. Ad hoc problem: (a) Problem domain,  $E = 1.0 \times 10^7 N / m^2$ , v = 0.3, (b) Mesh distortion types, (c) Regular quadrilateral and triangular meshes when N = 4, and (d) Quadrilateral and triangular meshes of the distortion type 2 when N = 8 (N : the number of element layers along an edge).

The following body forces that satisfy equilibrium equations are applied in the problem domain

$$f_x^B = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right), \quad f_y^B = -\left(\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x}\right), \tag{2.39}$$

in which the stress components are obtained from the in-plane displacements given by

 $u = (1 - x^2)^2 (1 - y^2)^2 e^{my} \cos mx, \quad v = (1 - x^2)^2 (1 - y^2)^2 e^{my} \sin mx \quad \text{with} \quad m = 5$ (2.40)

and material constants: Young's modulus  $E = 1.0 \times 10^7 N/m^2$  and Poisson's ratio v = 0.3. The body forces are applied to the finite element model through the load vector in Eq. (2.23) and the fixed boundary condition (u = v = 0) is imposed along the line v = -1.

We consider three quadratic elements (QUAD9, QUAD4-d1, TRI3-d1):

· QUAD9: standard 9-node quadrilateral element,

- · QUAD4-d1: new 4-node quadrilateral element enriched by linear covers,
- TRI3-d1: 3-node triangular element enriched by linear covers,

and three cubic elements (QUAD16, QUAD4-d2, TRI3-d2):

- QUAD16: standard 16-node quadrilateral element,
- QUAD4-d2: new 4-node quadrilateral element enriched by quadratic covers,
- TRI3-d2: 3-node triangular element enriched by quadratic covers.

Note that the order of the displacement interpolations of the enriched elements with linear and quadratic covers are quadratic and cubic, respectively. Therefore, the optimal order of convergence of the enriched elements with linear covers (QUAD4-d1, TRI3-d1) is the same as the standard quadratic element (QUAD9). The optimal order of convergence of the three cubic elements (QUAD4-d2, TRID-d2, QUAD16) is the same and is 6.

Fig. 2.12(c) shows the regular quadrilateral and triangular meshes used for the convergence studies when N = 4. We also consider the three different types (Types 1,2,3) of distorted meshes. Each square domain in Fig. 2.12(b) is subdivided into four quadrilateral subdomains by the lines AA and BB; then the edges of each quadrangle are subdivided into equal lengths to form the meshes. The distorted quadrilateral and triangular meshes of type 2 when N = 8 are shown in Fig. 2.12(d).

Figs. 2.12 and 2.13 show the convergence curves of the quadratic and cubic elements for the ad hoc problem. All 2D solid elements present similarly good convergence behaviors in regular and distorted meshes. Note that the convergence behavior of the new enriched 4-node element is better and less affected by mesh distortion than that of the enriched 3-node element.



Figure 2.13. Convergence curves of the quadratic elements for the ad hoc problem with the meshes shown in Fig. 2.12: The bold line represents the optimal convergence rate, which is 4.0 for quadratic elements.



Figure 2.14. Convergence curves of the cubic elements for the ad hoc problem with the meshes shown in Fig. 2.12: The bold line represents the optimal convergence rate, which is 6.0 for cubic elements.

### 2.4.2. Tool jig problem

A tool jig is subjected to a constant pressure on its top surface (line AB) and the fixed boundary condition is applied along the line AC as shown in Fig. 2.15(a) [59]. The standard 4-node element (QUAD4), the new enriched 4-node elements (QUAD4-d1 and QUAD4-d2), and the 4-node incompatible mode element (QUAD4+incompatible) in ADINA are considered with four different meshes, see Fig. 2.15(c) [60-63]. The reference solution is calculated using a fine mesh of 12,800 standard 9-node quadrilateral elements (QUAD9) leading to 103,518 DOFs, and the maximum stress occurs at point P.

As seen in Fig. 2.16, the new enriched element with the course mesh (1,866 DOFs, i.e., only about 22% of the DOFs of the standard element with fine-1 mesh) predicts the von Mises stress (effective stress) at the point P more accurately than does the standard element using the fine-1 mesh (8,638 DOFs). Fig. 2.17 shows the calculated *y*-displacement (v) and the von Mises stress (averaged at the nodes) along the line AB, when similar DOFs are used for finite element models of QUAD4 and QUAD4-d1. It is observed that QUAD4-d1 provides better solution accuracy compared to QUAD4.

The new enriched elements, QUAD4-d1 and -d2, are additionally compared with the 4-node incompatible mode element (QUAD4+incompatible) in ADINA. In Fig. 2.18, the calculated *y*-displacement (*v*) and the von Mises stress (averaged at the nodes) along the line AB are given when the coarse, medium, and fine-1 meshes are used. In all the cases, the results of the new enriched elements (QUAD4-d1 and -d2) and the 4-node incompatible mode element (QUAD4+incompatible) converge well to the reference value, and all three elements provide better solutions than the standard element (QUAD4) does.



Figure 2.15. The tool jig, material properties:  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3: (a) Problem description, (b) von Mises stress ( $\tau_v$ ) distribution of the reference solution, and (c) Coarse, medium, fine-1, and fine-2 meshes used.



Figure 2.16. von Mises stress distributions for the tool jig problem: The von Mises stress and its error at point P are presented for each solution. (DOFs = the number of degrees of freedom used, Error



$$= |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100 \%, \ \tau_{v,ref} = 13,193 N / m^2).$$

Figure 2.17. Comparison of results for the tool jig problem along the line AB when the standard 4-node element (QUAD4) and the new enriched element with linear covers (QUAD4-d1) are used for (a) medium and coarse meshes, respectively, and for (b) fine-2 and fine-1 meshes, respectively.



Figure 2.18. Comparison of numerical results for the tool jig problem along the line AB using: (a) Coarse mesh, (b) Medium mesh, and (c) Fine-1 mesh.

### 2.4.3. Slender beam problem

Here, we consider the slender cantilever beam problem proposed by MacNeal [64]. There are two load cases (a

shear force and a moment at the free tip), see Fig. 2.19. The beam has length L = 6m and width W = 0.2m. Its material properties are Young's modulus  $E = 1.0 \times 10^7 N/m^2$  and Poisson's ratio v = 0.3. The reference solutions of the y-displacement at point P are -0.1081 and -0.0054 for the shear force and moment load cases, respectively [64].

The finite element models are constructed using the standard 4-node element (QUAD4), the 4-node incompatible mode element (QUAD4+incompatible), and the new enriched 4-node element with linear and quadratic covers (QUAD4-d1 and -d2) for three different meshes, shown in Fig. 2.19. Tables 2.7 and 2.8 show the normalized y-displacements at the point P for shear force and moment load cases, respectively. Numerical results of the standard 4-node element and the 4-node incompatible mode element are affected by mesh distortion, while the new enriched 4-node elements show highly accurate solutions regardless of the mesh used.



Figure 2.19. Slender beam under two load cases: shear force (F = 1N) and moment ( $M = 0.2 N \cdot m$ ) at the free tip.

Table 2.7. Normalized y-displacement (v) at the point P for the slender beam subjected to shear force (F = 1N) at the free tip. The meshes (a-c) are shown in Fig. 2.19 ( $v_{ref} = -0.1081m$ ).

	Standard	4-node incompatible mode	The new enriched elements		
	4-node element	element	Linear covers	Quadratic covers	
	(QUAD4)	(QUAD4+incompatible)	(QUAD4-d1)	(QUAD4-d2)	
Mesh (a)	0.0933	0.9929	0.9821	0.9946	
Mesh (b)	0.0342	0.6318	0.9667	0.9948	
Mesh (c)	0.0269	0.0516	0.9628	0.9948	

	Standard	4-node incompatible mode	The new enriched elements		
	4-node elementelement(QUAD4)(QUAD4+incompatible)		Linear covers (QUAD4-d1)	Quadratic covers (QUAD4-d2)	
Mesh (a)	0.0933	1.0000	0.9916	0.9965	
Mesh (b)	0.0308	0.7250	0.9919	0.9966	
Mesh (c)	0.0223	0.0472	0.9920	0.9966	

Table 2.8. Normalized y-displacement (v) at the point P for the slender beam subjected to moment ( $M = 0.2 N \cdot m$ ) at the free tip. The meshes (a-c) are shown in Fig. 2.19 ( $v_{ref} = -0.0054 m$ ).

#### 2.4.4. Automotive wheel problem

We here illustrate the adaptive use of cover functions, an advantage of the enriched finite elements [19,20,60]. It is very effective to apply cover functions to nodes in the area where high stress gradients are observed. Numerical results obtained employing the enriched elements with quadratic covers (QUAD4-d2) and the adaptive use of no/linear/quadratic covers, are compared with results of the standard element (QUAD4) and the 4-node incompatible mode element (QUAD4+incompatible) in ADINA [38-40].

Consider an automotive wheel with a radius of 0.2 m, in which a lower part of the outer circle is subjected to a pressure load and the inner circle is fixed, as shown in Fig. 19(a). The 2D wheel structure is modeled using 3-node triangular and 4-node quadrilateral elements. Two different meshes are considered: coarse mesh (360 quadrilateral and 546 triangular elements, in total 906 elements) and fine mesh (2,289 quadrilateral and 198 triangular elements, in total 2,487 elements), see Figs. 2.20(b) and (c).

We perform the following six different cases of finite element analyses:

- (Case 1) No cover enrichment is adopted in the coarse mesh. That is, the standard 3- and 4-node finite elements (TRI3 and QUAD4) are used.
- (Case 2) In the fine mesh, the standard elements (TRI3 and QUAD4) are used without cover functions.
- (Case 3) The 4-node incompatible mode elements are adopted in the fine mesh and no cover enrichment is applied (QUAD4+incompatible).
- (Case 4) Quadratic covers are applied at entire nodes in the coarse mesh.
- (Case 5) No, linear and quadratic covers are adaptively used, see Fig. 2.20(d).
- (Case 6) As shown in Fig. 2.20(e), linear and quadratic covers are adaptively used.

The fine mesh is used only for Cases 2 and 3, and the coarse mesh is applied in all other cases. In Case 2, the standard 3- and 4-node elements are used for elements without cover enrichment. The 4-node incompatible mode elements and the standard 3-node elements are used for the fine mesh in Case 3. The adaptive use of cover functions in Cases 5 and 6 is determined by investigating the stress solutions obtained using the standard finite

elements in Case 1. Higher order covers are chosen for nodes where relatively higher von Mises stresses are predicted. The reference solutions are calculated using a mesh of standard 9-node quadratic finite elements, in which 9,058 elements and 74,414 DOFs are used.



Figure 2.20. The Automotive wheel problem: (a) Problem description,  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3, (b) and (c) Coarse and fine meshes, (d) No, linear, and quadratic covers adaptively used in Case 5, and (e) Linear and quadratic covers adaptively applied in Case 6.

We compare the strain energy and the von Mises stress (effective stress) at point P, shown in Fig. 2.20(a). Fig. 2.21(a) presents the distribution of the von Mises stress obtained from the reference solution. The number of DOFs used and errors in the results are summarized in Table 2.9. The von Mises stress fields obtained from Cases 1, 2, and 6 are presented in Fig. 21(b).

The solution accuracy is improved by using finer mesh or by applying the cover functions and the 4-node incompatible mode element. Comparing Cases 2, 3, and 5 using similar DOFs, it can be clearly observed that the adaptive use of the cover function is very effective in accurately predicting strain energy and von Mises stress.



Figure 2.21. von Mises stress distributions for the automotive wheel problem: (a) Reference solution obtained by using 9,058 standard 9-node quadratic elements, (b) von Mises stress distributions calculated in analysis Cases 1, 2, and 6. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$  %).

Table 2.9. Relative errors in von Mises stresses at point P and strain energies for the automotive wheel problem shown in Fig. 2.20(a). Relative error (%) in von Mises stresses,  $E_h^{\tau_v} = |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$ . Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_h| / e_{ref} \times 100$ .

	Standard linear element			Enriched element (coarse mesh)		
	Coarse mesh	Fine mesh		Quadratic	Adaptive use of interpolation covers	
		Standard 4-node element	4-node incompatible mode element	000013	No/linear/ quadratic covers	Linear/ quadratic overs
Analysis Case	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
DOFs	1,532	5,268	5,268	9,192	5,148	6,420
Relative error in von Mises stress	19.59%	3.75%	1.76%	0.08%	0.66%	0.44%
Relative error in strain energy	9.04%	2.66%	1.54%	0.99%	1.33%	1.02%
Reference		$\tau_{v,ref} = 1.384$	$\times 10^7 N/m^2$ , $e_{ref}$	$= 6.970 \times 10^{-1}$ M	$v \cdot m$	

## 2.4.5. Cantilever beam with fillets

A cantilever beam problem subjected to the line load is solved, see Fig. 2.22(a). In this example, quadrilateral and triangular elements are used together and different interpolation covers are applied over the domain considered. The mixed mesh pattern composed of 114 quadrilateral meshes and 168 triangular meshes and the von Mises stress contour of the reference solution using 1970 standard 9-node quadrilateral elements are shown in the Fig. 2.22, where the high stress gradients in the fillets can be seen.

Fig. 2.22(c) and (d) show two available cases that different interpolation covers are applied and these cases are referred to as Case 1 and 2, respectively. Note that the choices of interpolation cover are determined based on the results of standard elements. In addition, the cases where the linear and quadratic interpolation covers are applied to all nodes are also considered, and these results are summarized in the Table 2.10.

As expected, the accuracy of the solution increased with increasing the order of the interpolation cover. In particular, applying different interpolation covers, the strain energy is well predicted with a much smaller number of degrees of freedom.



Figure 2.22. The cantilever beam with large fillet radius: (a) problem description, material properties  $E = 7.2 \times 10^6 N / m^2$ , v = 0.3, (b) von Mises stress field of the reference solution, (c) analysis Case 1, and (d) analysis Case 2.

Table 2.10. Relative errors in strain energy in the cantilever beam with large fillets problem for the six cases. Relative error (%) =  $\left(E_{ref} - E_{h}\right) / E_{ref} \times 100$ .

	Standard element		Enriched e	Enriched element			
	9-node 3-node &	Linear	Quadratic	Mixed covers			
	(reference)	4-node	covers	covers	Case 1	Case 2	
DOFs	16204	468	1404	2808	1086	1842	
Relative error (%)	0.00	12.97	0.60	0.56	2.47	0.58	

## 2.4.6. Wrench problem

The wrench problem shown in Fig. 2.23(a) is solved. Wrench is subjected to a uniform pressure load on line AA. We first perform the linear static analysis using 143 4-node standard finite elements, and then we selectively enrich nodes that are in high stress gradient area as shown in Fig. 2.23(b). The number of degrees of freedom in each case are 360 and 666. Calculated effective stress along line BB are presented in Fig. 2.24. By applying enrichment functions adaptively, the accuracy of solution is improved effectively.



Figure 2.23. Wrench problem: (a) problem description and (b) mesh used ( $E = 1.0 \times 10^7 N / m^2$  and v = 0.3).



Figure 2.24. Effective stress ( $N/m^2$ ) distribution along line BB shown in Fig. 2.23(a).

### 2.4.5. Thick-walled cylinder problem

A thick-walled cylinder subjected to a internal pressure is under plane strain condition, see Fig. 2.25. Due to symmetry, only a one-quarter model is considered and the inner and outer radius ( $R_1$  and  $R_2$ ) are 3 and 9, respectively. When the Poisson's ratio v = 0.45, 0.49, 0.499, the corresponding radial displacement at  $r = R_1$  are  $4.9481 \times 10^{-3}$ ,  $5.0399 \times 10^{-3}$ ,  $5.0602 \times 10^{-3}$ , respectively. The corresponding strain energy are  $1.1659 \times 10^{-2}$ ,  $1.1875 \times 10^{-2}$ ,  $1.1923 \times 10^{-2}$ , respectively [65].

The standard 4-node and 9-node elements (QUAD4 and QUAD9), and the new enriched 4-node elements

(QUAD4-d1 and QUAD4-d2) are considered with mesh shown in Fig. 2.25. The normalized radial displacement at  $r = R_1$  and the normalized strain energy are summarized in Tables 2.10 and 2.11, respectively. It is shown that the volumetric locking can be avoided by using the new enriched elements



Figure 2.25. Thick-walled cylinder problem ( $R_1$ =3.0.  $R_2$ =9.0, thickness = 1.0, E = 1000, v = 0.45, 0.49, 0.499).

Table 2.11. Normalized radial displacement at  $r = R_1$  for a thick-walled cylinder problem shown in Fig. 2.25 [65].

		<i>v</i> = 0.45	v = 0.49	v = 0.499
Standard element	4-node	0.9616	0.8458	0.3592
	9-node	0.9997	0.9986	0.9864
Enriched element	Linear cover	0.9912	0.9819	0.9582
	Quadratic cover	0.9967	0.9973	0.9975

Table 2.12. Normalized strain energy for a thick-walled cylinder problem shown in Fig. 2.25 [65].

		v = 0.45	v = 0.49	v = 0.499
Standard element	4-node	0.9579	0.8425	0.3578
	9-node	0.9997	0.9986	0.9864
Enriched element	Linear cover	0.9963	0.9938	0.9762
	Quadratic cover	0.9974	0.9974	0.9970

### 2.4.7. Vibration analysis

In earlier studies, it has been reported that when using the previous enriched 4-node finite element, the global matrices become singular or ill-conditioned due to the linear dependence of some functions in the enriched interpolation function [38,41,46,48]. Therefore, the previous enriched element is inappropriate, especially for dynamic analyses. Here, a free vibration analysis is performed to show that the new enriched element can be used for dynamic analysis as well as for static analysis.

The free vibration analysis of a cantilever beam with length 1 m and width 0.1 m is considered, see Fig. 2.26(a). The fixed boundary condition is imposed along the line AB, and the enriched DOFs  $\hat{\mathbf{u}}_i$  of all nodes on the line AB are suppressed, as mentioned in Sections 2.2 and 2.3.

The generalized eigenvalue problem is given as

$$\mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \mathbf{M}\boldsymbol{\varphi}_i \quad \text{with} \quad \lambda_i = \omega_i^2 \quad \text{for} \quad i = 1, 2, \dots, n,$$
(2.41)

where **K** and **M** are the global stiffness and mass matrices, respectively;  $\omega_i$ ,  $\lambda_i$  and  $\varphi_i$  are the eigenfrequency, eigenvalue, and eigenvector corresponding to the ith mode, respectively; and n is the number of DOFs in the finite element model [1].

In Eq. (2.41), the global mass matrix is constructed in the same way as the global stiffness matrix in Eq. (2.23) as follows:

$$\mathbf{M} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \mathbf{M}^{(m)} \mathbf{L}^{(m)} = \sum_{m=1}^{e} \mathbf{L}^{(m)T} \int_{V^{(m)}} \rho \, \mathbf{H}^{(m)T} \mathbf{H}^{(m)} dV \, \mathbf{L}^{(m)},$$
(2.42)

in which  $\rho$  is the material density.

The finite element models are constructed using the previous and new enriched 4-node elements with quadratic covers for the coarse mesh (720 DOFs) in Fig. 2.26(b), and the 4-node incompatible mode element (QUAD4+incompatible) for the fine mesh (840 DOFs) shown in Fig. 2.26(c). We calculate the eigenfrequencies and eigenvectors corresponding to the  $1^{st}$ - $5^{th}$  modes. The reference solutions are obtained with a  $5 \times 40$  mesh of the standard 25-node quadrilateral finite elements (6,720 DOFs), see Fig. 2.26(d).

Table 2.13 and Fig. 2.27 show the calculated eigenfrequencies and mode shapes, respectively. The previous enriched 4-node element exhibits inaccurate eigenfrequencies and spurious energy modes. However, when the new enriched 4-node element is used, spurious energy modes do not occur. The results show that the new enriched element can be used not only for static analysis, but also for vibration analysis with good solution accuracy.

