박사학위논문 Ph.D.Dissertation

유한요소구조동역학모델의축소기법개발

On the finite element model reduction methods instructural dynamics



김진균(金鎭均Kim,Jin-Gyun)기계항공시스템학부,해양시스템공학전공 SchoolofMechanical,AerospaceandSystemsEngineeringDepartmentofOceanSystemsEngineering

KAIST

유한요소 구조동역학 모델의 축소기법 개발

On the finite element model reduction methods in structural dynamics



On the finite element model reduction methods in structural dynamics

Advisor : Professor Lee, Phill-Seung

by

Kim, Jin-Gyun

School of Mechanical, Aerospace and Systems Engineering Department of Ocean Systems Engineering KAIST

A thesis submitted to the faculty of KAIST in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the School of Mechanical, Aerospace and Systems Engineering, Department of Ocean Systems Engineering . The study was conducted in accordance with Code of Research Ethics¹.

> 2014. 5. 16. Approved by Professor Lee, Phill-Seung [Advisor]

¹Declaration of Ethical Conduct in Research: I, as a graduate student of KAIST, hereby declare that I have not committed any acts that may damage the credibility of my research. These include, but are not limited to: falsification, thesis written by someone else, distortion of research findings or plagiarism. I affirm that my thesis contains honest conclusions based on my own careful research under the guidance of my thesis advisor.

유한요소 구조동역학 모델의 축소기법 개발

김 진 균

위 논문은 한국과학기술원 박사학위논문으로 학위논문심사위원회에서 심사 통과하였음.



- 심사위원장 이 필승 (인)
 - 심사위원 오일권 (인)
 - 심사위원 조연우 (인)
 - 심사위원 정형조 (인)
 - 심사위원 박용화 (인)

DOSE 김 진 균. Kim, Jin-Gyun. On the finite element model reduction methods in structural dynamics. 유한요소 구조동역학 모델의 축소기법 개발. School of Mechanical, Aerospace and Systems Engineering, Department of Ocean Systems Engineering . 2014. 111p. Advisor Prof. Lee, Phill-Seung. Text in English.

ABSTRACT

Finite element model reduction methods have been widely used to reduce the computational costs of structural analysis and design. Using model reduction methods, global (original) models can be approximated by reduced models with much smaller matrix size. Although various model reduction methods such as dynamic condensation and component mode synthesis (CMS) have been proposed over the last several decades, many challenging issues still must still be addressed to improve the solution accuracy and computational efficiency of reduced models. The work in this thesis focuses on the development of enhanced model reduction, general mode selection, and accurate error estimation methods to overcome the known disadvantages and limitations of existing model reduction methods.

In this work, we first develop a new component mode synthesis enhancing the Craig-Bampton (CB) method, the most popular model reduction method. To develop the enhanced CB method, the original transformation matrix in the CB method is enhanced considering the residual flexibility that contains the residual substructural modal effect, and the unknown eigenvalue in the enhanced transformation matrix is approximated by using O'callahan's approach to Guyan reduction. Using the newly defined transformation matrix, global FE models can be more accurately approximated. We demonstrate its performance through numerical examples.

In model reduction methods, only a small proportion of the dominant degree of freedoms (DOFs) or the substructural modes is retained in the reduced model. Therefore, the accuracy of the reduced model highly depends on the choice of the retained dominant DOFs or substructural modes. In this work, we develop a new mode selection method for CMS methods. In contrast to the frequency cut-off mode selection method, in which substructural modes in sequence from the lowest substructural frequency to a cut-off frequency are retained, the proposed method selects the dominant substructural modes in accordance with the contribution of the substructural modes to the target global modes. Therefore, the new mode selection method enables the analyst to select substructural modes that can better represent the target global modes in the resulting reduced model. We then validate its performance and feasibility for both stiffness- and flexibility-based CMS (F-CMS) methods using a variety of numerical examples. We also provide a simple strategy to detect inaccurately approximated global modes in the reduced model, the correction of which leads to improved reduced models.

A major obstacle of model reduction methods has been the absence of a good methodology for estimating the reliability of reduced models. To resolve this issue, we develop a robust error estimator to accurately predict the relative eigenvalue errors. Derivation procedures show that the proposed error estimator is a direct approximation of the relative eigenvalue error. Therefore, using this new error estimator, the reliability of reduced models can be efficiently and precisely evaluated. In this work, we develop new error estimators for Guyan reduction, the CB method, and the F-CMS method.

Here, we also propose a high-fidelity formulation for interface reduction in the F-CMS method. Using the new formulation, we can construct more compact reduced models without significant loss of accuracy. Eigenvector relations between the global and reduced models are clearly defined in the interface reduction level. The performance of the present formulation is validated using numerical examples.

Contents

| Abstrac | t | i |
|------------|--|-----|
| Content | s | ii |
| List of 7 | Tables | iii |
| List of H | igures | iv |
| Chapter | 1. Introduction | 1 |
| Chapter | 2. Dynamic condensation | 3 |
| 2.1 | General description | 3 |
| 2.2 | Guyan reduction | 4 |
| 2.3 | Kidder's approach | 5 |
| 2.4 | Meirovitch's approach | 6 |
| 2.5 | Improved reduced system (IRS) | 7 |
| Chapter | 3. Component mode synthesis | 8 |
| 3.1 | General description | 8 |
| 3.2 | Craig-Bampton (CB) method | 10 |
| 3.3 | Flexibility-based component mode synthesis (F-CMS) | 12 |
| Chapter | 4. Enhanced CB method | 16 |
| 4.1 | Formulation details | 16 |
| 4.2 | Numerical examples | 19 |
| | 4.2.1 Simple plate problem | 19 |
| | 4.2.2 Hyperboloid shell problem | 22 |
| | 4.2.3 Stiffened plate problem | 25 |
| | 4.2.4 Ring solid problem | 28 |
| 4.3 | Closure | 31 |
| Chapter | 5. F-CMS method with interface reduction | 32 |
| 5.1 | Reduction of localized Lagrange multiplier | 33 |
| 5.2 | Reduction of interface DOFs | 34 |
| 5.3 | Numerical example | 36 |
| 5.4 | Closure | 38 |

| Chapter | 6. | Mode selection method for CMS methods | 39 |
|---------|-------|--|-----|
| 6.1 | Revie | w of Existing Mode Selection Methods | 40 |
| | 6.1.1 | Kammer's method | 40 |
| | 6.1.2 | The OMR method | 41 |
| | 6.1.3 | The CMS^P_χ method $\ \ldots \ $ | 41 |
| 6.2 | Prope | bed Mode selection method | 41 |
| | 6.2.1 | Modal relation from substructural to global eigenvector | 43 |
| | 6.2.2 | Mode selection procedure | 44 |
| 6.3 | Error | estimation and accuracy control $\ldots \ldots \ldots \ldots \ldots$ | 47 |
| 6.4 | Appli | cation in the F-CMS method | 48 |
| | 6.4.1 | A priori mode selection method | 48 |
| | 6.4.2 | A posteriori mode selection method $\ldots \ldots \ldots \ldots$ | 49 |
| 6.5 | Nume | erical examples | 49 |
| | 6.5.1 | Cylindrical shell problem | 49 |
| | 6.5.2 | Square plate problem | 57 |
| | 6.5.3 | Hemisphere shell problem | 60 |
| 6.6 | Closu | re | 62 |
| Chapter | 7. | Error estimation method for model reduction methods | 63 |
| 7.1 | Error | estimation method for Guyan reduction | 63 |
| | 7.1.1 | Derivation procedures | 64 |
| | 7.1.2 | Numerical examples | 65 |
| 7.2 | Error | estimation method for CB method $\ldots \ldots \ldots \ldots$ | 74 |
| | 7.2.1 | Derivation procedures | 74 |
| | 7.2.2 | Numerical examples | 77 |
| 7.3 | Error | estimation method for the F-CMS method | 91 |
| | 7.3.1 | Derivation procedures | 91 |
| | 7.3.2 | Numerical examples | 92 |
| 7.4 | Closu | re | 94 |
| Chapter | 8. | Conclusions | 102 |
| Appendi | ces | | 104 |
| Chapter | Α. | Derivation of F_{rs} and F_{rm} | 105 |
| Chapter | B. | Woodbury matrix identity | 106 |
| Chapter | C. | Specific derivation of Equation 7.4 | 107 |

References

Summary (in Korean)

108 112



List of Tables

| 4.1 | Comparison between the CB and enhanced CB methods. | 18 |
|-----|--|----|
| 4.2 | Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem | 20 |
| 4.3 | Retained substructural mode numbers $N_d^{(k)}$ in the hyperboloid shell problem | 22 |
| 4.4 | Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem | 25 |
| 5.1 | Numbers of retained modes and DOFs in the rectangular plate problem | 38 |
| 6.1 | The present mode selection procedure | 47 |
| 6.2 | Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the cylindrical shell problem 1 | 50 |
| 6.3 | Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the cylindrical shell problem 2 | 54 |
| 6.4 | Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the square plate problem | 57 |
| 6.5 | Selected substructural modes in the square plate problem using the CB method. Note | |
| | that $C_j^{(k)}$ is the value of the substructural modal contribution in Equation 6.14 | 59 |
| 6.6 | Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the hemisphere shell problem. | 60 |
| 7.1 | Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem | 77 |
| 7.2 | Exact and estimated eigenvalue errors in the rectangular plate problem (The frequency | |
| | cut-off mode selection method is used and $N_d = 15$) | 80 |
| 7.3 | Retained substructural mode numbers $N_d^{(k)}$ in the shaft-shaft interaction problem | 81 |
| 7.4 | Retained substructural mode numbers $N_d^{(k)}$ in the hemisphere shell problem | 84 |
| 7.5 | Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem | 87 |
| 7.6 | Computation times for the exact and estimated relative eigenvalue errors. \ldots . | 90 |
| 7.7 | Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem | 93 |
| 7.8 | Retained substructural mode numbers $N_d^{(k)}$ in the hemisphere shell problem | 93 |
| 7.9 | Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem | 94 |

List of Figures

| 3.1 | Interface boundary treatment in component mode synthesis: (a) The global (non-partitioned) structure Ω , (b) The global structure Ω partitioned into two substructures, $\Omega = \Omega_1 \cup \Omega_2$, $\Gamma = \Omega_1 \cap \Omega_2$, (c) Interface boundary treatment in the CB method, (d) Interface boundary treatment in the F-CMS method, \mathbf{u}_{δ} ; interface boundary DOFs, $\boldsymbol{\lambda}_{\ell}$; localized Lagrange | |
|-----|--|------------|
| | multiplier vector. | 9 |
| 4.1 | Simple plate problem. | 20 |
| 4.2 | Relative eigenvalue errors in the simple plate problem. (a) $N_d = 10$, (b) $N_d = 20$ | 21 |
| 4.3 | Hyperboloid shell problem. | 23 |
| 4.4 | Relative eigenvalue errors in the hyperboloid shell problem. (a) $N_d = 40$, (b) $N_d = 80$ | 24 |
| 4.5 | Stiffened plate problem. | 26 |
| 4.6 | Relative eigenvalue errors in the stiffened plate problem. (a) $N_d = 50$, (b) $N_d = 80$ | 27 |
| 4.7 | Solid ring problem | 29 |
| 4.8 | Relative eigenvalue errors in the solid ring problem. (a) $N_d = 40$, (b) $N_d = 80$ | 30 |
| 5.1 | Reduction procedures. (a) Fully retained model. (b) Level 1: reduction of internal DOFs. | |
| | (c) Level 2: reduction of localized Lagrange multipliers. (d) Level 3: reduction of interface | |
| | DOFs | 32 |
| 5.2 | Rectangular plate problem. | 36 |
| 5.3 | Relative eigenvalue error in the rectangular plate problem. (a) Case 1 and (b) Case 2 | 37 |
| 6.1 | Relation between global and substructural mode shapes in a cylindrical shell. (a) Global | |
| | modes 1 to 9, (b) Global modes 10 to 39. \ldots | 42 |
| 6.2 | Three eigenvalue problems for (a) the global eigenvalue problem, (b) the intermediate | |
| | reduced eigenvalue problem, and (c) the final reduced eigenvalue problem. N_d and N_d are | |
| | the numbers of the substructural modes retained in the intermediate and final reduced | |
| | models, respectively. | 45 |
| 6.3 | Cylindrical shell problem 1 | 50 |
| 6.4 | Relative natural frequency error of the present mode selection method depending on γ in | F 1 |
| 05 | the cylindrical shell problem I. (a) CB method, (b) F-CMS method | 51 |
| 0.5 | Retained substructural mode numbers N_d in the cylindrical shell problem 1. (a) CB method (b) E-CMS method | 52 |
| 66 | Belative natural frequency error in the cylindrical shell problem 1 (a) CB method (b) | 02 |
| 0.0 | F-CMS method. | 53 |
| 6.7 | Cylindrical shell problem 2 | 54 |
| 6.8 | Relative natural frequency error in the cylindrical shell problem 2. (a) CB method, (b) | |
| | F-CMS method. | 55 |
| 6.9 | Error estimation and accuracy control in the cylindrical shell problem 1 using the F-CMS | |
| | method. (a) Error estimation using Equation 6.22, (b) Accuracy control for the global | |
| | modes 5, 6, 15 and 16 | 56 |

| 6.10 | Square plate problem. | 57 |
|------------|--|-----|
| 6.11 | Relative natural frequency error in the square plate problem. (a) CB method, (b) F-CMS method | 50 |
| 0.10 | | 00 |
| 6.12 | Hemisphere shell problem. | 60 |
| 6.13 | F-CMS method | 61 |
| 7.1 | Cylindrical panel. | 66 |
| 7.2 | Selected nodes of the cylindrical panel. (a) 35 nodes, (b) 59 nodes | 66 |
| 7.3 | Exact and estimated relative errors in eigenvalues in the cylindrical panel. (a) 35 nodes, | ~ - |
| | (b) 59 nodes | 67 |
| 7.4 | Hyperboloid panel. | 68 |
| 7.5 | Selected nodes of the hyperboloid panel. (a) 35 nodes, (b) 59 nodes | 69 |
| 7.6 | Exact and estimated relative errors in eigenvalues in the hyperboloid panel. (a) 35 nodes, | 70 |
| 77 | (b) 59 nodes | 70 |
| 1.1 | Shart-shart interaction problem. | 71 |
| 7.8 7.0 | Selected nodes of the shart-shart interaction problem. (a) 39 nodes, (b) 70 nodes | 72 |
| 7.9 | Exact and estimated relative errors in eigenvalues in the shart-shart interaction problem. | 79 |
| 7 10 | (a) 59 hodes, (b) 70 hodes. | 75 |
| 1.10 | Exact and estimated relative eigenvalue errors in the simple plate problem. The frequency $aut = aut $ | 70 |
| 7.11 | Exact and estimated relative eigenvalue errors in the simple plate problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 15$, $\tilde{N}_s = 30$, (b) $N_s = 30$ | 10 |
| | $\tilde{N}_{d} = 60$ | 70 |
| 7 12 | Shaft-shaft interaction problem | 81 |
| 7.13 | Exact and estimated relative eigenvalue errors in the shaft-shaft interaction problem. The | 01 |
| | frequency cut-off mode selection method is used. (a) $N_d = 20$, (b) $N_d = 40$ | 82 |
| 7.14 | Exact and estimated relative eigenvalue errors in the shaft-shaft interaction problem. The | |
| | mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 20, N_d = 40$, (b) | 0.0 |
| - 1 - | $N_d = 40, N_d = 80. \dots \dots$ | 83 |
| 7.15 | Hemisphere shell problem. | 84 |
| 1.10 | Exact and estimated relative eigenvalue errors in the hemisphere shell problem. The fraction set off mode selection method is used (a) $N = 25$ (b) $N = 80$ | OF |
| 717 | irequency cut-on mode selection method is used. (a) $N_d = 25$, (b) $N_d = 80$ | 00 |
| 1.11 | selection method proposed by Kim et al. [67] is used. (a) $N_d = 25$, $\tilde{N}_d = 50$, (b) $N_d = 80$, | |
| | $\tilde{N}_d = 160.$ | 86 |
| 7.18 | Exact and estimated relative eigenvalue errors in the stiffened plate problem. The fre- | |
| | quency cut-off mode selection method is used. (a) $N_d = 25$, (b) $N_d = 75$ | 88 |
| 7.19 | Exact and estimated relative eigenvalue errors in the stiffened plate problem. The mode | |
| | selection method proposed by Kim et al. [67] is used. (a) $N_d = 25$, $\tilde{N}_d = 50$, (b) $N_d = 75$, | |
| | $\tilde{N}_d = 150.$ | 89 |
| 7.20 | Exact and estimated relative errors in eigenvalues in the simple plate problem using the | |
| | CB method. The frequency cut-off mode selection method is used. (a) $N_d = 10$, (b) | |
| | $N_d = 20$ | 95 |