Mode	Reference	Coarse mesh		Fine mesh
number		Enriched 4-node element with quadratic covers (720 DOFs)		4-node incompatible mode element
		Previous	New	(840 DOFs)
1	5.0857E2	5.0879E2	5.0884E2	5.0881E2
2	3.0509E3	3.0526E3	3.0529E3	30560E3
3	7.9327E3	7.9339E3	7.9339E3	7.9340E3
4	8.0360E3	7.9650E3	8.0418E3	8.0640E3
5	1.4603E4	8.0412E3	1.4615E4	1.4687E4

Table 2.13. Eigenfrequencies (rad / s) corresponding to the 1<sup>st</sup>~5<sup>th</sup> modes for the cantilever beam problem in Fig. 2.26.



Figure 2.26. Cantilever beam problem: (a) Problem description,  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3, and  $\rho = 7860 kg / m^3$ , (b) Coarse mesh, (c) Fine mesh, and (d) Mesh used for reference solution.



Figure 2.27. Mode shapes corresponding to the 1<sup>st</sup>~5<sup>th</sup> modes for the cantilever beam problem in Fig. 2.26.

## 2.5. Computational efficiency

In this section, numerical costs are compared considering the standard 9- and 16-node quadrilateral finite elements and the enriched 4-node finite elements with linear and quadratic covers. In all the cases, symmetric stiffness matrices are generated. To obtain valuable insight into the computational cost needed in the respective solutions, we consider the process of constructing the stiffness matrix (including the calculation of elemental stiffness matrices) and of obtaining a solution of the linear equations. The computational cost is tested considering the regular meshes shown in Fig. 2.12(c).

We first compare the number of numerical integration points used for the standard finite elements and the new enriched 4-node elements. The 9- and 16-integration points ( $3 \times 3$  and  $4 \times 4$  Gauss integrations) are adopted for the standard 9-node and 16-node elements, respectively. For the new enriched element with linear and quadratic covers, the 12 (3 integration points  $\times$  4 sub-triangles) and 24 (6 integration points  $\times$  4 sub-triangles) integration points are used, respectively. The new enriched element with linear and quadratic covers require approximately 1.3 and 1.5 times, respectively, the number of integration points as the standard finite elements of the same order.

We also investigate how the size of the stiffness matrices increases as a function of the number of element layers. Fig. 2.28 shows that, considering the same displacement interpolation order, the new enriched 4-node elements have fewer DOFs than do the standard finite elements.



Figure 2.28. The total number of DOFs when increasing the number of element layers, N, along an edge: p denotes the number of solution variables considered, hence p = 2 (u and v) for the plane stress problem.

Tables 2.14 and 2.15 list the sparseness of the stiffness matrices and the total number of DOFs, respectively. Fig. 2.29 shows the structures of the stiffness matrices with meshes used when N = 16. In the comparison between the standard finite elements and the new enriched elements of the same order, it can be seen that the half-bandwidth of the stiffness matrix using the enriched elements is smaller, but the number of non-zero entries in the stiffness matrices is larger than that of the standard finite elements. The results show that the enriched finite elements generate the stiffness matrices with a smaller half-bandwidth and size than the standard finite elements.



Figure 2.29. Meshes used and stiffness matrix structures when N = 16: Non-zero entries are colored in black.

We measured the actual calculation time for constructing the stiffness matrix and solving the linear equations using direct Gauss elimination, in which the factorization of the stiffness matrices represents the major expense. A quad-core machine (Intel(R) Core i7-3770 CPU@ 3.40 GHz, 32 GB RAM, Windows 10 64bit) was used for all solution cases and the results are summarized in Tables 2.16 and 2.17.

As expected, in the comparison between finite elements of the same order, the new enriched elements take more time to construct the stiffness matrix than do the standard finite elements, while solving the linear equations generally takes less time. Also, as the number of elements used increases, the solving time becomes dominant. When a mesh of more than  $32 \times 32$  elements is used, the new enriched finite elements require less computational cost than do the standard finite elements.

Ν	Standard 9-noo (QUAD9)	Standard 9-node finite element (QUAD9)			New enriched 4-node element with linear covers (QUAD4-d1)		
	DOFs	HB	NNZ	DOFs	HB	NNZ	
8	544	73	1.5E4	432	65	1.9E4	
16	2,112	137	6.4E4	1,632	113	8.1E4	
32	8,320	265	2.6E5	6,336	209	3.3E5	
64	33,024	521	1.0E6	24,960	401	1.3E6	
128	131,584	1,033	4.2E6	99,072	785	5.3E6	

Table 2.14. Stiffness matrices when using the quadratic elements (QUAD9 and QUAD4-d1) for the ad hoc problem shown in Fig. 2.12 (DOFs: degrees of freedoms, HB: half-bandwidth, NNZ: number of non-zero entries).

Table 2.15. Stiffness matrices information when using the cubic elements (QUAD16 and QUAD4-d2) for the ad hoc problem shown in Fig. 2.12 (DOFs: degrees of freedoms, HB: half-bandwidth, NNZ: number of non-zero entries).

Ν	Standard 16-node finite element (QUAD16)		nt	New enriched 4-node element with quadratic covers (QUAD4-d2)			
	DOFs	HB	NNZ	DOFs	HB	NNZ	
8	1,200	157	5.5E4	864	131	7.9E4	
16	4,704	301	2.3E5	3,264	227	3.2E5	
32	18,624	589	9.1E5	12,672	419	1.3E6	
64	74,112	1,165	3.7E6	49,920	803	5.3E6	
128	295,680	2,317	1.5E7	195,072	1,559	2.1E7	

Ν	Standard 9-node finite element (QUAD9)			New enriched 4-node element with linear covers (QUAD4-d1)			
	Stiffness construction	Equation solver	Total	Stiffness construction	Equation solver	Total	
8	0.02	0.00	0.02	0.02	0.02	0.03	
16	0.05	0.06	0.11	0.08	0.05	0.13	
32	0.39	0.81	1.20	0.39	0.56	0.95	
64	1.09	12.33	13.42	1.17	8.38	9.55	
128	3.66	175.50	179.16	5.69	115.30	120.99	

Table 2.16. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations when using the quadratic elements (QUAD9 and QUAD4-d1) for the ad hoc problem shown in Fig. 2.12.

Table 2.17. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations when using the cubic elements (QUAD16 and QUAD4-d2) for the ad hoc problem shown in Fig. 2.12.

N Standard 16-node finite element (QUAD16)				New enriched 4-node element with quadratic covers (QUAD4-d2)			
	Stiffness construction	Equation solver	Total	Stiffness construction	Equation solver	Total	
8	0.05	0.03	0.08	0.09	0.03	0.13	
16	0.28	0.45	0.74	0.38	0.34	0.72	
32	0.77	6.87	7.64	2.52	4.77	7.28	
64	3.95	100.20	104.15	6.97	69.28	76.25	
128	13.74	1,571.00	1,584.74	26.69	999.90	1,026.59	

### 2.6. Closure

In this chapter, we proposed a new enriched 4-node 2D solid finite element free from the linear dependence (LD) problem. Piecewise linear shape functions were introduced and used for the geometry and enriched displacement interpolations. The rank deficiency was not observed with various mesh patterns and excellent convergence behaviors were observed, even when distorted meshes were used. The new enriched 4-node element can be used with the enriched 3-node element, and the degrees of cover functions can be chosen arbitrarily to increase the solution accuracy without mesh refinement or introducing additional nodes.

The piecewise linear shape functions are also applicable for the enriched 4-node plate and shell finite elements. Based on the observed features, we expect that enriched 4-node plate and shell finite elements will likely also be free from the LD problem. They will also show good convergence performance with distorted meshes, if shear locking and membrane locking are properly alleviated as in the enriched MITC3 shell finite element [36,37] using the concept of the MITC method [66-72].

# Chapter 3. The enriched 3D solid finite elements

In this chapter, the enriched 3D solid finite elements (8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements) for linear analysis are presented. To alleviate the linear dependence problem, sets of piecewise linear shape functions for each element are proposed and adopted for geometry and displacement interpolations. We verify that the linear dependence problem is resolved in various mesh patterns and show element performance and effectiveness through numerical examples.

### 3.1. Formulation of the enriched 3D solid finite elements

The geometry interpolation of the enriched 3D solid finite elements is

$$\mathbf{x}(r,s,t) = \sum_{i=1}^{n} h_i(r,s,t) \mathbf{x}_i \quad \text{with} \quad \mathbf{x}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^T$$
(3.1)

where *n* is the number of the nodes in each element,  $\mathbf{x}_i$  is the position vector of node *i* in the global Cartesian coordinate system shown in Fig. 3.1(a), and  $h_i(r, s, t)$  are the shape functions of standard isoparametric procedure corresponding to node *i* defined in the natural coordinate system in Fig. 3.1(b). The shape functions of the 8-node hexahedral, 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral elements are summarized in Appendix A.

The 3D shape functions,  $h_i$  satisfy the partition of unity requirement,  $\sum_{i=1}^{n} h_i = 1$ . Therefore, the displacement interpolation of the enriched 3D solid finite element is obtained by multiplying the shape functions with cover functions defined in the cover area  $C_i$  as follows [11,14,19]:

$$\mathbf{u}(r,s,t) = \sum_{i=1}^{n} h_i(r,s,t) \tilde{\mathbf{u}}_i \quad \text{with} \quad \tilde{\mathbf{u}}_i = \begin{bmatrix} \tilde{u}_i & \tilde{v}_i & \tilde{w}_i \end{bmatrix}^T , \qquad (3.2)$$

in which  $\tilde{u}_i$ ,  $\tilde{v}_i$  and  $\tilde{w}_i$  are cover functions corresponding to the displacements in the *x*-, *y*- and *z*-directions, respectively, and the cover  $C_i$  is the union of elements attached to node *i*, see Fig. 3.2.

The cover functions are

$$\widetilde{u}_{i} = \mathbf{p}_{i}(\mathbf{x})\mathbf{u}_{i}^{u}, \quad \widetilde{v}_{i} = \mathbf{p}_{i}(\mathbf{x})\mathbf{u}_{i}^{v}, \quad \widetilde{w}_{i} = \mathbf{p}_{i}(\mathbf{x})\mathbf{u}_{i}^{w} \text{ in } C_{i}$$
(3.3)
with
$$\mathbf{p}_{i}(\mathbf{x}) = \begin{bmatrix} 1 \quad \zeta_{i} \quad \eta_{i} \quad \zeta_{i} \quad \zeta_{i}^{2} \quad \zeta_{i} \eta_{i} \quad \cdots \quad \zeta_{i}^{d} \quad \cdots \end{bmatrix}, \quad \zeta_{i} = \frac{(x - x_{i})}{\chi_{i}}, \quad \eta_{i} = \frac{(y - y_{i})}{\chi_{i}}, \quad \zeta_{i} = \frac{(z - z_{i})}{\chi_{i}},$$

$$\mathbf{u}_{i}^{u} = \begin{bmatrix} u_{i}^{1} \quad u_{i}^{\xi} \quad u_{i}^{\eta} \quad u_{i}^{\zeta} \quad u_{i}^{\xi^{2}} \quad u_{i}^{\xi\eta} \quad \cdots \quad u_{i}^{\xi^{d}} \quad \cdots \end{bmatrix}^{T},$$

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$$\mathbf{u}_{i}^{v} = \begin{bmatrix} v_{i}^{1} & v_{i}^{\xi} & v_{i}^{\eta} & v_{i}^{\xi} & v_{i}^{\xi\eta} & \cdots & v_{i}^{\xi^{d}} & \cdots \end{bmatrix}^{T},$$
  
$$\mathbf{u}_{i}^{w} = \begin{bmatrix} w_{i}^{1} & w_{i}^{\xi} & w_{i}^{\eta} & w_{i}^{\xi} & w_{i}^{\xi^{2}} & w_{i}^{\xi\eta} & \cdots & w_{i}^{\xi^{d}} & \cdots \end{bmatrix}^{T},$$
  
(3.4)

where  $\mathbf{p}(\mathbf{x})$  is a complete polynomial basis vector for node *i*, *d* is the degree of complete polynomial bases shown in Fig. 3.3,  $\chi_i$  is the largest edge length of elements attached to node *i*, and  $\mathbf{u}_i^u$ ,  $\mathbf{u}_i^v$  and  $\mathbf{u}_i^w$  are the degrees of freedom (DOFs) vectors corresponding to the complete polynomial bases for the displacements *u*, *v* and *w*, respectively.



Figure 3.1. Coordinate systems for 3D solid finite elements: (a) 8-node hexahedral and 6-node prismatic solid elements in the global Cartesian coordinate system (x, y, z) and nodal local coordinate systems  $(\xi_i, \eta_i, \zeta_i)$ , and (b) 8-node hexahedral, 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral solid elements in the natural coordinate systems.

Substituting Eq. (3.3) into Eq. (3.2), the displacement interpolation of the enriched 3D solid elements is obtained  $\mathbf{u}(r,s,t) = \overline{\mathbf{u}}(r,s,t) + \hat{\mathbf{u}}(r,s,t) = \sum_{i=1}^{n} h_i(r,s,t)\overline{\mathbf{u}}_i + \sum_{i=1}^{n} \hat{\mathbf{H}}_i(r,s,t)\hat{\mathbf{u}}_i$ (3.5)

with

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$$\overline{\mathbf{u}}_{i} = \begin{bmatrix} \overline{u}_{i} \\ \overline{v}_{i} \\ \overline{w}_{i} \end{bmatrix}, \ \hat{\mathbf{u}}_{i} = \begin{bmatrix} \hat{\mathbf{u}}_{i}^{u} \\ \hat{\mathbf{u}}_{i}^{v} \\ \hat{\mathbf{u}}_{i}^{w} \end{bmatrix}, \ \hat{\mathbf{H}}_{i}(r,s,t) = \begin{bmatrix} \hat{\mathbf{h}}_{i}(r,s,t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{h}}_{i}(r,s,t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{h}}_{i}(r,s,t) \end{bmatrix},$$
(3.6)

where  $\mathbf{\bar{u}}_i$  is the standard nodal displacement vector at node *i* in the global Cartesian coordinate system, and  $\hat{\mathbf{u}}_i$ and  $\hat{\mathbf{H}}_i(r,s,t)$  are the enriched DOFs vector and the corresponding interpolation matrix, respectively.



Figure 3.2. Cover regions (shaded area) corresponding to node i when finite element models consist of (a) the 8-node hexahedral elements and (b) the 4-node tetrahedral elements.



Figure 3.3. Bases of complete polynomials up to degree 2.

For the linear cover functions used (i.e., d = 1), the components of the interpolation matrix and the enriched DOFs vector are

$$\hat{\mathbf{h}}_{i}(r,s,t) = h_{i}(r,s,t)[\boldsymbol{\xi}_{i} \quad \boldsymbol{\eta}_{i} \quad \boldsymbol{\zeta}_{i}],$$

$$\hat{\mathbf{u}}_{i}^{u} = [\hat{u}_{i}^{\boldsymbol{\xi}} \quad \hat{u}_{i}^{\eta} \quad \hat{u}_{i}^{\boldsymbol{\zeta}}]^{T}, \quad \hat{\mathbf{u}}_{i}^{v} = [\hat{v}_{i}^{\boldsymbol{\xi}} \quad \hat{v}_{i}^{\eta} \quad \hat{v}_{i}^{\boldsymbol{\zeta}}]^{T}, \quad \hat{\mathbf{u}}_{i}^{w} = [\hat{w}_{i}^{\boldsymbol{\xi}} \quad \hat{w}_{i}^{\eta} \quad \hat{w}_{i}^{\boldsymbol{\zeta}}]^{T}.$$

$$(3.7)$$

When the quadratic cover functions are used (i.e., d = 2), the following components and vectors are employed  $\hat{\mathbf{h}}_{i}(r,s,t) = h_{i}(r,s,t)[\xi_{i} \quad \eta_{i} \quad \zeta_{i} \quad \xi_{i}^{2} \quad \xi_{i}\eta_{i} \quad \eta_{i}^{2} \quad \eta_{i}\zeta_{i} \quad \zeta_{i}^{2} \quad \xi_{i}\zeta_{i}],$   $\hat{\mathbf{u}}_{i}^{u} = [\hat{u}_{i}^{\xi} \quad \hat{u}_{i}^{\eta} \quad \hat{u}_{i}^{\zeta} \quad \hat{u}_{i}^{\xi^{2}} \quad \hat{u}_{i}^{\xi\eta} \quad \cdots \quad \hat{u}_{i}^{\xi\zeta}]^{T},$   $\hat{\mathbf{u}}_{i}^{w} = [\hat{v}_{i}^{\xi} \quad \hat{v}_{i}^{\eta} \quad \hat{v}_{i}^{\zeta} \quad \hat{v}_{i}^{\xi^{2}} \quad \hat{v}_{i}^{\xi\eta} \quad \cdots \quad \hat{v}_{i}^{\xi\zeta}]^{T}.$ (3.8)

The static equilibrium equations for an enriched 3D finite element model can be obtained using the same way in Section 2.1.2. In order to alleviate the LD problem, we enforce both  $\overline{\mathbf{u}}_i = \mathbf{0}$  and  $\hat{\mathbf{u}}_i = \mathbf{0}$  when imposing the essential boundary condition at nodes. For a finite element model consisting of 4-node tetrahedral elements, such treatment derives a well-conditioned stiffness matrix regardless of mesh topology. However, if a finite element model contains 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements, the global stiffness matrix could be rank deficient depending on the mesh topology.

# 3.2. Sets of piecewise linear shape functions for 3D solid elements

The previous chapter presented that the use of piecewise linear shape functions resolve the LD problem of the enriched 4-node 2D solid element. The key idea is to apply piecewise linear shape functions to geometry and displacement interpolations of 3D solid elements.

To derive new shape functions, we use similar requirements in Section 2.2 as follows:

- Partition of unity:  $\sum_{i=1}^{n} \hat{h}_i(r, s, t) = 1$ .
- Kronecker delta property:  $\hat{h}_i(r,s,t) = \delta_{ij}$  at node j with  $i, j = 1, \dots, n$ ,
- $(\delta_{ij} = 1 \text{ if } i = j \text{ and } 0, \text{ otherwise}), \text{ see Fig. 3.1.}$
- Compatibility: continuous displacement interpolation across the element boundaries.
- · Completeness: displacement interpolations able to represent constant strain states and rigid body modes.

When constructing the displacement interpolation as in Eq. (3.2) by applying the enrichment scheme, the shape functions for the enriched finite elements should meet the partition of unity requirement [38]. Satisfying the Kronecker delta property, boundary conditions can easily applied in the manner described in Section 2.1 [1,34,48]. Compatibility and completeness are required for monotonic convergence of solutions [1-3].



Figure 3.4. Subdivision of (a) the 8-node hexahedral, (b) 6-node prismatic and (c) 5-node pyramidal elements.

To derive piecewise linear shape functions for enriched 3D solid elements, let us define a set of points for each element. The set of points contains in total  $n_p (= n + n_{quad} + 1)$  points including *n* nodes,  $n_{quad}$  centers of

quadrilateral faces, and an element center, see Fig. 3.4.

For an 8-node hexahedral element, 15 points are defined, including 8 nodes, centers of 6 quadrilateral faces and an element center. The 6-node prismatic element and the 5-node pyramidal element have 10 (=6+3+1) and 7 (=5+1+1) points, respectively (see Table B1 in Appendix B).

Those points are used to divide each element into tetrahedral sub-domains. Vertices of each tetrahedral sub-domain are defined by four points as shown in Fig. 3.4. The element center is shared as one vertex of all tetrahedral sub-domains, and the remaining three vertices consist of points excluding the element center. For the 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements, 24, 14 and 8 sub-domains are defined, respectively. The points used as vertices of sub-domains of each element are given in Appendix B.

In each tetrahedral sub-domain, the linear shape function can be given as

$$\hat{h}_i(r,s,t) = (\hat{a}_i + \hat{b}_i r + \hat{c}_i s + \hat{d}_i t) / n, \qquad (3.9)$$

in which should satisfy the Kronecker delta and partition of unity requirements at four points constituting each sub-domain. For each sub-domain, sets of piecewise linear shape functions can be derived by using the requirements defined at the four points.

At nodes, the linear shape function in Eq. (3.9) should meet the following Kronecker delta requirements

$$h_i(r_j, s_j, t_j) = \delta_{ij} \quad \text{with} \quad i, j = 1, \cdots, n ,$$
(3.10)

where i and j are is the node numbers.

At the centers of quadrilateral faces, the partition of unity requirements for the linear shape function  $h_i$ ( $i = 1, \dots, n$ ) are

$$\hat{h}_i(r_j, s_j, t_j) = \begin{cases} 0 & \text{if } i \notin \Omega_j \\ 1/4 & \text{if } i \in \Omega_j \end{cases} \text{ with } j = n+1, \cdots, n+n_{quad},$$
(3.11)

in which  $\Omega_j$  is a set of nodes consisting of the quadrilateral face whose center is considered as point j.

At the element center, the partition of unity requirement is

$$h_i(r_j, s_j, t_j) = 1/n \text{ with } j = n_p.$$
 (3.12)

For an example, consider the sub-domain T24 (defined by points 1, 2, 14, and 15) of the 8-node hexahedral element and the linear shape function  $\hat{h}_1$  corresponding to node 1, see Fig. 2.4(a). To obtain the coefficients of the linear shape function in Eq. (3.9) on sub-domain T24, the requirements in Eqs. (3.10) to (3.12) at points 1, 2, 14, and 15 are used.

The Kronecker delta requirements in Eq. (3.10) apply to points 1 and 2, because the points are nodes. Eq. (3.10)

gives the following two conditions

$$\hat{h}_1(1,1,-1) = 1, \quad \hat{h}_1(-1,1,-1) = 0.$$
 (3.13)

The points 14 and 15 are the center of quadrilateral face and the element center, respectively. The partition of unity requirements in Eqs. (3.11) and (3.12) are used at the points. At point 14 (a center of quadrilateral face), the requirement is given by

$$\hat{h}_{t}(0,0,-1) = 1/4,$$
(3.14)

because node 1 belongs to  $\Omega_{14}$ , that is, node 1 is positioned in the quadrilateral face defined by nodes 1, 2, 3 and 4, see Fig. 3.4(a).

At point 15 (element center), Eq. (3.12) gives the following requirement  

$$\hat{h}_1(0,0,0) = 1/8$$
. (3.15)

The linear shape function  $\hat{h}_1$  satisfying requirements in Eqs. (3.13) to (15) is obtained by

$$\hat{h}_1 = (1 + 4r + 2s - t)/8 \text{ on T24.}$$
 (3.16)

In this way, we can find the linear shape function corresponding to node 1 in other tetrahedral sub-domains.

For the 6-node prismatic element, the linear shape function  $\hat{h}_3$  on T4 should satisfy following requirements:

$$\hat{h}_3(0,1,-1) = 1, \quad \hat{h}_3(0,1,1) = 0, \quad \hat{h}_3(0,1/2,0) = 1/4, \quad \hat{h}_3(1/3,1/3,0) = 1/6,$$
(3.17)

and the resulting linear shape function is

$$\hat{h}_3 = (s-t)/2$$
 on T4. (3.18)

Similarly, the piecewise linear shape functions of the 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements in all sub-domains are obtained and the resulting coefficients are summarized in Appendix C.

Piecewise linear shape functions derived satisfy all the requirements previously described and also the element isotropy, that is, the element behavior does not depend on the element orientation [53]. The piecewise linear shape functions provide linear variation on triangular face and piecewise linear variation on quadrilateral face, see Fig. 3.5. Thus, the 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements based on the piecewise linear shape functions and the standard 4-node tetrahedral elements are compatible with each other.

Substituting the shape functions of standard isoparametric procedure in Eqs. (3.1) and (3.2) with the piecewise linear shape functions derived in this section, new enriched 3D solid elements can be simply constructed. Applying the piecewise linear shape functions for geometry and displacement interpolations resolves the LD problem. Analytical and numerical studies on this problem will be presented in Section 3.3.



Figure 3.5. Shape functions of the 8-node hexahedral, 6-node prismatic and 5-node pyramidal elements corresponding to node i: (a) Shape functions of the standard finite elements and (b) piecewise linear shape functions.

In order to evaluate the element stiffness matrix and load vector, the Gauss integration is separately applied for each tetrahedral sub-domain in an element because  $C^1$  continuity is only satisfied in each sub-domain, not between sub-domains. The tetrahedral Gaussian integrations of degree 2 (4-point integration) and 4 (11-point integration) are adopted in each sub-domain to evaluate element stiffness matrices with linear and quadratic cover functions, respectively [73].

## 3.3. Investigation of the linear dependence problem

In this section, we investigate the LD problem of the enriched 3D solid elements. Two different elements are considered: the previous enriched element using the shape functions of the standard finite element procedure and the new enriched element using the piecewise linear shape functions. We first analytically verify whether the enriched interpolation functions of a cube element are linearly independent. Then the rank deficiencies of the global stiffness matrix are numerically evaluated by counting the number of zero eigenvalues in finite element models composed of the 8-node hexahedral, 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral elements.

For the analytical study, we consider a single cube element shown in Fig. 3.6 and the essential boundary conditions are imposed for both  $\bar{\mathbf{u}}_i$  (standard DOFs) and  $\hat{\mathbf{u}}_i$  (enriched DOFs) at nodes 1, 2, and 4. When the previous element is enriched with the linear cover functions for the single cube element, the enriched displacement interpolation v on the shaded plane (r = 1) is

$$v(r = 1, s, t) = \alpha(1 - s)(1 - t)\overline{v}_{4} + \alpha(1 + s)(1 + t)\overline{v}_{5} - \beta(1 + s)(1 + t)(1 - s)\hat{v}_{5}^{\gamma} - \beta(1 + s)(1 + t)(1 - t)\hat{v}_{5}^{\zeta} , \qquad (3.19) + \alpha(1 - s)(1 + t)\overline{v}_{8} + \beta(1 - s)(1 + t)(1 + s)\hat{v}_{8}^{\gamma} - \beta(1 - s)(1 + t)(1 - t)\hat{v}_{8}^{\zeta}$$

in which  $\alpha = 0.25$ ,  $\beta = 0.125$  and the underlined functions corresponding to DOFs  $\hat{v}_5^{\eta}$  and  $\hat{v}_8^{\eta}$  are identical; that is, the functions are linearly dependent [35].



Figure 3.6. A single cube element for the investigation of the LD problem: (a) the single cube element and (b) r = 1 plane of the cube element.

On the other hand, with the new enriched element, the linear independent enriched displacement interpolation v(r=1) is given by

$$\begin{aligned} v(r = 1, s, t) &= \alpha(1-t)\overline{v}_{4} \\ &+ \alpha(1+2s+t)\overline{v}_{5} - \beta(1+2s+t)(1-s)\hat{v}_{5}^{r} - \beta(1+2s+t)(1-t)\hat{v}_{5}^{r} & \text{on } T1, \\ &+ \alpha(1-2s+t)\overline{v}_{8} + \beta(1-2s+t)(1+s)\hat{v}_{8}^{n} - \beta(1-2s+t)(1-t)\hat{v}_{8}^{r} \end{aligned}$$

$$v(r = 1, s, t) &= \alpha(1-s-2t)\overline{v}_{4} \\ &+ \alpha(1+s)\overline{v}_{5} - \beta(1+s)(1-s)\hat{v}_{5}^{n} - \beta(1+s)(1-t)\hat{v}_{5}^{r} & \text{on } T2, \\ &+ \alpha(1-s+2t)\overline{v}_{8} + \beta(1-s+2t)(1+s)\hat{v}_{8}^{n} - \beta(1-s+2t)(1-t)\hat{v}_{8}^{r} \end{aligned}$$

$$v(r = 1, s, t) = \alpha(1-2s-t)\overline{v}_{4} \\ &+ \alpha(1+t)\overline{v}_{5} - \beta(1+t)(1-s)\hat{v}_{5}^{n} - \beta(1+t)(1-t)\hat{v}_{5}^{r} & \text{on } T3, \\ &+ \alpha(1+t)\overline{v}_{8} + \beta(1+t)(1+s)\hat{v}_{8}^{n} - \beta(1+t)(1-t)\hat{v}_{8}^{r} \end{aligned}$$

$$v(r = 1, s, t) = \alpha(1-s)\overline{v}_{4} \\ &+ \alpha(1+s)\overline{v}_{5} - \beta(1+s+2t)(1-s)\hat{v}_{5}^{n} - \beta(1+s+2t)(1-t)\hat{v}_{5}^{r} & \text{on } T4, \\ &+ \alpha(1+s+2t)\overline{v}_{5} - \beta(1+s+2t)(1-s)\hat{v}_{5}^{n} - \beta(1+s+2t)(1-t)\hat{v}_{5}^{r} & \text{on } T4, \\ &+ \alpha(1-s)\overline{v}_{8} + \beta(1-s)(1+s)\hat{v}_{8}^{n} - \beta(1-s)(1-t)\hat{v}_{5}^{r} \end{aligned}$$

$$(3.20)$$

where the LD problem does not occur. Similarly, it can be identified that functions corresponding to each DOF are linear independent in all domain of the new enriched elements (including the 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements).

Let us then consider finite element models of various mesh patterns. Fig. 3.7 shows hexahedral mesh patterns, and prismatic, pyramidal, and tetrahedral meshes are created by dividing each hexahedron domains of hexahedral mesh patterns using the division method shown in Fig. 3.8. The enriched DOFs  $\hat{\mathbf{u}}_i$  are suppressed at  $P_1$ ,  $P_2$  and  $P_3$ . Note that when a hexahedron is divided into six pyramids, additional nodes are added to a center of each hexahedron.

Tables 3.1 and 3.2 show the calculated rank deficiencies (RD) for hexahedral and prismatic meshes shown in Fig. 3.7. When the previous enriched elements are used in meshes (a), (b), and (c) (i.e., meshes with parallel edges), the rank deficiency increases as the number of element layers, or the degree of the cover functions, increases [46,-48]. On the other hand, when the new enriched element is used, the rank deficiency is not observed regardless of mesh patterns, the degree of the cover functions, or the number of element layers.

Calculated rank deficiencies for pyramidal meshes shown in Fig. 7 is given in Table 3.3. When the number of element layers is more than 2, no rank deficiency is observed in both the previous and new elements. However, in meshes with single element layer and parallel edges, rank deficiencies is observed when the previous element is used, but not when a new element is used. In Table 3.4, the calculated rank deficiencies for tetrahedral meshes shown in Fig. 3.7 is summarized, and no rank deficiency is observed.

The new enriched 8-node hexahedral, 6-node prismatic, 5-node pyramidal elements and enriched 4-node tetrahedral element are compatible with each other and can be used together to construct a finite element model.

A finite element model based on a mixed mesh pattern shown in Fig. 3.9 is also considered. When the previous elements are used, the numbers of zero eigenvalues calculated are 24 and 93 for linear and quadratic covers, respectively. On the other hand, when the new enriched elements are used, the LD problem does not occur. As in the meshes shown in Figs. 3.7 and 3.9, no rank deficiency is observed in meshes used in the following sections, when using the new enriched elements.



Figure 3.7. Hexahedral meshes (solid lines) for the investigation of the LD problem: These hexahedral meshes are used as a base mesh to create prismatic, pyramidal, and tetrahedral meshes. The division methods are shown in Fig. 3.8 and the dotted lines denote prismatic meshes.


Figure 3.8. Division of hexahedron into (a) two prisms, (b) six pyramids and (c) six tetrahedrons.

Element	Number of	RD / Total	RD / Total DOFs						
	element lavers	Mesh (a)		Mesh (b)		Mesh (c)			
	lagers	<i>d</i> = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2		
Previous	1	15/62	57/152	15/62	57/152	0/62	0/152		
	2	60/290	228/722	60/290	228/722	0/290	0/722		
	4	204/1466	786/3662	204/1466	786/3662	0/1466	0/3662		
New	1	0/62	0/152	0/62	0/152	0/62	0/152		
	2	0/290	0/722	0/290	0/722	0/290	0/722		
	4	0/1466	0/3662	0/1466	0/3662	0/1466	0/3662		

Table 3.1. Rank deficiency (RD) of the global stiffness matrices when the enriched 8-node hexahedral elements are used with the meshes shown in Fig. 3.7 (d : degree of the cover functions).

Element	Number of	RD / Total DOFs						
	element lavers	Mesh (a)		Mesh (b)		Mesh (c)		
		d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	d = 1	d = 2	
Previous	1	12/62	45/152	12/62	45/152	0/62	0/152	
	2	36/290	138/722	36/290	138/722	0/290	0/722	
	4	102/1466	396/3662	102/1466	396/3662	0/1466	0/3662	
New	1	0/62	0/152	0/62	0/152	0/62	0/152	
	2	0/290	0/722	0/290	0/722	0/290	0/722	
	4	0/1466	0/3662	0/1466	0/3662	0/1466	0/3662	

Table 3.2. Rank deficiency (RD) of the global stiffness matrices when the enriched 6-node prismatic elements are used with the prismatic meshes shown in Fig. 3.7 (d: degree of the cover functions).

Table 3.3. Rank deficiency (RD) of the global stiffness matrices when the enriched 5-node pyramidal elements are used with the pyramidal meshes shown in Fig. 3.7 (d: degree of the cover functions).

Element	Number of	RD / Total DOFs							
	element lavers	Mesh (a)		Mesh (b)	Mesh (b)				
	5	d = 1	d = 2	d = 1	d = 2	d = 1	d = 2		
Previous	1	3/74	15/182	3/74	15/182	0/74	0/182		
	2	0/386	0/962	0/386	0/962	0/386	0/962		
	4	0/2234	0/5585	0/2234	0/5585	0/2234	0/5585		
New	1	0/74	0/182	0/74	0/182	0/74	0/182		
	2	0/386	0/962	0/386	0/962	0/386	0/962		
	4	0/2234	0/5585	0/2234	0/5585	0/2234	0/5585		

Table 3.4. Rank deficiency (RD) of the global stiffness matrices when the enriched 4-node tetrahedral elements are used with the tetrahedral meshes shown in Fig. 3.7 (d: degree of the cover functions).

Number of element layers	RD / Total	RD / Total DOFs									
	Mesh (a)		Mesh (b)		Mesh (c)	Mesh (c)					
	d = 1	<i>d</i> = 2	d = 1	<i>d</i> = 2	d = 1	d = 2					
1	0/62	0/152	0/62	0/152	0/62	0/152					
2	0/290	0/722	0/290	0/722	0/290	0/722					
4	0/1466	0/3662	0/1466	0/3662	0/1466	0/3662					



Figure 3.9. Finite element model with a mixed mesh pattern for the investigation of the LD problem: The finite element model consists of four 8-node hexahedral, four 6-node prismatic, five 5-node prismatic, and eight 4-node tetrahedral elements.

# 3.4. Numerical examples

The new enriched 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements pass the isotropy, zero energy mode, and patch tests for arbitrary enrichment [1,36]. Fig. 3.10(a) shows meshes used for the isotropy and zero energy mode tests and the hexahedral mesh shown in Fig. 3.10(b) is used for the patch test (prismatic and pyramidal meshes are generated from the hexahedral mesh). In all the tests, the enriched DOFs  $\hat{\mathbf{u}}_i$  are suppressed at three boundary nodes (nodes 1, 2 and 3) in Fig. 10 [34,48].

In this section, we investigate the performance and effectiveness of the new enriched 3D elements. Convergence behaviors of the new enriched 8-node hexahedral and 6-node prismatic elements are explored using the ad hoc and tool jig problems. The straight and curved beam problems are solved using 8-node hexahedral, 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral elements. We also demonstrate the adaptive use of cover functions in gear and connecting rod problems. The essential boundary conditions are imposed as described in previous sections, and the nodal loads corresponding to both the standard DOFs  $\mathbf{\bar{u}}_i$  and the enriched DOFs  $\mathbf{\hat{u}}_i$  are considered. In addition, it is shown that the new enriched elements are suitable not only for static analysis but also for dynamic analysis through free vibration analysis of a cantilever beam and the connecting rod.



Figure 3.10. Finite element models used for isotropy, zero energy mode and patch tests: (a) Single hexahedral, pyramidal, and prismatic elements for isotropy and zero energy mode tests and (b) hexahedral mesh for patch tests.

## 3.4.1. Ad hoc problem

Considering the ad hoc 3D solid problem shown in Fig. 3.11 [1,34], we investigate the convergence behaviors with regular and distorted meshes.

The following body forces that satisfy equilibrium equations are applied in the problem domain

$$f_x^B = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right), \quad f_y^B = -\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right), \quad f_z^B = -\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right), \quad (3.21)$$

in which the stress components are obtained from the displacements given by

$$u = (1 - x^{2})^{2} (1 - y^{2})^{2} (1 - z^{2})^{2} e^{my} \cos(mx) \sin(my) \cos(mz),$$

$$v = (1 - x^{2})^{2} (1 - y^{2})^{2} (1 - z^{2})^{2} e^{my} \sin(mx) \cos(my) \cos(mz),$$

$$w = (1 - x^{2})^{2} (1 - y^{2})^{2} (1 - z^{2})^{2} e^{my} \cos(mx) \cos(my) \sin(mz) \text{ with } m = 5,$$
and material constants: Young's modulus  $E = 2.0 \times 10^{11} N/m^{2}$  and Poisson's ratio  $v = 0.3$ . The body forces are applied to the finite element model through the load vector in Eq. (2.23) and the fixed boundary condition

(u = v = w = 0) is imposed on the surface at v = -1.

We consider five quadratic elements (HEX27, HEX8-d1, PRI18, PRI6-d1, TET4-d1):

- HEX27: standard 27-node hexahedral element,
- HEX8-d1: new 8-node hexahedral element enriched by linear covers,
- PRI18: standard 18-node prismatic element,
- PRI6-d1: new 6-node prismatic element enriched by linear covers,
- TET4-d1: 4-node tetrahedral element enriched by linear covers,

and five cubic elements (HEX64, HEX8-d2, PRI40, PRI6-d2, TET4-d2):

- HEX64: standard 64-node hexahedral element,
- · HEX8-d2: new 8-node hexahedral element enriched by quadratic covers,
- PRI40: standard 40-node prismatic element,
- PRI6-d2: new 6-node prismatic element enriched by quadratic covers,
- TET4-d2: 4-node tetrahedral element enriched by quadratic covers.

The problem domain in Fig. 3.11(a) is subdivided into eight hexahedral sub-domains by three planes A, B and C as shown in Fig. 3.11(b); then edges of each sub-hexahedron are subdivided into equal lengths to form the meshes as shown in Fig. 3.11(c). The distorted hexahedral meshes of types 2 and 3 when N=4 are shown in Fig. 3.11(d). Fig. 3.11(e) shows the regular hexahedral mesh used for the convergence studies when N=4. Distorted meshes of three different types (Types 1, 2, 3) are also used. Prismatic and tetrahedral meshes are generated from the hexahedral mesh by dividing each hexahedron element as shown in Fig.8.

To perform the convergence study, we use the s-norm and normalized relative error  $E_h$  described in Section 2.4. If an element is uniformly optimal, the *k* represents the optimal order of convergence: k = 2, 4, and 6 for linear, quadratic and cubic elements. Note that the order of the displacement interpolations of the enriched elements with linear and quadratic covers are quadratic and cubic, respectively. Therefore, the optimal order of convergence of the enriched elements with linear covers (HEX8-d1, PRI6-d1, TET4-d1) is the same as the standard quadratic elements (HEX27, PRI18). The optimal order of convergence of the five cubic elements (HEX64, HEX8d-1, PRI40, PRI6-d2, TET4-d2) is the same and is 6. The convergence curves of the quadratic and cubic elements are shown in Figs. 3.12 and 3.13, respectively. All 3D solid elements provide similarly good convergence behaviors in both regular and distorted meshes.



Figure 3.11. Ad hoc problem: (a) problem domain, (b) planes A, B and C for mesh distortion, (c) mesh distortion types, (d) distorted hexahedral meshes when N=4, and (e) regular hexahedral mesh when N=4 (N: the number of element layers along an edge).