| 7.21 | Exact and estimated relative errors in eigenvalues in the simple plate problem. The |
|------|---|
| | frequency cut-off mode selection method is used. (a) $N_d = 10$, (b) $N_d = 20$ |
| 7.22 | Exact and estimated relative errors in eigenvalues in the simple plate problem. The mode |
| | selection method proposed by Kim et al. [67] is used. (a) $N_d = 10$, $\tilde{N}_d = 20$, (b) $N_d = 20$, |
| | $\tilde{N}_d = 40.\dots$ |
| 7.23 | Exact and estimated relative errors in eigenvalues in the hemisphere shell problem. The |
| | frequency cut-off mode selection method is used. (a) $N_d = 54$, (b) $N_d = 76$ |
| 7.24 | Exact and estimated relative errors in eigenvalues in the hemisphere shell problem. The |
| | mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 54$, $\tilde{N}_d = 108$, (b) |
| | $N_d = 76, \tilde{N}_d = 152.$ |
| 7.25 | Exact and estimated relative errors in eigenvalues in the stiffened plate problem. The |
| | frequency cut-off mode selection method is used. (a) $N_d = 32$, (b) $N_d = 52$ 100 |
| 7.26 | Exact and estimated relative errors in eigenvalues in the stiffened plate problem. The |
| | mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 32$, $\tilde{N}_d = 128$, (b) |
| | $N_d = 52, \ \tilde{N}_d = 208$ |



Chapter 1. Introduction

Along with enormous improvement of computer models and technologies, the size of finite element models has also rapidly increased. For this reason, model reduction methods (sometimes also referred to as reduced-order modeling techniques) remain important in computational mechanics. In structural dynamics communities, model reduction methods are categorized in two groups, DOFs based and mode based.

In DOFs based reduction methods, only a small proportion of the dominant DOFs, known as "master", is retained in the reduced model, and the remaining DOFs, known as "slave", are eliminated. Therefore, the size of the global (original) model can be significantly reduced.

In contrast, when using mode based reduction methods, a global structure is partitioned into multiple substructures according to the substructuring strategy. After obtaining a small proportion of dominant substructural modes by solving individual substructural eigenvalue problems, a reduced model is constructed by retaining only a select group of the dominant substructural modes.

Although the formulation details of model reduction methods are quite different, such methods all aim to construct reduced models with better accuracy and smaller size. To satisfy these requirements, various model reduction methods have been proposed over the last several decades. However, many important issues remain to be resolved. In this thesis, we have focused on the following three issues:

- Model reduction methods should lead to precisely reduced modeling. At the same time, these methods should allow computational efficiency and easy implementation. Indeed, there does not yet exist an "optimal" model reduction method that satisfies both requirements at the same time.
- Since the reduced models that result from model reduction methods are constructed using only master DOFs or dominant substructural modes, their accuracy highly depends on the choice of DOFs or substructural modes retained. A key challenge is then to develop high-fidelity DOFs or mode selection methods able to construct a reliable reduced model, in which the global model is accurately reflected, and in which the size of the reduced model is as small as possible.
- The reliability of the reduced model can be assessed by errors in its approximated global eigenvalues. For this reason, the relative eigenvalue error is generally used to evaluate the reliability of model reduction methods. However, since the calculation of the relative eigenvalue error requires exact eigenvalues that are obtained from global eigenvalue problems, this measurement technique is not easy to use in real engineering applications.

In Chapter 2, we briefly review conventional DOFs based model reduction methods. After explaining the general description of the DOFs based model reduction, we present formulations of Guyan reduction, Kidder's approach, Meirovitch's approach and the improved reduced system (IRS) method.

In Chapter 3, we review component mode synthesis (CMS), a family of mode based reduction methods popular in the structural dynamics community. We here introduce the Craig-Bampton (CB) and the flexibility based CMS (F-CMS) methods.

In chapter 4, we propose a new CMS method by enhancing the CB method. Formulation details and a newly defined transformation matrix are presented. Its excellent performance is validated through various numerical examples. In Chapter 5, a high-fidelity formulation for interface reduction in the F-CMS method is presented. The proposed formulation consists of three reduction levels for internal DOFs, localized Lagrange multipliers, and interface boundary DOFs. Using the present formulation, we can construct more compact reduced models. The numerical results show the performance of the new formulation.

In Chapter 6, we propose a new mode selection method for CMS methods. The present mode selection method can select the dominant substructural modes in accordance with the substructural modal contributions to the target global modes. Therefore, the new mode selection method enables the analyst to select substructural modes that can better represent the target global modes in the reduced model. Its excellent performance is tested for both stiffness- and flexibility-based CMS methods using various structural FE models. A simple accuracy control strategy for locally fluctuated errors is also proposed and tested.

In Chapter 7, we introduce an excellent error estimation method for Guyan reduction, a representative DOFs based reduction method. The proposed error estimator can accurately predict the relative eigenvalue errors without knowing the exact eigenvalues. To derive the present method, the exact eigensolutions are decomposed into approximated and error parts and Kidder's transformation matrix is used to approximate the exact eigenvector. Then, the error estimator can be derived from the global eigenvalue problem. Its feasibility and performance are demonstrated by various numerical examples.

We also propose error estimation methods for the CB and F-CMS methods that are popular CMS methods. Since CMS methods share the similar general description with DOFs based reduction methods, the concept of the error estimation method used in Guyan reduction may also be employed for the CB and F-CMS methods. However, since the formulation details differ for different model reduction methods, the final form and specific derivation procedures of error estimators also differ. In Sections 7.2 and 7.3, we present adequate error estimators for the CB and F-CMS methods, respectively.

Finally, the conclusions of the present study are given in Chapter 8.

Chapter 2. Dynamic condensation

Reduced-order modeling of large finite element (FE) models is essential in many engineering fields such as ocean, mechanical and aerospace engineering. Dynamic condensation is widely used a model reduction technique. In dynamic condensation, a small proportion of the dominant DOFs, known as "master", is only retained for the reduced model, and the other DOFs, known as "slave", are eliminated. Therefore, the order of original model can be dramatically reduced. Since the slave DOFs are directly eliminated, the dynamic condensation method can be categorized by the DOFs based model reduction method.

In 1960s, Guyan [1] proposed the most widely used DOFs based reduction method, and then similar approaches were introduced at the same time [2, 3]. These are generally referred as Guyan reduction (sometimes also referred to as eigenvalue economizer, mass condensation). Since then, various extended methods have been proposed [4, 5, 6, 7], and recent studies have focused on the developments of iterative procedures [8, 9, 10, 11, 12] or substructuring approaches [13, 14, 15].

In this section, we introduce Guyan reduction which is a representative DOFs based reduction method [1] and two different approaches of Guyan reduction by Kidder [4] and Meirovitch [16]. We also explain the improved reduced system (IRS) method that is known as the most precise DOFs based reduction technique [7].



2.1 General description

Neglecting the damping matrix, the linear dynamics equations can be expressed by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f},\tag{2.1}$$

where \mathbf{M} and \mathbf{K} are the mass and stiffness matrices, respectively, and \mathbf{u} and \mathbf{f} are the displacement and force vectors, respectively. Then, the generalized eigenvalue problem is

$$\mathbf{K}(\boldsymbol{\phi})_i = \lambda_i \mathbf{M}(\boldsymbol{\phi})_i, \quad i = 1, 2, ..., N, \quad \text{with } \mathbf{u} = \boldsymbol{\Phi} \mathbf{q},$$
(2.2)

where λ_i and $(\phi)_i$ are the eigenvalue and eigenvector, respectively, and N is the number of DOFs in the original structure. $\mathbf{\Phi}$ and \mathbf{q} are the eigenvector matrix and generalized coordinate vector, respectively. Here, λ_i and $(\phi)_i$ satisfy the following relations

$$(\boldsymbol{\phi})_i^T \mathbf{M}(\boldsymbol{\phi})_j = \delta_{ij} \quad \text{for } i \text{ and } j = 1, 2, ..., N,$$

$$(2.3a)$$

$$(\boldsymbol{\phi})_i^T \mathbf{K}(\boldsymbol{\phi})_j = \lambda_j \delta_{ij} \quad \text{for } i \text{ and } j = 1, 2, ..., N,$$

$$(2.3b)$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if i = j, otherwise $\delta_{ij} = 0$). Equations 2.3(a) and 2.3(b) are called as 'mass-orthonormality' and 'stiffness-orthogonality', respectively.

In the DOFs reduction method, we retain the dominant DOFs and others are eliminated. Irons named these as "master" and "slave" DOFs [2, 3], and those are denoted as subscripts 1 and 2, respectively. After eliminating the slave DOFs, the reduced eigenvalue problem with master DOFs $(N_1 \ll N)$ is obtained

$$\mathbf{K}_1(\boldsymbol{\phi}_1)_i = \bar{\lambda}_i \mathbf{M}_1(\boldsymbol{\phi}_1)_i, \quad i = 1, 2, \dots, N_1, \quad \text{with } \mathbf{u}_1 = \boldsymbol{\Phi}_1 \mathbf{q}_1, \tag{2.4}$$

where \mathbf{M}_1 and \mathbf{K}_1 are the reduced mass and stiffness matrices, respectively, and $\bar{\lambda}_i$ and $(\phi_1)_i$ are the eigenvalue and eigenvector obtained from the reduced matrices, respectively. \mathbf{u}_1 is the reduced displacement vector, and $\boldsymbol{\Phi}_1$ and \mathbf{q}_1 are the reduced eigenvector matrix and the corresponding generalized coordinate vector, respectively. Then, $\bar{\lambda}_i$ and $(\phi_1)_i$ satisfy the following relations

$$(\phi_1)_i^T \mathbf{M}_1(\phi_1)_j = \delta_{ij}$$
 for *i* and *j* = 1, 2, ..., N₁, (2.5a)

$$(\phi_1)_i^T \mathbf{K}_1(\phi_1)_j = \bar{\lambda}_j \delta_{ij}$$
 for *i* and *j* = 1, 2, ..., N₁. (2.5b)

Here, using $(\phi_1)_i$, we calculate the approximated eigenvector denoted by $(\bar{\phi})_i$, and an overbar $(\bar{\cdot})$ denotes the approximated quantities. While the formulation details may differ considerably among various DOFs based reduction methods, the general descriptions are similar.

2.2 Guyan reduction

Guyan reduction was developed based on the reduction technique of the stiffness matrix in static analysis [17]. The linear static equations of the original structural model are

$$\mathbf{K}\mathbf{u} = \mathbf{f}.$$
 (2.6)

and then, it can be partitioned

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \quad (2.7)$$

where \mathbf{K}_{ij} is submatrix of the stiffness matrix. When we assume that \mathbf{f}_2 is to be zero, \mathbf{u}_2 becomes

$$\mathbf{u}_2 = -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{u}_1. \tag{2.8}$$

Using Equations 2.7 and 2.8, we obtain the reduced stiffness matrix

$$\mathbf{K}_1 = \mathbf{K}_{11} - \mathbf{K}_{12} \mathbf{K}_{22}^{-1} \mathbf{K}_{21}, \tag{2.9}$$

and then, the displacement vector \mathbf{u} is approximated

$$\mathbf{u} \approx \bar{\mathbf{u}} = \mathbf{T}_G \mathbf{u}_1, \quad \mathbf{T}_G = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{22}^{-1} \mathbf{K}_{21} \end{bmatrix},$$
 (2.10)

where \mathbf{T}_G is original transformation matrix of Guyan reduction. Here, **I** is an identity matrix of the same order as the dimension of \mathbf{u}_1 .

This reduction technique can be applied to the mass matrix. Using Equation 2.10, the strain energy V and kinetic energy T can be written

$$V = \frac{1}{2} \mathbf{u}_1^T \mathbf{T}_G^T \mathbf{K} \mathbf{T}_G \mathbf{u}_1, \quad T = \frac{1}{2} \dot{\mathbf{u}}_1^T \mathbf{T}_G^T \mathbf{M} \mathbf{T}_G \dot{\mathbf{u}}_1, \tag{2.11}$$

and then, the reduced matrices are defined

$$\mathbf{K}_1 = \mathbf{T}_G^T \mathbf{K} \mathbf{T}_G, \quad \mathbf{M}_1 = \mathbf{T}_G^T \mathbf{M} \mathbf{T}_G.$$
(2.12)

Using Equation 2.7 in Equation 2.12, we can calculate the reduced stiffness matrix \mathbf{K}_1 , and it is same with Equation 2.9. Similarly, \mathbf{M} is also partitioned

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix},$$
(2.13)

where \mathbf{M}_{ij} is submatrix of the mass matrix, and then the reduced mass matrix \mathbf{M}_1 is

$$\mathbf{M}_{1} = \mathbf{M}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{M}_{21} - \mathbf{M}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} + \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{M}_{22}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}.$$
 (2.14)

These reduced mass and stiffness matrices can be used for the reduced eigenvalue problem in Equation 2.4. Then, **u** is expressed by the eigenvector matrix $\mathbf{\Phi}$ and the generalized coordinate vector **q**, see Equation 2.2. Comparing this expression with $\mathbf{\bar{u}}$ in Equation 2.10, we find

$$\mathbf{\Phi} \approx \bar{\mathbf{\Phi}} = \mathbf{T}_G \mathbf{\Phi}_1. \tag{2.15}$$

Note that the columns of $\overline{\Phi}$ are eigenvectors $(\overline{\phi})_i$, and these are approximated vectors of original eigenvector $(\phi)_i$.