Figure 3.12. Convergence curves of the quadratic elements for the ad hoc problem with the meshes shown in Fig. 3.11: The bold line represents the optimal convergence rate, which is 4.0 for quadratic elements.



Figure 3.13. Convergence curves of the quadratic elements for the ad hoc problem with the meshes shown in Fig. 3.11: The bold line represents the optimal convergence rate, which is 6.0 for cubic elements.

#### 3.4.2. Tool jig problem

We here consider a tool jig structure subjected to a constant pressure on its top surface and the fixed boundary condition is applied on left surface, see Fig. 3.14(a) [59]. The standard 8-node hexahedral and 6-node prismatic elements (HEX8 and PRI6), the new enriched 8-node hexahedral and 6-node prismatic elements (HEX8-d1 and PRI6-d1), and the 8-node incompatible mode element (HEX8+incompatible) in ADINA [60-63] are considered with four different meshes shown in Fig. 14(b). Meshes for the prismatic elements (PRI6 and PRI6-d1) are generated by dividing the hexahedral meshes in Fig. 14(b). The reference solution is obtained using a fine mesh of 57,344 standard 27-node hexahedral elements leading to 1,457,181 DOFs, and the maximum stress occurs at point P.



Figure 3.14. The tool jig, material properties:  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3: (a) Problem description and (b) coarse, medium, fine-1, and fine-2 meshes used.

Figs. 3.15 and 3.16 show the von Mises stress results obtained using hexahedral and prismatic elements, respectively. The new enriched elements (HEX8-d1 and PRI6-d1) more accurately predict the von Mises stress at the point P than the standard elements do. Comparing the calculated z-displacements (w) on the line AB shows again that, although less new enriched elements are used with smaller DOFs, the new enriched elements provide better solution accuracy compared to the standard elements, see Fig. 3.17.

We additionally compare the new enriched elements (HEX8-d1 and PRI6-d1) with the 8-node incompatible mode element (HEX8+incompatible) in ADINA. Fig. 3.18 shows the calculated z-displacement (*w*) and the von Mises stress (averaged at the nodes) along the line AB when the medium and fine-1 meshes are used. In the both cases, the results of the new enriched elements (HEX8-d1 and PRI6-d1) and the 8-node incompatible mode element (HEX8+incompatible) converge well to the reference value, and all three elements provide better solutions than the standard elements (HEX8 and PRI6) do.



Figure 3.15. von Mises stress distributions and von Mises stress at the point P for the tool jig problem when using hexahedral elements : The von Mises stress and its error at the point P are presented for each solution. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_v|/\tau_{v,ref} \times 100\%$ ,  $\tau_{v,ref} = 832.4N/m^2$ .)



Figure 3.16. von Mises stress distributions and von Mises stress at the point P for the tool jig problem when using pyramidal elements : The von Mises stress and its error at the point P are presented for each solution. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_v|/\tau_{v,ref} \times 100\%$ ,  $\tau_{v,ref} = 832.4N/m^2$ .)



Figure 3.17. Comparison of results for the tool jig problem along the line AB when the standard elements (HEX8 and PRI6) and the new enriched elements with linear covers (HEX8-d1 and PRI6-d1) are used: (a) Results of the hexahedral elements and (b) results of the prismatic elements.



Figure 3.18. Comparison of numerical results for the tool jig problem along the line AB using: (a) medium mesh, and (b) fine-1 mesh.

#### 3.4.3. Straight beam problem

The straight beam problem proposed by MacNeal [64] under four load cases (tension, in-plane and out-of-plane shears, and moment at the free tip) is considered, see Fig. 3.19. Young's modulus E is  $1.0 \times 10^7 N/m^2$  and Poisson's ratio v is 0.3. Length (L), width (W), and depth (D) of the beam are 6 m, 0.2 m, and 0.1 m, respectively. The reference solutions at point P are  $u = 3.0 \times 10^{-5} m$ , v = -0.1081m, w = -0.4321m, and v = -0.0054m for the tension force, in-plane shear force, out-of-plane shear force, and moment load cases, respectively [64].

The standard elements (HEX8, PRI6, and PYR5), the 8-node incompatible mode element (HEX8+incompatible),

and the new enriched elements with linear covers (HEX8-d1, PRI6-d1, and PYR5-1) are considered with three different meshes shown in Fig. 3.19. As shown in Fig. 3.8, the hexahedral meshes are transformed into prismatic and pyramidal meshes using the prismatic and pyramidal elements (PRI6, PRI6-d1, PYR5, and PYR5-d1). Tables 3.5, 3.6, 3.7, and 3.8 show the normalized displacements at the point P for four load cases, respectively. Numerical results of the standard elements and the 8-node incompatible mode element are affected by mesh distortion, while the new enriched elements provide highly accurate solutions regardless of the mesh used.



Figure 3.19. Straight beam under four load cases: tension force ( $F_x = 1.0 N$ ), in-plane shear force ( $F_y = 1.0 N$ ), out-of-plane shear force ( $F_z = 1.0 N$ ) and moment ( $M_z = 0.2 N \cdot m$ ). The solid and dotted lines denote hexahedral and prismatic meshes, respectively.

Table 3.5. Normalized x-displacement (u) at the point P for the straight beam subjected to tension force  $(F_x = 1.0 N)$  at the free tip. The meshes (a-c) are shown in Fig. 3.19 ( $u_{ref} = 0.00003 m$ ).

-	Standard elements			HEX8	Enriched elements		
	HEX8	PRI6	PYR5	+incompatible	HEX8-d1	PRI6-d1	PYR5-d1
Mesh (a)	0.9856	0.9809	0.9838	0.9876	0.9935	0.9935	0.9939
Mesh (b)	0.9884	0.9797	0.9853	0.9791	0.9935	0.9932	0.9940
Mesh (c)	0.9835	0.9825	0.9827	0.9769	0.9932	0.9940	0.9936

	Standard elements		HEX8	Enriched elements			
-	HEX8	PRI6	PYR5	+incompatible	HEX8-d1	PRI6-d1	PYR5-d1
Mesh (a)	0.0929	0.0311	0.0542	0.9782	0.9559	0.9587	0.9603
Mesh (b)	0.0256	0.0144	0.0205	0.0473	0.9375	0.9361	0.9430
Mesh (c)	0.0315	0.0214	0.0227	0.0315	0.9404	0.9476	0.9464

Table 3.6. Normalized y-displacement (v) at the point P for the straight beam subjected to in-plane shear force  $(F_y = 1.0N)$  at the free tip. The meshes (a-c) are shown in Fig. 3.19 ( $v_{ref} = -0.1081m$ ).

Table 3.7. Normalized z-displacement (w) at the point P for the straight beam subjected to out-of-plane shear force ( $F_z = 1.0 N$ ) at the free tip. The meshes (a-c) are shown in Fig. 3.19 ( $w_{ref} = -0.4321 m$ ).

	Standard elements		HEX8	Enriched elements			
	HEX8	PRI6	PYR5	+incompatible	HEX8-d1	PRI6-d1	PYR5-d1
Mesh (a)	0.0252	0.0255	0.0187	0.9729	0.9519	0.9511	0.9622
Mesh (b)	0.0105	0.0111	0.0088	0.0302	0.9376	0.9355	0.9481
Mesh (c)	0.0143	0.0150	0.0110	0.5279	0.9394	0.9412	0.9500

Table 3.8. Normalized y-displacement (v) at the point P for the straight beam subjected to moment  $(M_z = 0.2 N \cdot m)$  at the free tip. The meshes (a-c) are shown in Fig. 3.19 ( $v_{ref} = -0.0054 m$ ).

	Standard elements		HEX8	Enriched elements			
-	HEX8	PRI6	PYR5	+incompatible	HEX8-d1	PRI6-d1	PYR5-d1
Mesh (a)	0.0930	0.0306	0.0540	0.9902	0.9736	0.9758	0.9765
Mesh (b)	0.0211	0.0130	0.0175	0.0438	0.9747	0.9754	0.9774
Mesh (c)	0.0280	0.0194	0.0200	0.7178	0.9744	0.9780	0.9772

## 3.4.4. Curved beam problem

The curved beam problem proposed by MacNeal [64] under two load cases (in plane and out-of-plane shears at the free tip) is considered, see Fig. 3.20. Young's modulus E is  $1.0 \times 10^7 N/m^2$  and Poisson's ratio v is 0.3. The reference solutions at the point P are v = -0.08734m and w = 0.5022m for the in-plane shear force and out-of-plane shear force cases, respectively [64].

The standard elements (HEX8, PRI6, and TET4), the 8-node incompatible mode element (HEX8+incompatible), and the new enriched elements with linear covers (HEX8-d1, PRI6-d1, and TET4-d1) are considered with three different meshes shown in Fig. 3.20. As shown in Fig. 3.8, the hexahedral meshes are transformed into prismatic and tetrahedral meshes using the prismatic and tetrahedral elements (PRI6, PRI6-d1, TET4, and TET4-d1). Tables 3.9 and 3.10 show the normalized displacements at the point P for two load cases, respectively. The enriched elements provide more accurate results than the standard elements and the 8-node incompatible mode element.



Figure 3.20. Curved beam under two load cases:, in-plane shear force ( $F_y = 1.0 N$ ) and out-of-plane shear force ( $F_z = 1.0 N$ ):  $r_{in} = 4.12 m$ ,  $r_{out} = 4.32 m$ , t = 0.1 m.

Table 3.9. Normalized x-displacement (v) at the point P for the curved straight beam subjected to in-plane shear force ( $F_v = 1.0 N$ ) at the free tip. ( $v_{ref} = 0.08734 m$ ).

Standard elements			HEX8	En	riched eleme	ents
HEX8	PRI6	TET4	+incompatible	HEX8-d1	PRI6-d1	TET4-d1
0.0732	0.0252	0.0251	0.8796	0.9908	0.9861	0.9841

Standard elements			HEX8	En	riched eleme	ents
HEX8	PRI6	TET4	+incompatible	HEX8-d1	PRI6-d1	TET4-d1
0.2282	0.0200	0.0067	0.8195	0.9132	0.9084	0.9032

Table 3.10. Normalized x-displacement (w) at the point P for the curved straight beam subjected to out-of-plane shear force ( $F_w = 1.0 N$ ) at the free tip. ( $v_{ref} = 0.5022 m$ ).

## 3.4.5 Gear problem

In Sections 3.4.5 and 3.4.6, we illustrate the adaptive use of cover functions, an advantage of the enriched finite elements [35,37]. It is very effective to apply cover functions to nodes in the area where solution accuracy needs to be improved. Numerical results obtained employing the adaptive use of no/linear/quadratic covers is compared with results of the standard linear elements.

Let us consider a gear, in which the inner cylinder (colored in green) is fixed and 1000 N load in the *y*-direction is applied on side of a gear teeth (colored in red), see Fig. 3.21(a). The 3D gear structure is modeled using 8-node hexahedral and 6-node prismatic elements. Two different meshes are considered: coarse mesh (1310 hexahedral and 328 prismatic elements, in total 1638 elements) and fine mesh (5340 hexahedral and 1062 prismatic elements, in total 6402 elements), see Figs. 3.21(b) and (c).

We perform the following three different cases of finite element analysis:

- (Case 1) No cover enrichment is adopted in the coarse mesh. That is, the standard 8- and 6-node finite elements (HEX8 and PRI6) are used.
- (Case 2) In the fine mesh, the standard elements (HEX8 and PRI6) are used without cover functions.
- (Case 3) No, linear and quadratic covers are adaptively used as shown in Fig. 3.21(d).

The fine mesh is used only for Cases 2, and the coarse mesh is applied in other two cases. The adaptive use of cover functions in Cases 3 is determined by investigating the stress solutions obtained using the standard finite elements in Case 1. Higher order covers are chosen for nodes where relatively higher von Mises stresses are predicted. The reference solution is obtained using a mesh of standard 10-node tetrahedral finite elements, in which 77227 elements and 358689 DOFs are used.

We compare the von Mises stress (effective stress) at the point P shown in Fig. 3.21(a). The reference solution and the calculated von Mises stress distributions of Cases 1, 2, and 3 are shown in Fig. 3.22, and the number of DOFs used and errors in the results are also summarized. The solution accuracy is improved by using the finer mesh or by applying the covers adaptively. Comparing Cases 2, and 3, it can be clearly observed that the adaptive use of cover functions is very effective in improving solution accuracy with small DOFs.



Figure 3.21. The gear problem: (a) problem description,  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3, (b) and (c) coarse and fine meshes, and (d) no, linear, and quadratic covers adaptively used in Case 3.



Figure 3.22. von Mises stress distributions for the gear problem: (a) Reference solution obtained by using 77227 standard 10-node tetrahedral elements, (b), (c), and (d) von Mises stress distributions calculated in analysis Cases 1, 2, and 3. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_v|/\tau_{v,ref} \times 100\%$ ,  $\tau_{v,ref} = 39.55 MPa$ ).

### 3.4.6. Connecting rod problem

We consider the connecting rod, where loads of 100 N are applied on inside of the cylinder (colored in red, see Fig. 3.23(a)) in the x and y-directions, respectively. The fixed boundary conditions are imposed on the green colored surfaces shown in Fig. 3.23(a). Finite elements models for the 3D connecting rod are constructed using 8-node hexahedral, 6-node prismatic, 5-node pyramidal, and 4-node tetrahedral elements. Two different meshes are considered: coarse mesh (428 hexahedral, 86 prismatic elements, and 1044 tetrahedral elements, in total 1558 elements) and fine mesh (2540 hexahedral, 8 prismatic, 246 pyramidal, and 3369 tetrahedral elements, in total

6163 elements), see Figs. 3.23(b) and (c).

The following three cases of finite element analysis are performed:

- (Case 1) No cover enrichment is adopted in the coarse mesh. That is, the standard 8-node, 6-node and 4-node finite elements (HEX8, PRI6, TET4) are used.
- (Case 2) In the fine mesh, the standard elements (HEX8, PRI6, PYR5, TET4) are used without cover functions.
- (Case 3) No, linear and quadratic covers are adaptively used, see Fig. 3.23(d).

In Cases 1 and 3, the coarse mesh is used and Case 2 applies the fine mesh. The adaptive use of cover functions in Cases 3 is determined in the same manner as the gear problem in Section 3.4.4. The reference solution is calculated using a mesh of standard 10-node tetrahedral finite elements, in which 18569 elements and 94953 DOFs are used.

Fig. 3.24 shows the reference solution and the calculated von Mises stress distributions of Cases 1, 2, and 3. The number of DOFs used and errors in the von Mises stress at the point P are also summarized in Fig. 3.24. It is also shown that the solution accuracy can be improved by using the finer mesh or applying the cover functions adaptively, and the adaptive use of cover functions is very efficient.



Figure 3.23. The connecting rod problem: (a) problem description,  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3,  $\rho = 7850 kg / m^3$  (b) and (c) coarse and fine meshes, and (d) no, linear, and quadratic covers adaptively used in Case 3.



Figure 3.24. von Mises stress distributions for the connecting rod problem: (a) Reference solution obtained by using 18569 standard 10-node tetrahedral elements, (b), (c), and (d) von Mises stress distributions calculated in analysis Cases 1, 2, and 3. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_v|/\tau_{v,ref} \times 100\%$ ,  $\tau_{v,ref} = 98.718 MPa$ ).

# 3.4.7. Vibration analysis

In this section, we perform the free vibration analysis of a cantilever beam shown in Fig. 3.25 and the connecting rod in the previous section. The generalized eigenvalue problem for the free vibration analysis in Eq. (2.39) is solved and we obtain the eigenpairs (eigenvalues and eigenvectors) corresponding to the to the 1<sup>st</sup>~4<sup>th</sup> modes of the cantilever beam and the connecting rod.

For the cantilever beam, the previous and new enriched elements with quadratic covers (HEX8-d2 and PRI6-d2) are used to construct the finite element models for the coarse mesh (1800 DOFs) in Fig. 3.25. (b). Figs. 3.26 and 3.27 show the mode shapes calculated, and the eigenvalues are given in Table 3.11. The previous enriched element

exhibits inaccurate eigenmodes and eigenvalues. However, accurate modes and eigenvalues are calculated when the new enriched elements are used.



Figure 3.25. Cantilever beam problem: (a) Problem description, L = 1.0 m, W = 0.1 m, D = 0.2 m,  $E = 2.0 \times 10^{11} N/m^2$ , v = 0.3,  $\rho = 7860 kg/m^3$ , (b) and (c) Hexahedral meshes used for enriched elements and reference solution., respectively. The dotted line represents prismatic mesh.

Mode Number	Reference	Enriched 8-n with quadratic co	Enriched 8-node elements with quadratic covers (HEX8-d2)		ode elements vers (PRI6-d2)
		Previous	New	Previous	New
1	2.6266E+05	2.6415E+05	2.6414E+05	2.6389E+05	2.6422E+05
2	9.9843E+05	1.0022E+06	1.0035E+06	8.8175E+05	1.0038E+06
3	9.4397E+06	3.8753E+06	9.4913E+06	1.0017E+06	9.4966E+06
4	1.3835E+07	9.4970E+06	1.3964E+07	9.4830E+06	1.3979E+07

Table 3.11. Eigenvalues corresponding to  $1^{st} - 4^{th}$  modes for the cantilever beam problem in Fig. 3.25.



Figure 3.26. Mode shapes corresponding to the  $1^{st} \sim 4^{th}$  modes for the cantilever beam problem in Fig. 3.25 when the enriched 8-node hexahedral elements with quadratic covers are used.



Figure 3.27. Mode shapes corresponding to the  $1^{st} \sim 5^{th}$  modes for the cantilever beam problem in Fig. 3.25 when the enriched 6-node prismatic elements with quadratic covers are used.

For the connecting rod, the previous and new enriched elements with linear covers (HEX8-d1, PRI6-d1 and TET4-d1) are used to construct the finite element models for the coarse mesh (13740 DOFs) shown in Fig. 3.23(b). Fig. 3.28 and Table 3.12 show the mode shapes and the eigenvalues calculated, respectively. The previous enriched element exhibits wrong modes and eigenvalues. However, the new enriched elements produce correct solutions. The results show that the new enriched elements can be used not only for static analysis, but also for dynamic analysis.



Figure 3.28. Mode shapes corresponding to the  $1^{st} 4^{th}$  modes for the connecting rod problem in Fig. 3.23 when the enriched elements with linear covers are used with the coarse mesh.

Mode Number	Reference	Enriched 3D solid elements with linear covers		
		Previous	New	
1	2.7199E+03	6.7027E+02	2.8925E+03	
2	4.7079E+03	2.8764E+03	4.8871E+03	
3	3.7226E+04	4.8734E+03	4.1950E+04	
4	8.4975E+04	7.2706E+03	9.0300E+04	

Table 3.12. Eigenvalues corresponding to  $1^{st} - 4^{th}$  modes for the connecting rod problem in Fig. 3.23.

# 3.5. Computational efficiency

In this section, numerical costs of the standard elements and the new enriched elements are compared. We consider four quadratic elements (HEX27, HEX8-d1, PRI18, PRI6-d1):

- HEX27: standard 27-node hexahedral element,
- HEX8-d1: new 8-node hexahedral element enriched by linear covers,
- PRI18: standard 18-node prismatic element,
- PRI6-d1: new 6-node prismatic element enriched by linear covers,

and four cubic elements (HEX64, HEX8-d2, PRI40, PRI6-d2):

- HEX64: standard 64-node hexahedral element,
- HEX8-d2: new 8-node hexahedral element enriched by quadratic covers,
- PRI40: standard 40-node prismatic element,
- PRI6-d2: new 6-node prismatic element enriched by quadratic covers.

In all the cases, symmetric stiffness matrices are generated. To obtain valuable insight into the computational cost needed in each solution, the number of Gauss points used in each element and the size of the global stiffness matrix are considered. Then, the computational cost is tested considering the regular meshes shown in Fig. 3.11(e).