2.3 Kidder's approach

Since the same transformation matrix \mathbf{T}_G is used to reduce the mass matrix, Guyan reduction shares the same assumption ($\mathbf{f}_2 = \mathbf{0}$) with the stiffness reduction method, and it may lead to inaccurate results in approximation procedure to obtain the displacement vector \mathbf{u} from the reduced displacement vector \mathbf{u}_1 .

To solve this problem, Kidder proposed the another derivation procedure for the reduced order modeling from the structural eigenvalue problem [4]. Using the partitioned matrices in Equations 2.7 and 2.13, the structural eigenvalue problem can be written

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}.$$
(2.16)

Using the second row in Equation 2.16, we have

$$\mathbf{u}_{2} = \left[\mathbf{K}_{22} - \lambda \mathbf{M}_{22}\right]^{-1} \left[\lambda \mathbf{M}_{21} - \mathbf{K}_{21}\right] \mathbf{u}_{1}.$$
 (2.17)

It should be noted that Equation 2.17 is exact form of \mathbf{u}_2 .

Using Equation 2.17 in the first row in Equation 2.16, we obtain

$$\left[\mathbf{K}_{11} - \lambda \mathbf{M}_{11} - \left[\mathbf{K}_{12} - \lambda \mathbf{M}_{12}\right] \left[\mathbf{K}_{22} - \lambda \mathbf{M}_{22}\right]^{-1} \left[\mathbf{K}_{21} - \lambda \mathbf{M}_{21}\right]\right] \mathbf{u}_{1} = \mathbf{0}.$$
(2.18)

Equation 2.18 is the exact reduced eigenvalue problem, and it can be solved using an iterative solution technique. However, since λ is unknown, the inverse of $[\mathbf{K}_{22} - \lambda \mathbf{M}_{22}]$ might be calculated in each iteration step. To reduce the computational cost, the inverse term can be expanded

$$[\mathbf{K}_{22} - \lambda \mathbf{M}_{22}]^{-1} = \mathbf{K}_{22}^{-1} + \lambda \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} + O(\lambda^2) + O(\lambda^3) + \cdots .$$
(2.19)

Neglecting higher order terms of λ in Equation 2.19 and using it in Equation 2.18, we have

$$[\mathbf{K}_1 - \lambda \mathbf{M}_1] \,\mathbf{u}_1 = \mathbf{0},\tag{2.20}$$

and also

$$\mathbf{u}_{2} = \left[\mathbf{K}_{22}^{-1} + \lambda \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1}\right] \left[\lambda \mathbf{M}_{21} - \mathbf{K}_{21}\right] \mathbf{u}_{1}.$$
(2.21)

Then, the reduced matrices, \mathbf{M}_1 and \mathbf{K}_1 , are same as Guyan reduction, see Equations 2.14 and 2.9. Therefore, the final reduced eigenvalue problem by Kidder's approach is same with Guyan reduction. However, since the slave DOFs \mathbf{u}_2 in Equation 2.21 is more general formulation than the one by Guyan reduction, the displacement vector \mathbf{u} is more accurate approximated

$$\mathbf{u} \approx \bar{\mathbf{u}} = \mathbf{T}_{K} \mathbf{u}_{1}, \quad \mathbf{T}_{K} = \begin{bmatrix} \mathbf{I} \\ \begin{bmatrix} \mathbf{K}_{22}^{-1} + \lambda \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \end{bmatrix} [\lambda \mathbf{M}_{21} - \mathbf{K}_{21}] \end{bmatrix}, \quad (2.22)$$

where \mathbf{T}_K is a transformation matrix of Kidder's approach. When we use \mathbf{T}_K instead of \mathbf{T}_G in Equation 2.15, $\bar{\mathbf{\Phi}}$ is differently defined

$$\mathbf{\Phi} \approx \bar{\mathbf{\Phi}} = \mathbf{T}_G \mathbf{\Phi}_1. \tag{2.23}$$

Kidder's approach is also referred to "mass condensation" [16], and mathematically, Guyan reduction belongs in this approach.

2.4 Meirovitch's approach

We here introduce another approach to obtain the reduced matrices. As mentioned above, Equation 2.18 is obtained using the exact form of \mathbf{u}_2 in Equation 2.17, and then Equation 2.18 can be expanded

$$[\mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}]\mathbf{u}_{1} = \lambda[\mathbf{M}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{M}_{21} - \mathbf{M}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} + \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{M}_{22}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}]\mathbf{u}_{1} + O(\lambda^{2}) + O(\lambda^{3}) + \cdots .$$
(2.24)

Ignoring second and higher order terms of λ in Equation 2.24, the terms of left- and right-hand side in 2.24 are \mathbf{K}_1 and \mathbf{M}_1 defined in Equations 2.9 and 2.14, respectively, and then, using Equation 2.17, the displacement vector \mathbf{u} is redefined

$$\mathbf{u} = \mathbf{T}_M \mathbf{u}_1, \quad \mathbf{T}_M = \begin{bmatrix} \mathbf{I} \\ [\mathbf{K}_{22} - \lambda \mathbf{M}_{22}]^{-1} [\lambda \mathbf{M}_{21} - \mathbf{K}_{21}] \end{bmatrix}, \quad (2.25)$$

where \mathbf{T}_M is an exact transformation matrix. When we use \mathbf{T}_M instead of \mathbf{T}_G in Equation 2.15, $\boldsymbol{\Phi}$ is also redefined

$$\mathbf{\Phi} = \mathbf{T}_2 \mathbf{\Phi}_1. \tag{2.26}$$

Here, Equation 2.26 is the exact form of Φ . However, since Φ_1 is generally an approximated solution obtained from the reduced eigenvalue problem, Equation 2.26 can be regarded as an approximated form $\overline{\Phi}$.

Consequently, above three approaches lead to same eigensolutions $(\bar{\lambda}_i, (\phi_1)_i)$ from Equation 2.4. However, using Equations 2.23 and 2.26, we can obtain the more appropriately approximated eigenvector matrix $\bar{\Phi}$, and then eigenvectors $(\phi)_i$ have also better solution accuracies. In particular, the approximated eigenvector $(\bar{\phi})_i$ does not satisfy the mass-orthonormality property in Equation 2.3, and then it can be used to develop the robust error estimation method. Its detail will be presented in Chapter 7.

2.5 Improved reduced system (IRS)

To improve Guyan reduction, a new dynamic condensation method was proposed by O'callahan [7]. As we mentioned, since Kidder's transformation matrix T_K contains the unknown eigenvalue λ , it cannot be directly used for model reduction without handling this unknown terms. Expanding Equation 2.22 and neglecting λ^2 terms, we have

$$\mathbf{u} \approx \bar{\mathbf{u}} = \mathbf{T}_{3} \mathbf{u}_{1}, \quad \mathbf{T}_{3} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{22}^{-1} \mathbf{K}_{21} + \lambda \left[\mathbf{K}_{22}^{-1} \mathbf{M}_{21} - \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \mathbf{K}_{21} \right] \end{bmatrix}.$$
(2.27)

Using Equation 2.20, following relation is obtained

$$\lambda \mathbf{u}_1 = \mathbf{M}_1^{-1} \mathbf{K}_1 \mathbf{u}_1. \tag{2.28}$$

Substituting Equation 2.28 into Equation 2.29, \mathbf{T}_3 can be redefined without λ as

$$\mathbf{T}_{3} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21} + \begin{bmatrix} \mathbf{K}_{22}^{-1}\mathbf{M}_{21} - \mathbf{K}_{22}^{-1}\mathbf{M}_{22}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} \end{bmatrix} \mathbf{M}_{1}^{-1}\mathbf{K}_{1} \end{bmatrix}.$$
 (2.29)

The newly defined transformation matrix \mathbf{T}_3 can be directly employed for model reduction as

$$\mathbf{K}_{1}^{IRS} = \mathbf{T}_{3}^{T} \mathbf{K} \mathbf{T}_{3}, \quad \mathbf{M}_{1}^{IRS} = \mathbf{T}_{3}^{T} \mathbf{M} \mathbf{T}_{3}.$$
(2.30)

Since \mathbf{T}_3 is more accurate transformation matrix than the original transformation matrix in Guyan reduction, it leads to more accurate reduced models than previous dynamic condensation techniques. It is known as the improved reduced system (IRS) method, and it has been widely used in various engineering fields such as FE model updating, optimal sensor positioning, experimental model verification, etc. Also, extended researches of the IRS method have been performed [18, 19, 20].