Table 3.13 lists the number of numerical integration points used for the standard elements and the new enriched elements. For standard elements, line and triangular integrations are used [1,50]. On the other hand, the tetrahedral integrations are adopted for each sub-domain of the new enriched elements [73]. The new enriched element with linear and quadratic covers require approximately from 2.6 to 4.1 times the number of integration points compared to the standard finite elements of the same order and shape.

Fig. 3.29 shows how the number of degrees of freedom increases as the number of element layers increases. Considering the same displacement interpolation order, the new enriched element have fewer DOFs than the standard finite elements do.

	Stand	lard elements	New em	New enriched elements		
Element order	Element	# of integration points	Element	# of integration points		
Quadratic	HEX27	$27(=3\times3\times3)$	HEX8-d1	96(=24×4)		
	PRI18	21(=3×7)	PRI6-d1	56(=14×4)		
Cubic	HEX64	$64(=4 \times 4 \times 4)$	HEX8-d2	264(=24×11)		
	PRI40	48(=4×12)	PRI6-d2	154(=14×11)		

Table 3.13. Number of numerical integration points used for the standard elements and the new enriched elements.



Figure 3.29. The total number of DOFs when increasing the number of element layers, N, along an edge: p denotes the number of solution variables considered, hence p = 3 (u, v, and w) for the 3D problem.

The total number of DOFs and the sparseness of the stiffness matrices are listed in Tables 3.14 and 3.15 when hexahedral elements are used, and the structures of the stiffness matrices with meshes (N=4) used are shown in Fig. 3.30. Comparing the standard finite elements and the new enriched elements of the same order, it can be observed that the half-bandwidth and the number of non-zero entries of the stiffness matrix using the enriched elements are smaller than that of the standard finite elements. These results show that the stiffness matrices of the new enriched elements have a smaller half-bandwidth, non-zero entries and size than the standard finite elements.

	Stand	ard 27-node	element	New enriche	New enriched 8-node element with linear		
	(HEX27)			С	overs (HEX8-	d1)	
N	DOF	HB	NNZ	DOF	HB	NNZ	
2	300	159	3.1E+04	216	132	2.7E+04	
4	1944	495	2.7E+05	1200	324	2.3E+05	
8	13872	1743	2.3E+06	7776	996	1.8E+06	
16	104544	6543	1.9E+07	55488	3492	1.5E+07	

Table 3.14. Stiffness matrices when using hexahedral quadratic elements (HEX27 and HEX8-d1) for the ad-hoc problem shown in Fig. 3.11 (DOFs: degrees of freedom, HB: half-bandwidth, NNZ: number of non-zero entries).

	Stand	lard 64-node e	element	New enriched 8	8-node eleme	nt with quadratic
		(HEX64)		cc	overs (HEX8-	d2)
N	DOF	HB	NNZ	DOF	HB	NNZ
2	882	453	2.1E+05	540	330	1.8E+05
4	6084	1533	1.8E+06	3000	810	1.5E+06
8	45000	5637	1.5E+07	19440	2490	1.2E+07
16	345744	21681	1.2E+08	138720	8730	9.9E+07

Table 3.15. Stiffness matrices when using hexahedral cubic elements (HEX64 and HEX8-d2) for the ad-hoc problem shown in Fig. 3.11 (DOFs: degrees of freedom, HB: half-bandwidth, NNZ: number of non-zero entries).



Figure 3.30. Meshes used and stiffness matrix structures when N=4; non-zero entries are colored in black.

Actual calculation times are measured when the stiffness matrix is constructed and the linear equations are solved using the direct Gauss elimination. A machine (Intel(R) Xeon E5-2667 CPU@ 3.2 GHz, 128 GB RAM, Linux 64bit) was used for all cases. The measured times are shown in Tables 3.16, 17, 18, and 19.

As expected, when comparing the standard finite elements and the new enriched elements of the same order, the new enriched elements take more time to construct the stiffness matrices, and the standard elements requires more

time to solve the linear equations. When the number of element layer is more than 8, the equation solving time becomes major computational cost, and much less computational cost is required for the new enriched elements.

Table 3.16. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations with the direct Gauss elimination when using the hexahedral quadratic elements (HEX27 and HEX8-d1) for the ad hoc problem shown in Fig. 3.11.

	HEX27			HEX8-d1		
	Stiffness	Equation	Tatal	Stiffness	Equation	Tatal
Ν	construction	Solving	Iotal	construction	Solving	Total
2	0.00	0.00	0.00	0.02	0.00	0.02
4	0.03	0.04	0.07	0.14	0.01	0.15
8	0.28	3.09	3.37	1.13	0.93	2.05
16	2.97	738.76	741.72	9.25	181.91	191.16

Table 3.17. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations with the direct Gauss elimination when using the hexahedral cubic elements (HEX64 and HEX8-d2) for the ad hoc problem shown in Fig. 3.11.

	HEX64			HEX8-d2		
	Stiffness	Equation	T-4-1	Stiffness	Equation	T-4-1
N	construction	Solving	Total	construction	Solving	Total
2	0.05	0.01	0.06	0.29	0.00	0.29
4	0.39	0.87	1.26	2.30	0.20	2.50
8	3.34	183.05	186.40	18.49	20.47	38.96
16	35.73	23368.91	23404.64	149.17	2836.93	2986.10

Table 3.18. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations with the direct Gauss elimination when using the prismatic quadratic elements (PRI18 and PRI6-d1) for the ad hoc problem shown in Fig. 3.11.

	PRI18				PRI6-d1	
	Stiffness	Equation	Total	Stiffness	Equation	Total
N	construction	Solving	Total	construction	Solving	Total
2	0.00	0.00	0.00	0.01	0.00	0.01
4	0.04	0.04	0.07	0.10	0.01	0.11
8	0.31	3.12	3.42	0.79	0.91	1.69
16	3.11	804.47	807.58	6.52	181.67	188.18

	PRI40			PRI6-d2		
	Stiffness	Equation	T-4-1	Stiffness	Equation	T-4-1
Ν	construction	Solving	Total	construction	Solving	Total
2	0.03	0.01	0.04	0.20	0.00	0.20
4	0.24	0.86	1.11	1.58	0.19	1.77
8	2.14	182.46	184.60	12.71	19.79	32.50
16	23.37	22865.38	22888.74	103.07	2826.91	2929.98

Table 3.19. Solution times (in seconds) for constructing the stiffness matrix and solving the linear equations with Gauss direction elimination when using the prismatic cubic elements (PRI40 and PRI6-d2) for the ad hoc problem shown in Fig. 3.11.

# 3.6. Closure

In this chapter, new enriched 8-node hexahedral, 6-node prismatic, and 5-node pyramidal finite elements were presented for 3D analysis in solid mechanics problems. The linear dependence (LD) problem is resolved by adopting sets of piecewise linear shape functions for geometry and enriched displacement interpolations. The new enriched elements pass all basic tests and show good solution accuracy even when distorted meshes are used. The new enriched 3D solid finite elements and the enriched 4-node tetrahedral element are compatible with each other and can be used together to construct a finite element model. Since the LD problem was resolved, the cover functions can be adaptively applied to 3D solid finite element models to effectively improve the solution accuracy without introducing additional nodes or mesh refinement.

# Chapter 4. Procedure to improve finite element solutions automatically by the adaptive use of cover functions.

The most important advantage of the enriched finite element method is that it can increase the accuracy of the finite element solutions without remeshing or adding nodes by applying the cover function to the mesh used for finite element analysis. These advantages can be used for procedure that improves the accuracy of finite element analysis results automatically. This procedure is to improve finite element solution with the adaptive use of cover functions after performing finite element analysis. That is, the error for each node is estimated with the analysis result. Based on the calculated error, the appropriate orders of cover functions are selected. Then, the finite element analysis with the cover functions is performed again to obtain an improved solution. Fig. 4.1 shows an example of implementing this procedure using enriched MITC3 shell finite elements with linear cover functions [36].

In this chapter, we propose error indicator and scheme that determines the order of cover function. The error indicator and the scheme are used for procedure to improve finite element solutions automatically by the adaptive use of cover functions. We demonstrate the automatic procedure through several 2D problems. Note that we consider the 3-node triangular and 4-node quadrilateral elements mesh, and the order of cover function up to quadratic.

# 4.1. Error indicator and scheme for the adaptive use of cover functions.

The procedure covered in this chapter employs the adaptive use of cover function to improve finite element analysis results. This procedure consists of three steps (see Fig. 4.1), and a brief description of the procedure is as follows. First, a finite element model consisting of 3-node triangle and 4-node quadrilateral elements is constructed, and a finite element analysis is performed. Based on the analysis results, the order of cover function for each node is determined using an error indicator. Then, an analysis is performed on the finite element model to which the cover function is applied adaptively according to the error indicator, thereby obtaining the improved analysis result.



Figure 4.1. Description of the automatic procedure to improve finite element solution with the adaptive use of cover functions [35].

For the procedure described above, error indicator is very important. Among various studies for estimating the error of the finite element solution [74-76], Kim and Bathe [35] suggested the error indicator and the scheme that choose the order of cover functions. They solved several examples to illustrate the performance of the procedure. In Kim and Bathe's study, following requirements that they would like to fulfill with the error indicator are considered [35]:

- The indicator should be simple and computationally efficient.
- The indicator should asymptotically converge as the actual error converges.
- The indicator should directly tell where covers are best applied and what cover orders are best used, and that for a large range of problems.
- No parameter should be used in the definition of the error indicator.

In addition, it is determined that a local error measure rather than a global energy-based error measure is suitable for the adaptive use of cover functions

Based on the above descriptions, the error indicator was presented as follows using the stress jump for a scalar stress quantity of interest (say  $\tau$ ) at each node

$$\bar{M}_{i}^{\tau} = \frac{J_{i}^{\tau}}{\gamma_{e}\tau_{mean}} \left(\frac{h}{L_{c}}\right)^{\beta} \text{ with } J_{i}^{\tau} = \max(\tau_{i}) - \min(\tau_{i}), \qquad (4.1)$$

where  $\tau_i$  and  $J_i$  are the stress and largest stress jump at node *i*, respectively,  $\tau_{mean}$  is the mean stress over the finite element model, *h* and  $L_c$  are a mesh size and a characteristic length, and  $\gamma_e$  and  $\beta$  are artificial coefficients [35]. They suggested the scheme that determine the order of cover functions using the error indicator in Eq. (4.1) as follows:

$$d(i) = \begin{cases} 0 & if \qquad M_i^{\tau} < \gamma_0 \\ 1 & if \qquad \gamma_0 \le \bar{M}_i^{\tau} < \gamma_1 \\ 2 & if \qquad \gamma_1 \le \bar{M}_i^{\tau} < \gamma_2 \\ 3 & if \qquad \gamma_2 \le \bar{M}_i^{\tau} \end{cases},$$
(4.2)

in which  $\gamma_k$ , k = 0, 1, 2 denotes 'adaptivity threshold constants' to be set by the user. However, unlike the requirements mentioned above, the error indicator contains many coefficients, and only 2D 3-node triangular and 3D 4-node tetrahedral elements, of which linear dependence problem can be resolved by suppressing enriched DOFs at essential boundary, were considered [35,48].

In this study, referring to the previous error indicator in Eq. (4.1), a new error indicator based on the stress jump is given by

$$M_{i}^{\tau} = \left\{ \frac{J_{i}^{\tau}}{\tau_{mean}} + \left( \frac{J_{mean}}{\tau_{mean}} \right) \frac{\tau_{i}}{\tau_{mean}} \right\} \left( \frac{\chi_{i}}{L_{c}} \right)^{1/2}, \tag{4.3}$$

where  $J_{mean}$  is the mean value of  $J_i$  over the finite element model,  $\chi_i$  is the diameter of the largest finite element sharing the node *i*. The previous error indicator  $\overline{M}_i^{\tau}$  in Eq. (4.1) includes artificial constants,  $\gamma_e$  and  $\beta$ ,

while the new error indicator  $M_i^{\tau}$  in Eq. (4.2) does not have any artificial coefficients. Note that, for the error indicators in Eqs. (4.1) and (4.3), any stress quantity of interest can be employed, but we apply the jump of the von Mises stress for the new error indicator.

Since the suggested error indicator in Eq. (4.3) is based on the stress jump as in the previous error indicator in Eq. (4.1), the suggested error indicator also converges asymptotically as the actual error. For example, if we calculate the error indicator with analysis results of the patch test shown in Figure 2.11, there is no stress jump, so the error indicator is calculated as zero. On the other hand, when the stress jump is large, the error indicator is also calculated in proportion to the stress jump.

To check whether the indicator asymptotically converges as the actual error converges, we consider the ad-hoc problem in Section 2.4.1 and regular quadrilateral meshes are used when  $N = 8, 16, \dots, 128$ . For each finite element model, von Mises stress jump value ( $J_i = \max(\tau_i^v) - \min(\tau_i^v)$ ), von Mises stress error value ( $e_i = |\tau_{v,i}^{ref} - \tau_{v,i}^h|$ ), and suggested error indicator in Eq. (4.3) are calculated. The average values of all nodes are evaluated and shown in Fig. 4.2. As the error converges, the proposed error indicator also converges, and the proposed error indicator is simple to calculate. The examples in the next section confirm that the proposed error indicator selects appropriate cover orders.

The order of cover functions for each node is determined by following scheme:

$$d(i) = \begin{cases} 0 & if \qquad M_i^{\tau} < 0.1 \\ 1 & if \qquad 0.1 \le M_i^{\tau} < 0.2 \\ 2 & if \qquad 0.2 \le M_i^{\tau} \end{cases}$$
(4.4)

The coefficients used in this scheme are determined through the tool jig problem in Section 4.2.1. The error indicator range for each order of cover function is provided, but these can be changed according to the purpose of the user.



Figure 4.2. Ad hoc problem: convergence of the averaged stress jump values, stress error values and error indicators in the von Mises stress.

In the following sections, we demonstrate the procedure to improve finite element solutions automatically by the adaptive use of cover functions. The tool jig problem, the wrench problem, and the wheel problem are considered, and the 3-node triangular and 4-node quadrilateral elements are used for the finite element model. We apply the scheme in Eq. (4.4) without any change of the error indicator range to all problems. Note that, to avoid the linear dependence problem, the new enriched 4-node element proposed in Section 2 is used for analysis.

# 4.2. Numerical examples

#### 4.2.1. Tool jig

Let's consider a tool jig problem shown in Fig. 4.3(a). The tool jig is subjected to a constant pressure on its top surface (the line AB) and the fixed boundary condition is applied along the line AC. Three mesh patterns (Mesh  $1\sim3$ ) shown in Fig. 4.3(c) are used.

Fig. 4.4 shows the calculated error indicator in Eq. (4.3) when the standard 4-node quadrilateral elements are used with three mesh patterns in Fig. 4.3(c). On the horizontal axis, 0 indicates the node with the smallest error indicator, and 1 represents the node whose error indicator is the maximum value. The vertical axis represents the error indicator value. Using finer mesh (from Mesh 1 to Mesh 3), the error indicator is calculated to a smaller value. That is, as the error decreases, the error indicator value also tends to decrease.



Figure 4.3. The tool jig problem: (a) problem description,  $E = 2.0 \times 10^{11} N/m^2$ , v = 0.3, (b) von Mises stress distribution of the reference solution, and (c) Meshes 1, 2, and 3 used.


Figure 4.4. The error indicator for the tool jig problem with three different meshes.

For Meshes 1-3, first we perform analysis with the standard 4-node element. Then, based on the error indicator in Eq. (4.3) and the scheme in Eq. (4.4), the cover functions are adaptively applied and an improved finite element solutions are obtained. Fig. 4.5 and Table 4.1 show how the cover functions are applied and analysis results for each quadrilateral mesh. The adaptive scheme with Mesh 1 uses quadratic covers on almost all nodes, while the adaptive scheme with Mesh 3 applied linear and quadratic covers to less than half of nodes. Comparing the solutions obtained by the adaptive scheme with the solutions of the standard 4-node element, it is clearly observed that the adaptive use of cover function is very effective in predicting strain energy and von Mises stress.

We also consider triangular meshes obtained by dividing quadrilateral elements shown in Fig. 4.4(c) into two triangular elements. Analysis results of the standard 3-node elements and the adaptive use of cover functions are summarized in Fig. 4.6 and Table 4.2. Similar to the quadrilateral mesh results shown in Fig. 4.5, the automatic procedure provides much more accurate solutions.



Figure 4.5. von Mises stress results of the tool jig problem shown in Fig. 4.4(a): (a) results of the standard 4-node element, (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100\%$ )

Table 4.1. Computational results for the tool jig problem shown in Fig. 4.4 when 4-node quadrilateral elements are used. Relative error (%) in von Mises stress,  $E_h^{\tau_v} = |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$ . Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_{v,h}| / |e_{ref} \times 100$ 

		Standard 4-node element (QUAD4)	Adaptive use of interpolation covers
Mesh 1	Relative error in strain energy	64.06 %	6.59 %
	Relative error in von Mises stress at the point P	51.00 %	14.43 %
	DOFs	182	962
Mesh 2	Relative error in strain energy	38.24 %	3.96 %
	Relative error in von Mises stress at the point P	42.03 %	0.88 %
	DOFs	655	2606
Mesh 3	Relative error in strain energy	15.67 %	5.13 %
	Relative error in von Mises stress at the point P	18.94 %	1.80 %
	DOFs	2270	5324
Reference	$\tau_{v,ref} = 13193N / m^2$ , $e_{ref} = 3.4$	$538 \times 10^{-7} Nm$	



Figure 4.6. von Mises stress results of the tool jig problem when triangular elements are used: (a) results of the standard 3-node element, (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100\%$ )

Table 4.2. Computational results for the tool jig problem shown in Fig. 4.4(a) when 3-node triangular elements are used. Relative error (%) in von Mises stress,  $E_h^{\tau_v} = |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$ . Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_{v,h}| / |e_{ref} \times 100$ 

		Standard 3-node element (TRI3)	Adaptive use of interpolation covers
Mesh 1	Relative error in strain energy	69.29 %	4.93 %
	Relative error in von Mises stress at the point P	72.15 %	12.32 %
	DOFs	182	1008
Mesh 2	Relative error in strain energy	48.56 %	1.92 %
	Relative error in von Mises stress at the point P	54.29	1.72 %
	DOFs	655	3068
Mesh 3	Relative error in strain energy	26.02 %	4.72 %
	Relative error in von Mises stress at the point P	30.93 %	0.56 %
	DOFs	2270	6414
Reference	$\tau_{v,ref} = 13193N / m^2, \ e_{ref} = 3.4$	$538 \times 10^{-7} Nm$	

#### 4.2.2. Wrench

The wrench problem subjected to a uniform pressure load is solved, see Fig. 4.7(a). We first perform the linear static analysis using Meshes 1-2 shown in Fig. 4.7(b), and then using the error indicator in Eq. (4.3) and the scheme in Eq. (4.4), the cover functions are applied adaptively. Analysis results are summarized in Fig. 4.8 and Table 4.3. Like the tool jig problem in the previous section, smaller cover functions are applied as finer mesh is used. It is also shown that the solution accuracy can be improved through the automatic procedure described in Section 4.1



Figure 4.7. The tool jig problem: (a) Problem description ( $E = 1.0 \times 10^7 N / m^2$ , v = 0.3) and (c) Meshes 1 and 2 used.



Figure 4.8. von Mises stress results of the wrench problem when quadrilateral elements are used: (a) results of the standard 4-node element, (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100\%$ )

Table 4.3. Computational results for wrench problem shown in Fig. 4.7(a) when 4-node quadrilateral elements are used. Relative error (%) in von Mises stress,  $E_h^{\tau_v} = |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$ . Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_{v,h}| / |e_{ref} \times 100$ 

		Standard 4-node element (QUAD4)	Adaptive use of interpolation covers
Mesh 1	Relative error in strain energy	5.88 %	2.45 %
	Relative error in von Mises stress at the point P	11.04 %	1.20 %
	DOFs	360	900
Mesh 2	Relative error in strain energy	2.48 %	2.39 %
	Relative error in von Mises stress at the point P	12.74 %	4.43 %
	DOFs	1102	1722
Reference	$\tau_{v,ref} = 93.86 N / m^2, \ e_{ref} = 1.87$	$V64 \times 10^{-7} Nm$	

#### 4.2.3. Wheel

Let's consider a 2D automotive wheel with a radius of 0.2m, in which a lower part of the outer circle is subjected to a pressure and the inner circle is fixed, see Fig. 4.9(a). The wheel structure is modeled using the 3-node triangular and 4-node quadrilateral elements and two difference meshes are considered: Mesh 1 (coarse mesh, 360 quadrilateral and 546 triangular elements) and Mesh 2 (fine mesh, 2,289 quadrilateral and 18 triangular elements), see Fig. 4.9(b).