Chapter 3. Component mode synthesis

Component mode synthesis (CMS) methods have been widely used for FE model reduction in structural dynamics. In the CMS methods, the original large structural FE model is partitioned into smaller substructural FE models connected at interface boundary. A small proportion of substructural modes (dominant substructural modes) and interface constraint conditions are used to reduce the original structural model. Since, instead of the original large structural model, we handle the reduced model constructed using the small substructural models, CMS methods can dramatically reduce the computational cost. The accurately approximated reduced models are valuable indeed in structural systems design, system identification, and experimentally verified model development. Unlike the DOFs based reduction method, CMS methods belong in the mode based reduction method because substructural modes are truncated.

In the 1960s, based on Hurty and Guyan's idea [21, 1], Craig and Bampton proposed a CMS method referred to as the Craig-Bampton (CB) method [22]. Since then, various related studies have been done to develop robust CMS methods [23, 24, 25, 26, 27, 28, 29, 30]. The CMS methods can be categorized by their interface handling techniques such as fixed interface, free interface and hybrid type. Reviews of the CMS methods can be founded in Refs. [31, 32, 33].

In this section, we introduce two CMS methods: the CB and flexibility based CMS (F-CMS) methods. The CB method is still the most popular and widely used CMS method because of its simplicity and reliability. The CB method is based on the fixed interface condition. To more precise reduced-order modeling, the F-CMS method has been recently developed by Park and Park [28]. In the F-CMS method, substructures are connected by free interface and interface constraint conditions using localized Lagrange multipliers. Previous researches show that the F-CMS method makes better solution accuracy than the CB method [28, 30].

3.1 General description

Considering the global (non-partitioned) structure Ω modeled by finite element discretization in Figure 3.1(a), the linear dynamics equations can be expressed by

$$\mathbf{M}_g \ddot{\mathbf{u}}_g + \mathbf{C}_g \dot{\mathbf{u}}_g + \mathbf{K}_g \mathbf{u}_g = \mathbf{f}_g, \tag{3.1}$$

where \mathbf{M}_g and \mathbf{K}_g are the global mass and stiffness matrices, respectively, and \mathbf{C}_g is the global damping matrix. \mathbf{u}_g is the global displacement vector, and \mathbf{f}_g is the global force vector. The subscript g denotes the global structure.

Then, the generalized eigenvalue problem of the global structure is

$$\mathbf{K}_g(\boldsymbol{\phi}_g)_j = \lambda_j \mathbf{M}_g(\boldsymbol{\phi}_g)_j, \quad j = 1, 2, \dots, N_g, \quad \text{with } \mathbf{u}_g = \boldsymbol{\Phi}_g \mathbf{q}_g, \tag{3.2}$$

where λ_j and $(\phi_g)_j$ are the eigenvalue and eigenvector directly calculated in the global structure, respectively, and N_g is the number of DOFs in the global structure. Φ_g and \mathbf{q}_g are the global eigenvector matrix and generalized coordinate vector, respectively.



Figure 3.1: Interface boundary treatment in component mode synthesis: (a) The global (non-partitioned) structure Ω , (b) The global structure Ω partitioned into two substructures, $\Omega = \Omega_1 \cup \Omega_2$, $\Gamma = \Omega_1 \cap \Omega_2$, (c) Interface boundary treatment in the CB method, (d) Interface boundary treatment in the F-CMS method, \mathbf{u}_b : interface boundary DOFs, λ_ℓ : localized Lagrange multiplier vector.

Here, λ_j and $(\phi_g)_j$ satisfy the following relations

$$(\boldsymbol{\phi}_g)_i^T \mathbf{M}_g(\boldsymbol{\phi}_g)_j = \delta_{ij}$$
 Mass-orthonormality, (3.3a)

$$(\boldsymbol{\phi}_g)_i^T \mathbf{K}_g(\boldsymbol{\phi}_g)_j = \lambda_j \delta_{ij}$$
 Stiffness-orthogonality, (3.3b)

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if i = j).

In CMS methods, the global structure is partitioned into substructures as shown Figure 3.1(b), eigenvalue analyses of individual substructures are carried out, and dominant substructural eigenvectors (modes) and their eigenvalues are chosen. Then, the reduced eigenvalue problem with \bar{N}_p DOFs ($\bar{N}_p \ll N_q$) is then obtained using the selected substructural modes

$$\bar{\mathbf{K}}_{p}(\bar{\boldsymbol{\phi}}_{p})_{j} = \bar{\lambda}_{j}\bar{\mathbf{M}}_{p}(\bar{\boldsymbol{\phi}}_{p})_{j}, \quad j = 1, 2, ..., \bar{N}_{p}, \quad \text{with } \bar{\mathbf{u}}_{p} = \bar{\boldsymbol{\Phi}}_{p}\bar{\mathbf{q}}_{p}, \tag{3.4}$$

where $\bar{\mathbf{M}}_p$ and $\bar{\mathbf{K}}_p$ are the reduced mass and stiffness matrices, respectively, and $\bar{\lambda}_j$ and $(\bar{\boldsymbol{\phi}}_p)_j$ are the approximated eigenvalue and eigenvector, respectively. $\bar{\mathbf{u}}_p$ is the approximated displacement vector, and $\bar{\boldsymbol{\Phi}}_p$ and $\bar{\mathbf{q}}_p$ are the approximated eigenvector matrix and the corresponding generalized coordinate vector, respectively. Note that the subscript p denotes the partitioned structure, and an overbar ($\bar{\cdot}$)

denotes the approximated quantities. In the similar with Equation 3.3, $\bar{\lambda}_j$ and $(\bar{\phi}_p)_j$ also satisfy the following relations

$$(\bar{\boldsymbol{\phi}}_p)_i^T \bar{\mathbf{M}}_p (\bar{\boldsymbol{\phi}}_p)_j = \delta_{ij}$$
 Mass-orthonormality, (3.5a)

$$(\bar{\boldsymbol{\phi}}_p)_i^T \bar{\mathbf{K}}_p (\bar{\boldsymbol{\phi}}_p)_j = \bar{\lambda}_j \delta_{ij}$$
 Stiffness-orthogonality. (3.5b)