As in the previous sections, the finite element analyses are performed along the automatic procedure. The results of the standard linear elements and automatic procedure, and how the cover functions are applied for Meshes 1-2 are summarized in Fig. 4.10 and Table 4.4. When the automatic scheme is applied to Meshes 1-2, it can be seen that the cover functions are applied to only local areas where high stress gradient occurs.

The automatic procedure provides much more accurate solutions than standard linear element. In addition, when the cover functions are applied adaptively on the Mesh 1, a more accurate solution is obtained with smaller DOFs than when the standard linear elements used on the Mesh 2.



Figure 4.9. The wheel problem: (a) Problem description ( $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3), (b) Meshes 1 and 2 used.



Figure 4.10. von Mises stress results obtained by applying interpolation covers adaptively: (a) results of the standard 3-node and 4-node elements, (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, Error =  $|\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100\%$ ).

		Standard 4-node and 3-node elements (QUAD4, TRI3)	Adaptive use of interpolation covers
Mesh 1	Relative error in strain energy	9.04 %	2.02 %
	Relative error in von Mises stress at the point P	19.59 %	1.90 %
	DOFs	1532	3714
Mesh 2	Relative error in strain energy	2.66 %	2.09 %
	Relative error in von Mises stress at the point P	3.75 %	0.14 %
	DOFs	5268	7240
Reference	$\tau_{v,ref} = 1.384 \times 10^7  N / m^2 , \ e_{ref} =$	$= 6.970 \times 10^{-1} Nm$	

Table 4.4. Computational results for the wheel problem shown in Fig. 4.10(a). Relative error (%) in von Mises stress,  $E_h^{\tau_v} = |\tau_{v,ref} - \tau_{v,h}| / \tau_{v,ref} \times 100$ . Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_{v,h}| / e_{ref} \times 100$ 

# 4.3. Closure

In this chapter, feasibility of the adaptive use of cover functions to automatically improve solution accuracy was demonstrated through several problems. New error indicator based on stress jump and scheme that select the appropriate order of cover function for each node were presented. The new error indicator  $(M_i^r)$  doesn't have any coefficients and asymptotically converge as the actual error converges. To all problems in this chapter, the error indicator range in the scheme in Eq. (4.4) is applied identically, and the automatic procedure provides significantly improved solution accuracy.

Further research is required on large finite element models and 3D problems so that the automatic procedure can be applied to practical engineering problems in the future. In addition, the adaptive use of cover function and the local mesh refinement may be more effective if used properly at the same time. It would also be valuable to conduct a study on this to develop more effective ways to improve finite element analysis solutions using both the local mesh refinement and the adaptive use of cover functions appropriately.

#### Chapter 5. The strain-smoothed element enriched by interpolation covers

The 3-node triangular element has been widely used for analysis of 2D solid mechanics problem due to their simplicity and efficiency. However, the predictive capability of the standard 3-node triangular finite element is insufficient to be used for engineering practice [53]. Many researchers have been carried out to develop 3-node triangular element with improved accuracy while maintaining its advantages.

The smoothed finite element method (SFEM) was proposed by Liu et al. [77] and has been successfully applied to various mechanics problems [77-96]. In the SFEM, smoothing domains are defined and piecewise constant strain fields are constructed in each smoothing domain. The cell-based, node-based, edge-based, and face-based SFEM methods were developed, and their smoothing domains are defined based on cell, node, edge, and face, respectively. The SFEM does not require additional DOFs, and edge-based SFEM is generally known as to be most effective among the SFEM methods. Recently, Lee and Lee proposed the new strain-smoothed element (SSE) method for the 3-node triangular and 4-node tetrahedral elements which show improved convergence behavior compared to the standard and edge-based smoothed elements [97].

In this chapter, the polynomial enrichment scheme is applied to the strain-smoothed 3-node triangular element which shows improved convergence behavior than that of the standard element. The feasibility of improving the solution accuracy through the adaptive use of cover functions is demonstrated.

#### 5.1. Strain-smoothed 3-node triangular element

Here, we briefly review the formulation of the strain-smoothed element (SSE) method for the 3-node triangular element. The geometry interpolation of the standard 3-node triangular 2D solid element is described by

$$\mathbf{x}(r,s) = \sum_{i=1}^{3} h_i(r,s) \mathbf{x}_i \quad \text{with} \quad \mathbf{x}_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T,$$
(5.1)

in which  $\mathbf{x}_i$  is the position vector of node *i* in the global Cartesian coordinate system (see Fig. 2.3),  $h_i$  is the shape function of the standard isoparametic procedure corresponding to node *i* given by

$$h_1 = 1 - r - s, h_2 = r, h_3 = s$$
 (5.2)

The displacement of the standard 3-node triangular 2D solid element is interpolated by

$$\overline{\mathbf{u}}(r,s) = \sum_{i=1}^{3} h_i(r,s) \overline{\mathbf{u}}_i \quad \text{with} \quad \mathbf{u}_i = \begin{bmatrix} \overline{u}_i & \overline{v}_i \end{bmatrix}^T,$$
(5.3)

where  $\overline{\mathbf{u}}_i$  is the standard nodal displacement vector of node *i* in the global Cartesian coordinate system. Note that the standard nodal displacement vector,  $\overline{\mathbf{u}}_i$  is expressed using the upper bar to distinguish it from the enriched DOFs vector,  $\hat{\mathbf{u}}_i$  with upper hat, see Section 2.1.2.

Employing the standard isoparametric finite element procedure, the stain field of the standard 3-node triangular element m is obtained using

$$\overline{\boldsymbol{\varepsilon}}^{(m)} = \overline{\mathbf{B}}^{(m)} \overline{\mathbf{u}}^{(m)}$$
with  $\overline{\mathbf{B}}^{(m)} = [\overline{\mathbf{B}}_1 \quad \overline{\mathbf{B}}_2 \quad \overline{\mathbf{B}}_3]^T$ ,  $\overline{\mathbf{u}}^{(m)} = [\overline{\mathbf{u}}_1 \quad \overline{\mathbf{u}}_2 \quad \overline{\mathbf{u}}_3]^T$ , (5.4)

in which  $\overline{\mathbf{B}}^{(m)}$  is the strain-displacement matrix of an element m,  $\overline{\mathbf{u}}^{(m)}$  is the standard nodal displacement vector of an element m, and  $\overline{\mathbf{B}}_i$  is the standard strain-displacement matrix corresponding to node i. The upper bar in  $\overline{\mathbf{\varepsilon}}^{(m)}$  and  $\overline{\mathbf{B}}_i$  means that the strain and the strain-displacement matrix in Eq. (5.4) correspond to the standard DOFs.

With the strain-smoothed element (SSE) method, the strains of all neighboring elements are used in the strain smoothing process. In case of the 3-node triangular element, the strains of up to three neighboring elements can be utilized through element edge where  $\overline{\epsilon}^{(m)}$  is the strain of a target element *m* and  $\overline{\epsilon}^{(k)}$  is the strain of the *k*th neighboring element, see Fig. 5.1(a) [97].

Between the target element m and its neighboring elements, smoothed strain can be defined as follows:

$$\widehat{\mathbf{\varepsilon}}^{(k)} = \frac{1}{A^{(m)} + A^{(k)}} \left( A^{(m)} \overline{\mathbf{\varepsilon}}^{(m)} + A^{(k)} \overline{\mathbf{\varepsilon}}^{(k)} \right) \text{ with } k = 1, 2, 3,$$
(5.5)

where  $A^{(m)}$  and  $A^{(k)}$  are the areas of the target element *m* and the *k*th neighboring element, respectively, see Fig. 5.1(b). If there is no neighboring element for the *k*th edge of the target element, we use  $\hat{\epsilon}^{(m)} = \overline{\epsilon}^{(k)}$ . The smoothed strain in Eq. (5.1) can also expressed in a matrix and vector form

$$\widehat{\mathbf{\epsilon}}^{(k)} = \widehat{\mathbf{B}}^{(k)} \widehat{\mathbf{u}}^{(k)}$$

with 
$$\widehat{\mathbf{B}}^{(k)} = \begin{bmatrix} \widehat{\mathbf{B}}_1 & \widehat{\mathbf{B}}_1 & \widehat{\mathbf{B}}_1 & \widehat{\mathbf{B}}_{k+3} \end{bmatrix}, \ \widehat{\mathbf{u}}^{(k)} = \begin{bmatrix} \overline{\mathbf{u}}_1 & \overline{\mathbf{u}}_2 & \overline{\mathbf{u}}_3 & \overline{\mathbf{u}}_{k+3} \end{bmatrix}^T,$$
 (5.6)

in which  $\hat{\mathbf{B}}^{(k)}$  and  $\hat{\mathbf{u}}^{(k)}$  are the strain-displacement matrix and the corresponding displacement vector of the element for the smoothed strains  $\hat{\mathbf{\epsilon}}^{(k)}$ . The subscript *i* in  $\hat{\mathbf{B}}_i$  and  $\mathbf{u}_i$  denotes the neighboring node number, see Fig. 5.1(a).

The strain field of the strain-smoothed 3-node triangular element can be obtained by assigning the smoothed strain  $\hat{\epsilon}^{(k)}$  in Eq. (5.6) into the Gauss integration points (a, b, and c shown in Fig. 5.1(c)) of the target element *m*, shown in Fig. 5.1(a), using the following equations

$$\boldsymbol{\varepsilon}^{a} = \frac{1}{2} \left( \hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(3)} \right), \ \boldsymbol{\varepsilon}^{b} = \frac{1}{2} \left( \hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)} \right), \ \boldsymbol{\varepsilon}^{c} = \frac{1}{2} \left( \hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)} \right).$$
(5.7)

The obtained strain field of the target element m can be expressed as follows:

$$\tilde{\boldsymbol{\varepsilon}}^{(m)} = \left[1 - \frac{1}{q-p}(r+s-2p)\right] \boldsymbol{\varepsilon}^a + \frac{r-p}{q-p} \boldsymbol{\varepsilon}^b + \frac{s-p}{q-p} \boldsymbol{\varepsilon}^c , \qquad (5.8)$$

where p = 1/6 and q = 4/6.

When the target element m has three neighboring elements, th strain-displacement relation can be expressed in a

matrix and vector form as

$$\tilde{\boldsymbol{\varepsilon}}^{(m)} = \tilde{\mathbf{B}}^{(m)} \tilde{\mathbf{u}}^{(m)}$$
(5.9)

with

$$\tilde{\mathbf{B}}^{(m)} = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \tilde{\mathbf{B}}_2 & \tilde{\mathbf{B}}_3 & \tilde{\mathbf{B}}_4 & \tilde{\mathbf{B}}_5 & \tilde{\mathbf{B}}_6 \end{bmatrix},\tag{5.10}$$

$$\tilde{\mathbf{u}}^{(m)} = \begin{bmatrix} \overline{\mathbf{u}}_1 & \overline{\mathbf{u}}_2 & \overline{\mathbf{u}}_3 & \overline{\mathbf{u}}_4 & \overline{\mathbf{u}}_5 & \overline{\mathbf{u}}_6 \end{bmatrix}, \tag{5.11}$$

in which  $\tilde{\mathbf{B}}^{(m)}$  is the strain-displacement matrix of the strain-smoothed 3-node triangular element *m*, and  $\tilde{\mathbf{u}}^{(m)}$  is the corresponding displacement vector of element *m*.

The strain-smoothed 3-node triangular element passes the isotropy, zero energy mode, and patch tests and shows improved convergence behavior, when compared to the standard and edge-based smoothed elements [97].



Figure 5.1. Strain-smoothed element method for the 3-node triangular element: (a) Strains of a target element m and its neighboring elements. Node numbers are used for explaining the formulation. (b) Strain smoothing between the target and each neighboring element. (c) Three Gauss integration points in the natural coordinate system (r, s). (d) Construction of the smoothed strain field.

#### 5.2. Enriched strain-smoothed 3-node triangular element

In this section, we present an strain-smoothed 3-node triangular element enriched by polynomial cover functions. The geometry interpolation of the enriched strain-smoothed 3-node element is identical to that of the corresponding standard finite element in Eq. (5.1), while the strain field is modified using the strain-smoothed element (SSE) method.

The strain vector of the enriched strain-smoothed 3-node element *m* is divided into two strain vectors corresponding to the standard DOFs  $\overline{\mathbf{u}}^{(m)}$  and the enriched DOFs  $\hat{\mathbf{u}}^{(m)}$ , and the strain-smoothed element (SSE) method is applied to the strain vector corresponding to the standard DOFs.

The strain vector of the enriched 3-node triangular element is given by

$$\boldsymbol{\varepsilon}^{(m)} = \overline{\boldsymbol{\varepsilon}}^{(m)} + \hat{\boldsymbol{\varepsilon}}^{(m)} \quad \text{with}$$
(5.12)

$$\overline{\boldsymbol{\varepsilon}}^{(m)} = \overline{\mathbf{B}}^{(m)} \overline{\mathbf{u}}^{(m)}, \quad \overline{\mathbf{B}}^{(m)} = \begin{bmatrix} \overline{\mathbf{B}}_1 & \overline{\mathbf{B}}_2 & \overline{\mathbf{B}}_3 \end{bmatrix}, \quad \overline{\mathbf{u}}^{(m)} = \begin{bmatrix} \overline{\mathbf{u}}_1 & \overline{\mathbf{u}}_2 & \overline{\mathbf{u}}_3 \end{bmatrix}^T, \quad (5.13)$$

$$\hat{\boldsymbol{\varepsilon}}^{(m)} = \hat{\boldsymbol{B}}^{(m)} \hat{\boldsymbol{u}}^{(m)}, \quad \hat{\boldsymbol{B}}^{(m)} = \begin{bmatrix} \hat{\boldsymbol{B}}_1 & \hat{\boldsymbol{B}}_2 & \hat{\boldsymbol{B}}_3 \end{bmatrix}, \quad \hat{\boldsymbol{u}}^{(m)} = \begin{bmatrix} \hat{\boldsymbol{u}}_1 & \hat{\boldsymbol{u}}_2 & \hat{\boldsymbol{u}}_3 \end{bmatrix}^T, \quad (5.14)$$

in which  $\overline{\mathbf{\epsilon}}^{(m)}$  and  $\hat{\mathbf{\epsilon}}^{(m)}$  are the strain vectors corresponding to the standard DOFs and the enriched DOFs, respectively,  $\overline{\mathbf{u}}^{(m)}$  is the standard nodal displacement vector of an element *m*, and  $\overline{\mathbf{B}}^{(m)}$  is the corresponding strain-displacement matrix of an element *m*, respectively.  $\hat{\mathbf{u}}^{(m)}$  and  $\hat{\mathbf{B}}^{(m)}$  are the enriched DOFs vector and the corresponding strain-displacement matrix of an element *m*, respectively. Detail explanations for the enrichment scheme is given in Section 2.1.2.

By applying SSE method for the strain vector  $\overline{\mathbf{\epsilon}}^{(m)}$  corresponding to the standard DOFs, the strain field of the enriched strain-smoothed 3-node element is obtained:

$$\boldsymbol{\varepsilon}^{(m)} = \tilde{\boldsymbol{\varepsilon}}^{(m)} + \hat{\boldsymbol{\varepsilon}}^{(m)} \tag{5.15}$$

with

 $\tilde{\boldsymbol{\varepsilon}}^{(m)} = \tilde{\boldsymbol{B}}^{(m)} \tilde{\boldsymbol{u}}^{(m)}, \quad \tilde{\boldsymbol{B}}^{(m)} = \begin{bmatrix} \tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{B}}_3 & \tilde{\boldsymbol{B}}_4 & \tilde{\boldsymbol{B}}_5 & \tilde{\boldsymbol{B}}_6 \end{bmatrix}, \quad \tilde{\boldsymbol{u}}^{(m)} = \begin{bmatrix} \overline{\boldsymbol{u}}_1 & \overline{\boldsymbol{u}}_2 & \overline{\boldsymbol{u}}_3 & \overline{\boldsymbol{u}}_4 & \overline{\boldsymbol{u}}_5 & \overline{\boldsymbol{u}}_6 \end{bmatrix}, \quad (5.16)$ 

where  $\tilde{\mathbf{B}}^{(m)}$  is the strain-displacement matrix of the strain-smoothed 3-node triangular element *m*, and  $\tilde{\mathbf{u}}^{(m)}$  is the corresponding displacement vector of element *m*, see Fig. 5.1(a) and Eqs. (5.10) and (5.11).

Note that the strain vector of the enriched strain-smoothed 3-node element corresponding to the standard DOFs  $(\tilde{\boldsymbol{\epsilon}}^{(m)})$  is the same with that of the strain-smoothed 3-node element (see Section 5.1), and the strain vector of the enriched strain-smoothed 3-node element corresponding to the enriched DOFs  $(\hat{\boldsymbol{\epsilon}}^{(m)})$  is identical to that of the enriched 3-node element (see Section 2.1).

#### 5.3. Numerical examples

The enriched strain-smoothed 3-node triangular element, presented in previous section, passes the isotropy, zero energy mode, and patch tests for arbitrary cover enrichment, see Fig. 5.2 [1,36]. In all tests, to avoid the linear dependence problem, enriched DOFs  $\hat{\mathbf{u}}_i$  are suppressed at nodes on the essential boundary.



Figure 5.2. Finite element models for isotropy, zero energy mode and patch tests: (a) Single element for isotropy and zero energy mode tests and (b) Mesh for patch tests.

In the following section, we investigate the convergence and effectiveness of the enriched strain-smoothed 3-node triangular element. Convergence is explored using the cook beam problem. In the tool jig problem, the finite element analyses are performed through the automatic procedure, and the results of the adaptive scheme with the strain-smoothed 3-node element are compared with that of the standard 3-node element.

#### 5.3.1 Cook's skew beam problem

We solve Cook's skew beam problem shown in Fig. 5.3. The 2D structure is subjected to distributed shear force of total magnitude P = 1 at the right edge, and fixed boundary condition is applied on the left edge. The plane stress condition with  $E = 3 \times 10^7$  and v = 0.3 are applied.

We consider two linear elements (TRI3, SS-TRI3):

- TRI3: standard 3-node triangular element,
- SS-TRI3: strain-smoothed 3-node triangular element.

and two quadratic elements (TRI3-d1, SS-TRI3-d1):

- TRI3-d1: enriched 3-node triangular element by linear covers,
- SS-TRI3-d1: enriched strain-smoothed 3-node triangular element by linear covers,

and two cubic elements (TRI3-d2, SS-TRI3-d2):

- TRI3-d2: enriched 3-node triangular element by quadratic covers,
- SS-TRI3-d2: enriched strain-smoothed 3-node triangular element by quadratic covers,

The solutions are obtained with  $N \times N$  element meshes (N = 2, 4, 8, 16, and 32).



Figure 5.3. Cook's skew beam problem (2  $\times$  2 mesh,  $E = 3 \times 10^7$  and v = 0.3).

Fig. 5.4 and Table 5.1 show the strain energy value and convergence curves of the linear, quadratic, and cubic elements for the cook's skew beam problem. The strain-smoothed 3-node triangular element shows much better convergence behavior than do the standard 3-node triangular element. The convergence behavior of the enriched strain-smoothed elements (SS-TRI3-d1, SS-TRI3-d2) is similar to the convergence behavior of the enriched elements (TRI3-d1, TRI3-d2). In Section 2.4.1, it is shown that the enriched elements (TRI3-d1, TRI3-d2) present good convergence behaviors. Therefore, the enriched strain-smoothed elements (SS-TRI3-d1, SS-TRI3-d2) can be considered to show good convergence behavior.