Since $\mathbf{u}_g \approx \bar{\mathbf{u}}_g = \bar{\mathbf{\Phi}}_g \bar{\mathbf{q}}_g$ and $\bar{\mathbf{\Phi}}_g$ and $\bar{\mathbf{q}}_g$ can be obtained from $\bar{\mathbf{\Phi}}_p$ and $\bar{\mathbf{q}}_p$ in Equation 3.4, we finally obtain the reduced form of Equation 3.1 in the generalized coordinates

$$\ddot{\mathbf{q}}_{g} + \bar{\mathbf{C}}_{p} \dot{\mathbf{q}}_{g} + \bar{\mathbf{\Lambda}}_{p} \bar{\mathbf{q}}_{g} \approx \bar{\mathbf{f}}_{g},$$

$$\bar{\mathbf{\Phi}}_{g}^{T} \mathbf{M}_{g} \bar{\mathbf{\Phi}}_{g} \approx \mathbf{I}_{p}, \quad \bar{\mathbf{\Phi}}_{g}^{T} \mathbf{C}_{g} \bar{\mathbf{\Phi}}_{g} = \bar{\mathbf{C}}_{p}, \quad \bar{\mathbf{\Phi}}_{g}^{T} \mathbf{K}_{g} \bar{\mathbf{\Phi}}_{g} \approx \bar{\mathbf{\Lambda}}_{p},$$

$$\bar{\mathbf{\Lambda}}_{p} = \operatorname{diag}(\bar{\omega}_{1}^{2}, \bar{\omega}_{2}^{2}, ... \bar{\omega}_{i}^{2} ..., \bar{\omega}_{\bar{N}_{p}}^{2}), \quad \bar{\mathbf{f}}_{g} = \bar{\mathbf{\Phi}}_{g}^{T} \mathbf{f}_{g}, \qquad (3.6)$$

where \mathbf{I}_p is the $\bar{N}_p \times \bar{N}_p$ identity matrix, and $\bar{\mathbf{C}}_p$ is a real symmetric matrix, generally non-diagonal. $\bar{\omega}_i$ is the *i*th approximated natural frequency.

While the formulation and algorithmic details may differ considerably among various CMS methods, the general descriptions are similar. It should be noted that, in CMS methods, the direct computations of the eigensolutions by employing Equation 3.2 are avoided. Instead, we solve the reduced eigenvalue problem in Equation 3.4 constructed by only retaining the dominant substructural modes. Therefore, CMS methods can adopt a *divide-and-conquer* paradigm for tackling very large eigenvalue problems, which can be carried out in parallel computations (see, e.g., [34]). In addition, it can accommodate experimentally determined substructural modes and mode shapes in the synthesis of the global eigenvalue problems.

In this section, we review the formulations of two well-known CMS methods: the Craig-Bampton (CB) method (a stiffness-based CMS method)[22] and the flexibility-based CMS (F-CMS) method[28]. Figures 3.1(c) and 3.1(d) show the interface boundary treatments in the CB and F-CMS methods, respectively. In the CB method, the two substructures are connected with a fixed interface at the interface boundary Γ , see Figure 3.1(c). The F-CMS method, in contrast, uses the free interface boundary between substructures, and then the interface boundary is constructed by the interface DOFs \mathbf{u}_b and the localized Lagrange multiplier vector $\boldsymbol{\lambda}_{\ell}$ as shown in Figure 3.1(d). The subscript *b* denotes the interface boundary. The CB and F-CMS methods are derived and explained in the following sections.

3.2 Craig-Bampton (CB) method

The CB method has been widely used because it is accurate and simple to implement [22, 32]. The first step in the CB method is to separate the global equations of motion with substructural (or interior) and interface boundary DOFs after partitioning the global structure into N_s substructures. N_s is the number of substructures. Then, Equation 3.1 becomes

$$\begin{bmatrix} \mathbf{M}_s & \mathbf{M}_c \\ \mathbf{M}_c^T & \mathbf{M}_b \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_s \\ \ddot{\mathbf{u}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{K}_s & \mathbf{K}_c \\ \mathbf{K}_c^T & \mathbf{K}_b \end{bmatrix} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_b \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ \mathbf{f}_b \end{bmatrix},$$
(3.7)

where the subscript s denotes the substructural DOFs, and c denotes the coupling DOFs between substructures and interface boundary. Hence, \mathbf{M}_s and \mathbf{K}_s are the partitioned block diagonal mass and stiffness matrices of the substructures that consist of mass and stiffness matrices $(\mathbf{M}_s^{(k)} \text{ and } \mathbf{K}_s^{(k)})$ of each substructure. \mathbf{M}_b and \mathbf{K}_b are the mass and stiffness matrices corresponding to the interface boundary DOFs. \mathbf{M}_c and \mathbf{K}_c are the coupling matrices between the substructural and interface boundary DOFs.

From Equations 3.1 and C.2, the global displacement vector \mathbf{u}_g can be represented as

$$\mathbf{u}_{g} = \begin{bmatrix} \mathbf{u}_{s} \\ \mathbf{u}_{b} \end{bmatrix} = \mathbf{T}_{CB} \begin{bmatrix} \mathbf{q}_{s} \\ \mathbf{u}_{b} \end{bmatrix}, \quad \mathbf{T}_{CB} = \begin{bmatrix} \mathbf{\Phi}_{s} & -\mathbf{K}_{s}^{-1}\mathbf{K}_{c} \\ \mathbf{0} & \mathbf{I}_{b} \end{bmatrix}, \quad (3.8)$$

where \mathbf{q}_s is the generalized coordinate vector for the substructural modes, and \mathbf{T}_{CB} is the transformation matrix of the CB method. $\mathbf{\Phi}_s$ is a block diagonal eigenvector matrix calculated by the following substructural eigenvalue problems

$$[\mathbf{K}_{s}^{(k)} - \lambda_{j}^{(k)}\mathbf{M}_{s}^{(k)}](\boldsymbol{\phi}^{(k)})_{j} = \mathbf{0}, \quad j = 1, 2, ..., N_{q}^{(k)}, \text{ for } k = 1, 2, ..., N_{s},$$
(3.9)

where $N_q^{(k)}$ is the number of deformable modes in the k^{th} substructure. $\lambda_j^{(k)}$ and $(\phi^{(k)})_j$ are the eigenvalue and the corresponding eigenvector of the k^{th} substructure, respectively. Φ_s is a $N_q \times N_q$ matrix $(N_q = N_q^{(1)} + N_q^{(2)} + ... + N_q^{(N_s)})$.

Premultiplying \mathbf{T}_{CB}^{T} in Equation C.2, we can obtain the equations of motion for the partitioned structure

$$\mathbf{M}_{p}\ddot{\mathbf{u}}_{p} + \mathbf{K}_{p}\mathbf{u}_{p} = \mathbf{f}_{p},$$
$$\mathbf{M}_{p} = \begin{bmatrix} \mathbf{I}_{s} & \hat{\mathbf{M}}_{c} \\ \hat{\mathbf{M}}_{c}^{T} & \hat{\mathbf{M}}_{b} \end{bmatrix}, \ \mathbf{K}_{p} = \begin{bmatrix} \mathbf{\Lambda}_{s} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{K}}_{b} \end{bmatrix}, \ \mathbf{u}_{p} = \begin{bmatrix} \mathbf{q}_{s} \\ \mathbf{u}_{b} \end{bmatrix}, \ \mathbf{f}_{p} = \mathbf{T}_{CB}^{T} \begin{bmatrix} \mathbf{f}_{s} \\ \mathbf{f}_{b} \end{bmatrix}.$$
(3.10)

The component matrices in Equation 3.10 are defined by

$$\mathbf{I}_s = \mathbf{\Phi}_s^T \mathbf{M}_s \mathbf{\Phi}_s, \tag{3.11a}$$

$$\hat{\mathbf{M}}_c = \mathbf{\Phi}_s^T [\mathbf{M}_c - \mathbf{M}_s \mathbf{K}_s^{-1} \mathbf{K}_c], \qquad (3.11b)$$

$$\hat{\mathbf{M}}_{b} = \mathbf{M}_{b} + \mathbf{K}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} - \mathbf{M}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} - \mathbf{K}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{M}_{c}, \qquad (3.11c)$$

$$\mathbf{\Lambda}_s = \mathbf{\Phi}_s^T \mathbf{K}_s \mathbf{\Phi}_s, \tag{3.11d}$$

$$\hat{\mathbf{K}}_b = \mathbf{K}_b - \mathbf{K}_c^T \mathbf{K}_s^{-1} \mathbf{K}_c.$$
(3.11e)

Note that Equation 3.10 retains all the substructural modes. Here, the substructural displacement vector \mathbf{u}_s can be decomposed as the dominant and residual modes

$$\mathbf{u}_{s} = \mathbf{\Phi}_{s}\mathbf{q}_{s} - \mathbf{K}_{s}^{-1}\mathbf{K}_{c}\mathbf{u}_{b} = \begin{bmatrix} \mathbf{\Phi}_{d} & \mathbf{\Phi}_{r} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{d} \\ \mathbf{q}_{r} \end{bmatrix} - \mathbf{K}_{s}^{-1}\mathbf{K}_{c}\mathbf{u}_{b}, \qquad (3.12)$$

where subscripts d and r denote the dominant and residual terms, respectively. Φ_d and Φ_r are the dominant and residual substructural eigenvector matrices with N_d and N_r substructural modes, respectively, and \mathbf{q}_d and \mathbf{q}_r are the corresponding generalized coordinate vectors. N_d and N_r are the numbers of the dominant and residual modes, respectively.