Figure 5.4. Convergence curve of the Cook's skew beam problem shown in Fig. 5.3.

	Linear	elements	Quadrati	c elements	Cubic elements				
Ν	TRI3	SS-TRI3	TRI3-d1	SS-TRI3-d1	TRI3-d2	SS-TRI3-d2			
2	72.34	48.72	15.96	12.59	2.37	5.48			
4	53.27	11.48	4.17	2.02	0.92	0.14			
8	27.78	2.01	1.13	0.32	0.35	0.31			
16	10.09	0.50	0.38	0.08	0.16	0.09			
32	3.06	0.18	0.17	0.06	0.10	0.01			

Table 5.1. Computational results for the cook's skew beam problem shown in Fig. 5.3. Relative error (%) in strain energies,  $E_h^e = |e_{ref} - e_{v,h}| / |e_{ref}| \times 100 \cdot (|e_{ref}| = 3.9999 \times 10^{-7})$ 

# 5.3.2 Tool jig

We here illustrate the automatic procedure, presented in Chapter 4, with a tool jig problem shown in Fig. 5.5(a). The too jig is subjected to a constant pressure on its top surface (the line AB) and the fixed boundary condition is applied along the line AC. Four triangular meshes (Mesh  $1\sim4$ ) are considered.



Figure 5.5. The tool jig problem: (a) problem description,  $E = 2.0 \times 10^{11} N / m^2$ , v = 0.3, (b) von Mises stress distribution of the reference solution, and (c) triangular meshes used (Mesh 1-4)

Analysis results of the automatic procedure with the strain-smoothed element (SS-TRI3) and the enriched strainsmoothed elements (SS-TRI3-d1, SS-TRI3-d2) are shown in Fig. 5.6. In cases of Mesh 1, 2, and 3, improved solutions are obtained by applying the cover function adaptively. In particular, the accuracy of the strain energy is significantly improved.



Figure 5.6. von Mises stress and strain energy results of the tool jig problem shown in Fig. 5.5(a): (a) results of the strain-smoothed 3-node element, (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, von Mises stress error =  $|\tau_{v,ref} - \tau_{v,h}|/\tau_{v,ref} \times 100\%$ , strain energy error =  $|e_{v,ref} - e_{v,h}|/e_{v,ref} \times 100\%$ )

We additionally compare results of the automatic procedure when the standard 3-node element (TRI3) and strainsmoothed 3-node element (SS-TRI3) are used in the same mesh pattern (Mesh 3-4). The results obtained by applying the same automatic procedure for two elements (TRI3 and SS-TRI3) are shown in Figs 5.7 and 5.8. Note that the same scheme that determines the order of cover function for each node in Eq. (4.4) are applied to both elements.

Since the strain-smoothed element (SS-TRI3) provides a more accurate solution than the stand element (TRI3), the cover function is applied less in case of the strain-smoothed element (SS-TRI3). Therefore, the automatic procedure with the strain-smoothed element (SS-TRI3) uses smaller DOFs than the automatic procedure with the standard element (TRI3). Nevertheless, the strain-smoothed element with the adaptive use of cover function shows sufficiently accurate results.

## Mesh 3



Figure 5.7. Computation results of the tool jig problem shown in Fig. 5.5(a) when the automatic procedure applied for Mesh 3 with the standard 3-node element (TRI3) and the strain-smoothed 3-node element (SS-TRI3): (a) results of the standard 3-node element (TRI3) and the strain-smoothed 3-node element (SS-TRI3), (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, von Mises stress error =  $|\tau_{v,ref} - \tau_{v,h}|/\tau_{v,ref} \times 100\%$ , strain energy error =  $|e_{v,ref} - e_{v,h}|/e_{v,ref} \times 100\%$ )

### Mesh 4



Figure 5.8. Computation results of the tool jig problem shown in Fig. 5.5(a) when the automatic procedure applied for Mesh 4 with the standard 3-node element (TRI3) and the strain-smoothed 3-node element (SS-TRI3): (a) results of the standard 3-node element (TRI3) and the strain-smoothed 3-node element (SS-TRI3), (b) how the cover functions are applied, and (c) results of the adaptive use of cover functions. (DOFs = the number of degrees of freedom used, von Mises stress error =  $|\tau_{v,ref} - \tau_{v,h}|/\tau_{v,ref} \times 100\%$ , strain energy error =  $|e_{v,ref} - e_{v,h}|/e_{v,ref} \times 100\%$ )

# 5.4. Closure

In this chapter, we proposed an strain-smoothed 3-node triangular element enriched by polynomial cover functions. The strain field of the enriched 3-node element was divided into two parts, one corresponding to the standard DOFs and the other corresponding to the enriched DOFs. The strain-smoothed element (SSE) method, proposed by Lee and Lee [97], was applied to the strain field corresponding to the standard DOFs. In the cook's skew beam problem, the enriched strain-smoothed element shows good convergence behavior. In addition, the feasibility of improving the solution accuracy through the automatic procedure using the enriched strain-smoothed element was shown.

### Chapter 6. Conclusions and Future works

This dissertation focused on developing the 2D and 3D solid finite elements (4-node quadrilateral, 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements) enriched by interpolation covers. The linear dependence problem of the enriched elements was resolved in a simple and effective way. The developed elements pass the basic tests (the patch, zero energy mode, and isotropy tests for arbitrary enrichment) and show good convergence behaviors. In addition, the adaptive use of cover function, a most important advantage of the enriched elements, was demonstrated through several examples.

First, we proposed the new enriched 2D 4-node quadrilateral solid finite elements free from the linear dependence problem. To resolve the linear dependence problem, the piecewise linear shape functions for 4-node quadrilateral element were proposed and adapted for the geometry and displacement interpolations of the enriched element. The linear dependence problem was tested using various mesh patterns and no rank deficiency was observed. The effectiveness and performance of proposed element were demonstrated through several problems.

Second, the new enriched 3D 8-node hexahedral, 6-node prismatic, and 5-node pyramidal solid finite elements were proposed. The linear problem of the enriched 3D solid elements was avoid in the similar way of the new enriched 2D 4-node quadrilateral element. That is, the sets of piecewise linear shape functions for 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements were proposed and applied for the geometry and displacement interpolations. It was shown that the new enriched 3D solid element are free from the linear dependence problem. Through several problems, the performance and effectiveness of the developed elements were tested.

Third, the feasibility of improving solution accuracy with the adaptive use of cover functions was demonstrated. To apply the cover functions automatically, we presented the error indicator and the scheme that selects the appropriate order of cover functions for each node. The presented error indicator does not include any coefficients and asymptotically converges as the actual error converges. The automatic procedure using the error indicator and the scheme provided significantly improved solution for several 2D problems.

Finally, the enriched strain-smoothed 3-node triangular element was proposed. The strain components of the enriched 3-node element was divided into two parts, one corresponding to the standard DOFs and the other corresponding to the enriched DOFs, and the strain-smoothed element (SSE) method was applied to the strain part corresponding to the standard DOFs. The enriched strain-smoothed 3-node element shows good convergence performance. We also compared the results of the automatic procedure with the standard 3-node element and the strain-smoothed 3-node element. It was shown that the automatic procedure with the strain-smoothed element provides sufficiently accurate analysis results even with smaller DOFs than the automatic procedure with the standard element.

In the future work, it would be valuable to develop the enriched 4-node plate and shell finite elements free from the linear dependence problem by applying the piecewise linear shape function presented in Chapter 2. We expect that they will also show good convergence performance with distorted meshes, if shear and membrane locking are properly alleviated using the MITC method. It also expected to expand the new enriched 2D and 3D solid finite elements for nonlinear analysis because the enriched elements provide good accuracy in stress prediction.

To use the automatic procedure improving the solution accuracy with the adaptive use of cover functions practically, further research is needed. Various 2D and 3D problems and large finite element models should be considered to verify and improve the automatic procedure. In addition, the adaptive use of cover function and the local mesh refinement may be more effective if used properly at the same time. It would also be valuable to conduct a study on this to develop more effective ways to improve solution of the finite element analysis.

# Appendix

# Appendix A. The shape functions of the standard isoparametric procedure.

The shape functions of the 8-node hexahedral element shown in Fig. 3.1 are

$$h_{1}(r,s,t) = (1+r)(1+s)(1-t)/8, \quad h_{2}(r,s,t) = (1-r)(1+s)(1-t)/8,$$

$$h_{3}(r,s,t) = (1-r)(1-s)(1-t)/8, \quad h_{4}(r,s,t) = (1+r)(1-s)(1-t)/8,$$

$$h_{5}(r,s,t) = (1+r)(1+s)(1+t)/8, \quad h_{6}(r,s,t) = (1-r)(1+s)(1+t)/8,$$

$$h_{7}(r,s,t) = (1-r)(1-s)(1+t)/8, \quad h_{8}(r,s,t) = (1+r)(1-s)(1+t)/8,$$
(A.1)

where subscript means the node index corresponding to each shape function.

The shape functions of the 6-node prismatic element shown in Fig. 3.1 are

$$h_{1}(r,s,t) = (1-r-s)(1-t)/2, \quad h_{2}(r,t) = r(1-t)/2, \quad h_{3}(s,t) = s(1-t)/2,$$

$$h_{4}(r,s,t) = (1-r-s)(1+t)/2, \quad h_{5}(r,t) = r(1+t)/2, \quad h_{6}(s,t) = s(1+t)/2,$$
(A.2)

where subscript means the node index corresponding to each shape function.

The shape functions of the 5-node pyramidal element shown in Fig. 3.1 are

$$h_{1}(r,s,t) = (1+r)(1+s)(1-t)/8, \quad h_{2}(r,s,t) = (1-r)(1+s)(1-t)/8,$$

$$h_{3}(r,s,t) = (1-r)(1-s)(1-t)/8, \quad h_{4}(r,s,t) = (1+r)(1-s)(1-t)/8, \quad h_{5}(t) = (1+t)/2,$$
(A.3) where subscript means the node index corresponding to each shape function.

The shape functions of the 4-node tetrahedral element shown in Fig. 3.1 are

$$h_1(r,s,t) = 1 - r - s - t$$
,  $h_2(r) = r$ ,  $h_3(s) = s$ ,  $h_t(t) = t$ , (A.4)

where subscript means the node index corresponding to each shape function.

# Appendix B. The points used as vertices of each tetrahedral sub-domain of the 8-node hexahedral, 6-node prismatic and 5-node pyramidal elements.

8-node h	nexahe	dral ele	ment	6-node	prisma	tic elen	nent	5-node j	oyrami	dal elei	nent
Point j	$r_{j}$	$S_{j}$	$t_j$	Point j	$r_{j}$	$\boldsymbol{S}_{j}$	$t_j$	Point j	$r_{j}$	$\boldsymbol{S}_{j}$	$t_j$
1	1	1	-1	1	0	0	-1	1	1	1	-1
2	-1	1	-1	2	1	0	-1	2	-1	1	-1
3	-1	-1	-1	3	0	1	-1	3	-1	-1	-1
4	1	-1	-1	4	0	0	1	4	1	-1	-1
5	1	1	1	5	1	0	1	5	0	0	1
6	-1	1	1	6	0	1	1	6	0	0	-1
7	-1	-1	1	7	0	0.5	0	7	0	0	0
8	1	-1	1	8	0.5	0	0				
9	1	0	0	9	0.5	0.5	0				
10	-1	0	0	10	1/3	1/3	0				
11	0	1	0								
12	0	-1	0								
13	0	0	1								
14	0	0	-1								
15	0	0	0								

Table B.1. Coordinates of points of the 8-node hexahedral, 6-node prismatic, and 5-node pyramidal elements in the natural coordinate system shown in Fig. 3.1.

Table B.2. The points used as vertices of each tetrahedral sub-domain  $(T1 \sim T24)$  of the 8-node hexahedral element. The element center (point 15) is used for all sub-domains.

Sub-		Point		Sub-		Point		Sub-		Point	
domain	(1)	(2)	(3)	domain	(1)	(2)	(3)	domain	(1)	(2)	(3)
T1	5	8	9	Т9	5	1	11	T17	5	6	13
T2	8	4	9	T10	1	2	11	T18	6	7	13
Т3	4	1	9	T11	2	6	11	T19	7	8	13
T4	1	5	9	T12	6	5	11	T20	8	5	13
T5	6	2	10	T13	8	7	12	T21	1	4	14
T6	2	3	10	T14	7	3	12	T22	4	3	14
Τ7	3	7	10	T15	3	4	12	T23	3	2	14
Т8	7	6	10	T16	4	8	12	T24	2	1	14

Sub-		Point		Sub-		Point		Sub-		Point	
domain	(1)	(2)	(3)	domain	(1)	(2)	(3)	domain	(1)	(2)	(3)
T1	6	4	5	T6	1	4	7	T11	6	5	9
T2	1	3	2	T7	5	4	8	T12	5	2	9
Т3	4	6	7	T8	4	1	8	T13	2	3	9
T4	6	3	7	Т9	1	2	8	T14	3	6	9
T5	3	1	7	T10	2	5	8				

Table B.3. The points used as vertices of each tetrahedral sub-domain  $(T1 \sim T14)$  of the 6-node prismatic element. The element center (point 10) is used for all sub-domains.

Table B.4. The points used as vertices of each tetrahedral sub-domain  $(T1 \sim T8)$  of the 5-node pyramidal element. The element center (point 7) is used for all sub-domains.

Sub-		Point		Sub-		Point		Sub-		Point	
domain	(1)	(2)	(3)	domain	(1)	(2)	(3)	domain	(1)	(2)	(3)
T1	4	1	5	T4	3	4	5	Τ7	3	2	6
T2	1	2	5	Т5	1	4	6	T8	2	1	6
Т3	2	3	5	T6	4	3	6				

Sub-		i = 1			i = 2			<i>i</i> = 3			<i>i</i> = 4			<i>i</i> = 5			<i>i</i> = 6			i = 7			<i>i</i> = 8	
domain	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_i$	$\hat{c}_i$	$\hat{d}_i$	$\hat{b}_i$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_i$	$\hat{c}_i$	$\hat{d}_i$	$\hat{b}_i$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_i$
T1	1	0	-2	-1	0	0	-1	0	0	1	0	-2	1	4	2	-1	0	0	-1	0	0	1	-4	2
T2	1	2	0	-1	0	0	-1	0	0	1	-2	-4	1	2	0	-1	0	0	-1	0	0	1	-2	4
Т3	1	4	-2	-1	0	0	-1	0	0	1	-4	-2	1	0	2	-1	0	0	-1	0	0	1	0	2
T4	1	2	-4	-1	0	0	-1	0	0	1	-2	0	1	2	4	-1	0	0	-1	0	0	1	-2	0
T5	1	0	0	-1	2	-4	-1	-2	0	1	0	0	1	0	0	-1	2	4	-1	-2	0	1	0	0
T6	1	0	0	-1	4	-2	-1	-4	-2	1	0	0	1	0	0	-1	0	2	-1	0	2	1	0	0
Τ7	1	0	0	-1	2	0	-1	-2	-4	1	0	0	1	0	0	-1	2	0	-1	-2	4	1	0	0
Τ8	1	0	0	-1	0	-2	-1	0	-2	1	0	0	1	0	0	-1	4	2	-1	-4	2	1	0	0
Т9	2	1	-4	-2	1	0	0	-1	0	0	-1	0	2	1	4	-2	1	0	0	-1	0	0	-1	0
T10	4	1	-2	-4	1	-2	0	-1	0	0	-1	0	0	1	2	0	1	2	0	-1	0	0	-1	0
T11	2	1	0	-2	1	-4	0	-1	0	0	-1	0	2	1	0	-2	1	4	0	-1	0	0	-1	0
T12	0	1	-2	0	1	-2	0	-1	0	0	-1	0	4	1	2	-4	1	2	0	-1	0	0	-1	0
T13	0	1	0	0	1	0	0	-1	-2	0	-1	-2	0	1	0	0	1	0	-4	-1	2	4	-1	2
T14	0	1	0	0	1	0	-2	-1	-4	2	-1	0	0	1	0	0	1	0	-2	-1	4	2	-1	0
T15	0	1	0	0	1	0	-4	-1	-2	4	-1	-2	0	1	0	0	1	0	0	-1	2	0	-1	2
T16	0	1	0	0	1	0	-2	-1	0	2	-1	-4	0	1	0	0	1	0	-2	-1	0	2	-1	4
T17	0	0	-1	0	0	-1	0	0	-1	0	0	-1	4	2	1	-4	2	1	0	-2	1	0	-2	1
T18	0	0	-1	0	0	-1	0	0	-1	0	0	-1	2	0	1	-2	4	1	-2	-4	1	2	0	1
T19	0	0	-1	0	0	-1	0	0	-1	0	0	-1	0	2	1	0	2	1	-4	-2	1	4	-2	1
T20	0	0	-1	0	0	-1	0	0	-1	0	0	-1	2	4	1	-2	0	1	-2	0	1	2	-4	1
T21	2	4	-1	-2	0	-1	-2	0	-1	2	-4	-1	0	0	1	0	0	1	0	0	1	0	0	1
T22	0	2	-1	0	2	-1	-4	-2	-1	4	-2	-1	0	0	1	0	0	1	0	0	1	0	0	1
T23	2	0	-1	-2	4	-1	-2	-4	-1	2	0	-1	0	0	1	0	0	1	0	0	1	0	0	1
T24	4	2	-1	-4	2	-1	0	-2	-1	0	-2	-1	0	0	1	0	0	1	0	0	1	0	0	1

Table C.1. Coefficients of the piecewise linear shape functions of the 8-node hexahedral element.  $\hat{a}_i = 1$  for  $i = 1 \sim 8$  on all sub-domains.

Appendix. C. Coefficients of the piecewise linear shape functions in Eq. (3.9).

Sub-		<i>i</i> =	= 1			<i>i</i> =	= 2			<i>i</i> =	= 3			<i>i</i> =	= 4			<i>i</i> =	= 5			<i>i</i> =	= 6	
domain	$\hat{a}_i$	$\hat{b}_{l}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{a}_i$	$\hat{b}_{l}$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{a}_i$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{a}_i$	$\hat{b}_{l}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{a}_i$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{a}_i$	$\hat{b}_{i}$	$\hat{c}_i$	$\hat{d}_i$
T1	1.0	0.0	0.0	-1.0	1.0	0.0	0.0	-1.0	1.0	0.0	0.0	-1.0	5.0	-6.0	-6.0	1.0	-1.0	6.0	0.0	1.0	-1.0	0.0	6.0	1.0
T2	5.0	-6.0	-6.0	-1.0	-1.0	6.0	0.0	-1.0	-1.0	0.0	6.0	-1.0	1.0	0.0	0.0	1.0	1.0	0.0	0.0	1.0	1.0	0.0	0.0	1.0
T3	1.5	-1.5	0.0	-1.5	0.0	3.0	0.0	0.0	1.5	-1.5	0.0	-1.5	4.5	-4.5	-6.0	1.5	0.0	3.0	0.0	0.0	-1.5	1.5	6.0	1.5
T4	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	-3.0	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	3.0
T5	4.5	-4.5	-6.0	-1.5	0.0	3.0	0.0	0.0	-1.5	1.5	6.0	-1.5	1.5	-1.5	0.0	1.5	0.0	3.0	0.0	0.0	1.5	-1.5	0.0	1.5
T6	3.0	-3.0	-3.0	-3.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	0.0	3.0	-3.0	-3.0	3.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	0.0
T7	1.5	0.0	-1.5	-1.5	1.5	0.0	-1.5	-1.5	0.0	0.0	3.0	0.0	4.5	-6.0	-4.5	1.5	-1.5	6.0	1.5	1.5	0.0	0.0	3.0	0.0
T8	3.0	-3.0	-3.0	-3.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	0.0	3.0	-3.0	-3.0	3.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	0.0
Т9	4.5	-6.0	-4.5	-1.5	-1.5	6.0	1.5	-1.5	0.0	0.0	3.0	0.0	1.5	0.0	-1.5	1.5	1.5	0.0	-1.5	1.5	0.0	0.0	3.0	0.0
T10	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	-3.0	0.0	0.0	3.0	0.0	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	3.0	0.0	0.0	3.0	0.0
T11	3.0	-3.0	-3.0	0.0	0.0	1.5	1.5	-1.5	0.0	1.5	1.5	-1.5	3.0	-3.0	-3.0	0.0	0.0	4.5	-1.5	1.5	0.0	-1.5	4.5	1.5
T12	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	-3.0	0.0	0.0	3.0	0.0	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	3.0	0.0	0.0	3.0	0.0
T13	3.0	-3.0	-3.0	0.0	0.0	4.5	-1.5	-1.5	0.0	-1.5	4.5	-1.5	3.0	-3.0	-3.0	0.0	0.0	1.5	1.5	1.5	0.0	1.5	1.5	1.5
T14	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	-3.0	3.0	-3.0	-3.0	0.0	0.0	3.0	0.0	0.0	0.0	0.0	3.0	3.0

Table C.2. Coefficients of the piecewise linear shape functions of the 6-node prismatic element.

Sub-		<i>i</i> = 1			<i>i</i> = 2			<i>i</i> = 3			<i>i</i> = 4			<i>i</i> = 5	
domain	$\hat{b}_l$	$\hat{c}_i$	$\hat{d}_{i}$	$\hat{b}_l$	$\hat{c}_i$	$\hat{d}_i$	$\hat{b}_{l}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{b}_{l}$	$\hat{c}_i$	$\hat{d}_i$	$\hat{b}_l$	$\hat{c}_i$	$\hat{d}_i$
T1	0.50	2.50	-1.00	-2.00	0.00	-1.00	-2.00	0.00	-1.00	0.50	-2.50	-1.00	3.00	0.00	4.00
T2	2.50	0.50	-1.00	-2.50	0.50	-1.00	0.00	-2.00	-1.00	0.00	-2.00	-1.00	0.00	3.00	4.00
Т3	2.00	0.00	-1.00	-0.50	2.50	-1.00	-0.50	-2.50	-1.00	2.00	0.00	-1.00	-3.00	0.00	4.00
T4	0.00	2.00	-1.00	0.00	2.00	-1.00	-2.50	-0.50	-1.00	2.50	-0.50	-1.00	0.00	-3.00	4.00
T5	1.25	2.50	-0.25	-1.25	0.00	-0.25	-1.25	0.00	-0.25	1.25	-2.50	-0.25	0.00	0.00	1.00
T6	0.00	1.25	-0.25	0.00	1.25	-0.25	-2.50	-1.25	-0.25	2.50	-1.25	-0.25	0.00	0.00	1.00
Τ7	1.25	0.00	-0.25	-1.25	2.50	-0.25	-1.25	-2.50	-0.25	1.25	0.00	-0.25	0.00	0.00	1.00
T8	2.50	1.25	-0.25	-2.50	1.25	-0.25	0.00	-1.25	-0.25	0.00	-1.25	-0.25	0.00	0.00	1.00

Table C.3. Coefficients of the piecewise linear shape functions of the 5-node pyramidal element.  $\hat{a}_i = 1$  for  $i = 1 \sim 5$  on all sub-domains.

# Bibliography

[1] Bathe KJ. Finite Element Procedures. New York: Prentice Hall; 1996.

[2] Hughes TJR. The finite element method-Linear static and dynamic finite element analysis. New York: Dover Publications; 2000.

[3] Zienkiewicz OC. The finite element method. New York: McGraw-Hill; 1991.

[4] Fish J. The s-version of the finite element method. Comput Struct 1992;43:539-47.

[5] Oden JT, Duarte CA. Solution of singular problems using hp clouds. In: Whiteman JR, editor. The mathematics of finite elements and applications 96. New York, NY: Wiley, 1982. p. 35±54.

[6] Duarte CA. The hp cloud method. PhD dissertation, University of Texas at Austin, Austin, TX, USA, December 1996.

[7] Duarte CA, Oden JT. An hp adaptive method using clouds. Comput Methods in Appl Mech and Eng 1996;139:237±62.

[8] Oden JT, Duarte CA. Clouds, cracks and fem's. In: Reddy BD, editor. Recent developments in computational and applied mechanics. Barcelona, Spain: International Center for Numerical Methods in Engineering, CIMNE, 1997. p. 302±21.

[9] Oden JT, Duarte CA, Zienkiewicz OC. A new cloud-based hp finite element method. Comput Methods Appl Mech Eng 1998;153(1-2):117-26.

[10] de TR Mendonça P, de Barcellos CS, & Duarte CA. Investigations on the hp-Cloud Method by solving Timoshenko beam problems. Comput Mech 200;25(2-30:286-95

[11] Garcia O, Fancello EA, Barcellos CS, Duarte CA. Hp clouds in Mindlin's thick plate model. Int J Numer Method Eng 2000;47:1381–400.

[12] Chen GQ, Ohnishi Y, Ito T. Development of high-order manifold method. Int J Numer Methods Eng 1998;43(4):685–712.

[13] Cheng YM, Zhang YH, Chen WS. Wilson non-conforming element in numerical manifold method. Commun

Numer Methods Eng 2002;18(12):877-84.

[14] Zhang G, Sigiura Y, Hasegawa H, Wang G. The second order manifold method with six node triangle mesh. Struct Eng/Earthq Eng JSCE 2002;19: 1–9.

[15] Li S, Cheng Y, Wu Y-F. Numerical manifold method based on the method of weighted residuals. Comput Mech 2005;35(6):470–80.

[16] Ma G, An X, He L. The numerical manifold method: a review. Int J Comput Methods 2010;7(1):1–32.

[17] He L, Ma G. Development of 3D numerical manifold method. Int J Comput Methods 2010;7(1):107-29.

[18] Okazawa S, Terasawa H, Kurumatani M, Terada K, Kashiyama K. Eulerian finite cover method for solid dynamics. Int J Comput Methods 2010;7(1):33–54.

[19] Belytschko T, Black T. Elastic crack growth in finite elements with minimal remeshing. Int J Numer Methods Eng 1999;45(5):602–20.

[20] Moës N, Dolbow J, Belytschko T. A finite element method for crack growth without remeshing. Int J Numer Methods Eng 1999;46(1):131–50.

[21] Dolbow J, Moës N, Belytschko T. Discontinuous enrichment infinite elements with a partition of unity method. Finite Elem Anal Des 2000;36(3–4):235–60.

[22] Nagashima T, Omoto T, Tani S. Stress intensity factor analysis of interface cracks using X-FEM. Int J Numer Meth Eng 2003;56: 1151–73.

[23] Zi G, Chen H, Xu J, Belytschko T. The extended finite element method for dynamic fractures. Shock Vib 2005;12:9–23.

[24] Svahn PO, Torbjon E, Runesson K. Discrete crack modeling in a new X-FEM format with emphasis on dynamic response. Int J Numer Anal Meth Geomech 2007;31:261–83.

[25] Abdelaziz Y, Hamouine A. A survey of the extended finite element. Comput Struct 2008;86(11-12):1141-51.

[26] Bordas S, Nguyen P, Dunant C, Dang H, Guidoum A. An extended finite element library. Int J Numer Meth Eng 2006;2:1–33.

[27] Menk A, Bordas S. Numerically determined enrichment functions for the extended finite element method and applications to bi-material anisotropic fracture and polycrystals. Int J Numer Methods Eng 2010;83(7):805–

28.

[28] Fries TP, Belytschko T. The extended/generalized finite element method: An overview of the method and its applications. Int J Numer Methods Eng 2010;84(3):253–304.

[29] Strouboulis T,Babuska I,Copps K. The design and analysis of the generalized finite element method. Comput Methods Appl Mech Eng 2000;181(1-3):43–69.

[30] Strouboulis T,Copps K,Babuska I. The generalized finite element method: an example of its implementation and illustration of its performance. Int J Numer Methods Eng 2000;47(8):1401–17.

[31] Duarte C, Hamzeh O, Liszka T, Tworzyd W. A generalized finite element method for the simulation of threedimensional dynamic crack propagation. Comput Meth Appl Mech Eng 2001;190:227–62.

[32] Simone A, Duarte CA, Van der Giessen E. A generalized finite element method for polycrystals with discontinuous grain boundaries. Int J Numer Methods Eng 2006;67(8):1122–45.

[33] Strouboulis T, Babuška I, Hidajat R. The generalized finite element method for Helmholtz equation: theory, computation, and open problems. Comput Meth Appl Mech Eng 2006;195:4711–31.

[34] Kim J, Bathe KJ. The finite element method enriched by interpolation covers. Comput Struct 2013;116:35–49.

[35] Kim J, Bathe KJ. Towards a procedure to automatically improve finite element solutions by interpolation covers. Comput Struct 2014;131:81-97.

[36] Jeon HM, Lee PS, Bathe KJ. The MITC3 shell finite element enriched by interpolation covers. Comput Struct 2014;134:128–42.

[37] Jeon HM, Yoon KH, Lee PS, Bathe KJ. The MITC3+ shell element enriched in membrane displacements by interpolation covers. Comput Methods Appl Mech Eng 2018;337:458–480.

[38] Babuška I, Melenk JM. The partition of unity method. Int J Numer Methods Eng 1997;40(4):727–58.

[39] Duarte CA, Babuska I, Oden JT. Generalized finite element methods for three dimensional structural mechanics problems. Comput Struct 2000;77(2):215–32.

[40] Rabczuk T, Bordas S, Zi G. On three-dimensional modelling of crack growth using partition of unity methods. Comput Struct 2010;88(23–24):1391–411. [41] Ham S, Bathe KJ. A finite element method enriched for wave propagation problems. Comput Struct 2012;94–95:1–12.

[42] Bathe KJ, Almeida C. A simple and effective pipe elbow element – linear analysis. ASME J Appl Mech 1980;47:93–100.

[43] Bathe KJ, Almeida C, Ho LW. A simple and effective pipe elbow element – some nonlinear capabilities. Comput Struct 1983;17:659–67.

[44] Bathe KJ, Chaudhary A. On the displacement formulation of torsion of shafts with rectangular cross-sections. Int J Numer Meth Eng 1982;18:1565–8.

[45] Yoon KH, Lee YG, Lee PS. A continuum mechanics based 3-D beam finite element with warping displacements and its modeling capabilities. Struct Eng Mech 2012;43(4):411–37

[46] An XM, Li LX, Ma GW, Zhang HH. Prediction of rank deficiency in partition of unity-based methods with plane triangular or quadrilateral meshes. Comput Methods Appl Mech Eng 2011;200(5-8):665–74.

[47] An XM, Zhao ZY, Zhang HH, Li LX. Investigation of linear dependence problem of three-dimensional partition of unity-based finite element methods. Comput Methods Appl Mech Eng 2012;233:137–51.

[48] Tian R, Yagawa G, Terasaka H. Linear dependence problems of partition of unity-based generalized FEMs. Comput Methods Appl Mech Eng 2006;195(37-40):4768–82.

[49] Babuška I, Banerjee U. Stable generalized finite element method (SGFEM). Comput Methods Appl Mech Eng 2012;201:91-111.

[50] Zhang Q, Banerjee U, Babuška I. Higher order stable generalized finite element method. Numer Math 2014;128(1):1-29.

[51] Hong WT, Lee PS. Coupling flat-top partition of unity method and finite element method. Finite Elem Anal Des 2013;67:43–55.

[52] Hong WT, Lee PS. Mesh based construction of flat-top partition of unity. Appl Math Comput 2013;219:8687–704.

[53] Lee PS, Bathe KJ. Development of MITC isotropic triangular shell finite elements. Comput Struct 2004;82(11-12):945–62.

[54] Dunavant DA. High degree efficient symmetrical Gaussian quadrature rules for the triangle. Int J Numer

Methods Eng 1985;21(6):1129-48.

[55] Hiller JF, Bathe KJ. Measuring convergence of mixed finite element discretizations: an application to shell structures. Comput Struct 2003;81:639–54.

[56] Bathe KJ. The inf-sup condition and its evaluation for mixed finite element methods. Comput Struct 2001;79:243–52.

[57] Bathe KJ, Lee PS. Measuring the convergence behavior of shell analysis schemes. Comput Struct 2011;89:285–301.

[58] Lee PS, Bathe KJ. Insight into finite element shell discretizations by use of the "basic shell mathematical model". Comput Struct 2005;83:69–90.

[59] Bucalem ML, Bathe KJ. The mechanics of solids and structures – hierarchical modeling and the finite element solution. Springer; 2011.

[60] Wilson EL, Taylor RL, Doherty WP, Ghaboussi J. Incompatible displacement models: numerical and computer methods in structural mechanics. New York: Academic Press; 1973. p. 43–57.

[61] Wilson EL, Ibrahimbegovic A. Use of incompatible displacement models for the calculation of element stiffness or stresses. Finite Elements Anal Des 1990;7(3):229-241.

[62] Ibrahimbegovic A, Wilson EL. A modified method of incompatible modes. Int J Numer Method in Biomed Eng 1991;7(3):187-194.

[63] ADINA R&D Inc. ADINA theory and modeling guide, Watertown (MA): ADINA R&D Inc.; 2001.

[64] MacNeal RH, Harder RL. A proposed standard set of problems to test finite element accuracy. Finite Elem Anal Des 1985;1:3–20.

[65] Cen S, Zhou PL, Li CF, Wu CJ. An unsymmetric 4-node, 8-DOF plane membrane element perfectly breaking through MacNeal's theorem. Int J Numer Meth Eng 2015;103:469–500.

[66] Dvorkin EN, Bathe KJ. A continuum mechanics based four-node shell element for general nonlinear analysis. Eng Comput 1984;1(1):77–88.

[67] Bathe KJ, Lee PS, Hiller JF. Towards improving the MITC9 shell element. Comput Struct 2003;81(8):477–89.

[68] Lee PS, Bathe KJ. Development of MITC isotropic triangular shell finite elements. Comput Struct 2004;82(11):945–62.

[69] Lee Y, Yoon K, Lee PS. Improving the MITC3 shell finite element by using the Hellinger–Reissner principle. Comput Struct 2012;110–111:93–106.

[70] Lee Y, Lee PS, Bathe KJ. The MITC3+ shell element and its performance. Comput Struct 2014;138:12–23.

[71] Ko Y, Lee PS, Bathe KJ. The MITC4+ shell element and its performance. Comput Struct 2016;169:57-68.

[72] Ko Y, Lee PS, Bathe KJ. A new MITC4+ shell element. Comput Struct 2017;182:404-18.

[73] Xiao H, Gimbutas Z. A numerical algorithm for the construction of efficient quadrature rules in two and higher dimensions. Comput Math Appl 2010;59(2):663–76.

[74] Sussman T, Bathe KJ. Studies of finite element procedures—on mesh selection. Comput Struct 1985:21(1-2),257-264.

[75] Grätsch T, Bathe KJ. A posteriori error estimation techniques in practical finite element analysis. Comput Struct 2005;83(4-5):235-265.

[76] Bathe KJ, Zhang H. A mesh adaptivity procedure for CFD and fluid-structure interactions. Comput Struct 2009;87(11-12):604-617.

[77] Liu GR, Dai KY, Nguyen TT. A smoothed finite element method for mechanics problems. Comput. Mech 2007:39:859–877.

[78] Nguyen-Thanh N, Rabczuk, T, Nguyen-Xuan H, Bordas SP. A smoothed finite element method for shell analysis. Comput Methods Appl Mech Eng 2008;198(2):165-177.

[79] Bordas SP, Rabczuk T, Hung NX, Nguyen VP, Natarajan S, Bog T, Hiep NV. Strain smoothing in FEM and XFEM. Comput Struct 2010;88(23-24):1419-1443.

[80] Phung-Van P, Nguyen-Thoi T, Luong-Van H, Lieu-Xuan Q. Geometrically nonlinear analysis of functionally graded plates using a cell-based smoothed three-node plate element (CS-MIN3) based on the C0-HSDT. Comput Methods Appl Mech Eng 2014;270:15-36.

[81] Phung-Van P, Nguyen-Thoi T, Dang-Trung H, Nguyen-Minh N. A cell-based smoothed discrete shear gap method (CS-FEM-DSG3) using layerwise theory based on the C0-HSDT for analyses of composite plates. Compos Struct 2014;111:553-565.

[82] Hamrani A, Habib SH, Belaidi I. CS-IGA: A new cell-based smoothed isogeometric analysis for 2D computational mechanics problems. Comput Methods Appl Mech Eng 2017;315:671-690.

[83] Liu GR, Nguyen-Thoi T, Nguyen-Xuan H, Lam KY. A node-based smoothed finite element method (NS-FEM) for upper bound solutions to solid mechanics problems. Comput Struct 2009;87(1-2):14-26.

[84] Nguyen-Thoi T, Vu-Do HC, Rabczuk T, Nguyen-Xuan H. A node-based smoothed finite element method (NS-FEM) for upper bound solution to visco-elastoplastic analyses of solids using triangular and tetrahedral meshes. Comput Methods Appl Mech Eng 2010;199(45-48):3005-3027.

[85] Wang G, Cui XY, Feng H, Li GY. A stable node-based smoothed finite element method for acoustic problems. Comput Methods Appl Mech Eng 2015;297:348-370.

[86] Liu GR, Nguyen-Thoi T, Lam KY. An edge-based smoothed finite element method (ES-FEM) for static, free and forced vibration analyses of solids. J Sound Vib 2009;320(4-5),1100-1130.

[87] He ZC, Liu GR, Zhong ZH, Wu SC, Zhang GY, Cheng AG. An edge-based smoothed finite element method (ES-FEM) for analyzing three-dimensional acoustic problems. Comput Methods Appl Mech Eng 2009;199(1-4):20-33.

[88] Chen L, Rabczuk T, Bordas SPA, Liu GR, Zeng KY, Kerfriden P. Extended finite element method with edgebased strain smoothing (ESm-XFEM) for linear elastic crack growth. Comput Methods Appl Mech Eng 2012;209:250-265.

[89] Nguyen-Xuan H, Liu GR, Bordas S, Natarajan S, Rabczuk T. An adaptive singular ES-FEM for mechanics problems with singular field of arbitrary order. Comput Methods Appl Mech Eng 2013;253:252-273.

[90] Shin CM, Lee BC. Development of a strain-smoothed three-node triangular flat shell element with drilling degrees of freedom. Finite Elem Anal Des 2014;86:71-80.

[91] Lee C, Kim H, Im S. Polyhedral elements by means of node/edge-based smoothed finite element method. Int J Numer Meth Eng 2017;110(11):1069–1100.

[92] Lee C, Kim H, Kim J, Im S. Polyhedral elements using an edge-based smoothed finite element method for nonlinear elastic deformations of compressible and nearly incompressible materials. Comput. Mech 2017;60(4):659-682.

[93] Nguyen-Thoi T, Liu GR, Lam KY, Zhang GY. A face-based smoothed finite element method (FS-FEM) for 3D linear and geometrically non-linear solid mechanics problems using 4-node tetrahedral elements. Int J Numer Meth Eng 2009;78(3):324–353. [94] Nguyen-Thoi T, Liu GR, Vu-Do HC, Nguyen-Xuan H. A face-based smoothed finite element method (FS-FEM) for visco-elastoplastic analyses of 3D solids using tetrahedral mesh. Comput Methods Appl Mech Eng 2009;198(41-44):3479-3498.

[95] Sohn D, Han J, Cho YS, Im S. A finite element scheme with the aid of a new carving technique combined with smoothed integration. Comput Methods Appl Mech Eng 2013;254:42-60.

[96] Jin S, Sohn D, Im S. Node-to-node scheme for three-dimensional contact mechanics using polyhedral type variable-node elements. Comput Methods Appl Mech Eng 2016;304:217-242.

[97] Lee C, Lee PS. A new strain smoothing method for triangular and tetrahedral finite elements. Comput Methods Appl Mech Eng 2018;341:939-955.