To construct the reduced model from Equation C.2, the residual modes are truncated and then \mathbf{T}_{CB} and \mathbf{u}_g are approximated with only dominant substructural modes

$$\mathbf{u}_{g} = \begin{bmatrix} \mathbf{u}_{s} \\ \mathbf{u}_{b} \end{bmatrix} \approx \bar{\mathbf{u}}_{g} = \bar{\mathbf{T}}_{CB} \begin{bmatrix} \mathbf{q}_{d} \\ \mathbf{u}_{b} \end{bmatrix}, \quad \bar{\mathbf{T}}_{CB} = \begin{bmatrix} \mathbf{\Phi}_{d} & -\mathbf{K}_{s}^{-1}\mathbf{K}_{c} \\ \mathbf{0} & \mathbf{I}_{b} \end{bmatrix}.$$
(3.13)

Premultiplying $\bar{\mathbf{T}}_{CB}^T$ instead of \mathbf{T}_{CB}^T in Equation C.2, we can obtain the approximated equations of motion for the partitioned structure

$$\bar{\mathbf{M}}_{p}\ddot{\mathbf{u}}_{p} + \bar{\mathbf{K}}_{p}\bar{\mathbf{u}}_{p} = \bar{\mathbf{f}}_{p},$$

$$\bar{\mathbf{M}}_{p} = \begin{bmatrix} \mathbf{I}_{d} & \bar{\mathbf{M}}_{c} \\ \bar{\mathbf{M}}_{c}^{T} & \hat{\mathbf{M}}_{b} \end{bmatrix}, \ \bar{\mathbf{K}}_{p} = \begin{bmatrix} \mathbf{\Lambda}_{d} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{K}}_{b} \end{bmatrix}, \ \bar{\mathbf{u}}_{p} = \begin{bmatrix} \mathbf{q}_{d} \\ \mathbf{u}_{b} \end{bmatrix}, \ \bar{\mathbf{f}}_{p} = \bar{\mathbf{T}}_{CB}^{T} \begin{bmatrix} \mathbf{f}_{s} \\ \mathbf{f}_{b} \end{bmatrix},$$
(3.14)

in which the component matrices are defined by

$$\mathbf{I}_d = \mathbf{\Phi}_d^T \mathbf{M}_s \mathbf{\Phi}_d, \tag{3.15a}$$

$$\bar{\mathbf{M}}_c = \mathbf{\Phi}_d^T [\mathbf{M}_c - \mathbf{M}_s \mathbf{K}_s^{-1} \mathbf{K}_c], \qquad (3.15b)$$

$$\mathbf{\Lambda}_d = \mathbf{\Phi}_d^T \mathbf{K}_s \mathbf{\Phi}_d. \tag{3.15c}$$

Using Equation 3.14 with the free vibration condition ($\mathbf{f}_s = \mathbf{f}_b = 0$), the final reduced eigenvalue problem by the CB method can be obtained

$$\bar{\mathbf{K}}_{p}\bar{\mathbf{\Phi}}_{p} = \bar{\lambda}\bar{\mathbf{M}}_{p}\bar{\mathbf{\Phi}}_{p} \quad \text{with } \bar{\mathbf{u}}_{p} = \bar{\mathbf{\Phi}}_{p}\bar{\mathbf{q}}_{p},$$
$$\bar{\mathbf{\Phi}}_{p} = \begin{bmatrix} \bar{\mathbf{\Phi}}_{q_{d}}^{T} & \bar{\mathbf{\Phi}}_{u_{b}}^{T} \end{bmatrix}^{T}, \qquad (3.16)$$

where $\bar{\Phi}_p$ can be separated into the substructural part $\bar{\Phi}_{q_d}$ and the interface boundary part $\bar{\Phi}_{u_b}$. From Equations 3.13 to C.4, the global displacement vector \mathbf{u}_g can be approximated as

$$\mathbf{u}_g \approx \bar{\mathbf{u}}_g = \bar{\mathbf{\Phi}}_g \bar{\mathbf{q}}_p = \bar{\mathbf{T}}_{CB} \bar{\mathbf{u}}_p = \bar{\mathbf{T}}_{CB} \bar{\mathbf{\Phi}}_p \bar{\mathbf{q}}_p, \qquad (3.17)$$

and then, the approximated global eigenvector matrix $\bar{\mathbf{\Phi}}_g$ is rewritten as

$$\Phi_g \approx \bar{\Phi}_g = \begin{bmatrix} \Phi_d \bar{\Phi}_{q_d} - \mathbf{K}_s^{-1} \mathbf{K}_c \bar{\Phi}_{u_b} \\ \bar{\Phi}_{u_b} \end{bmatrix}.$$
(3.18)

3.3 Flexibility-based component mode synthesis (F-CMS)

Recently, the F-CMS method has been developed using the localized Lagrange multipliers λ_{ℓ} [28, 35]. The derivation of the F-CMS method starts from the displacement-based discrete energy functional $\Pi(\mathbf{u}_g)$

$$\Pi(\mathbf{u}_g) = \mathbf{u}_g^T \left(\frac{1}{2} \mathbf{K}_g \mathbf{u}_g - \mathbf{f}_g + \mathbf{M}_g \ddot{\mathbf{u}}_g \right),$$
$$\mathbf{M}_g = \mathbf{L}^T \mathbf{M}_s \mathbf{L}, \mathbf{K}_g = \mathbf{L}^T \mathbf{K}_s \mathbf{L},$$
(3.19)

where the global (non-partitioned) structure and partitioned substructures are related by an assembly Boolean matrix **L**. $\delta \Pi(\mathbf{u}_g)$ yields the linear dynamics equations of the global structure in Equation 3.1.

Here, $\Pi(\mathbf{u}_g)$ can be redefined as the sum of substructural energy functionals and the constraint energy functionals

$$\Pi(\mathbf{u}_s, \boldsymbol{\lambda}_{\boldsymbol{\ell}}, \mathbf{u}_b) = \mathbf{u}_s^T \left(\frac{1}{2} \mathbf{K}_s \mathbf{u}_s - \mathbf{f}_s + \mathbf{M}_s \ddot{\mathbf{u}}_s \right) + \boldsymbol{\lambda}_{\boldsymbol{\ell}}^T \left(\mathbf{B}^T \mathbf{u}_s - \mathbf{L}_b \mathbf{u}_b \right),$$
(3.20)

where **B** denotes the interface Boolean matrix and \mathbf{L}_b is obtained by yanking out the rows with zero entries of $\mathbf{B}^T \mathbf{L}$ (see, e.g., [35]).

Then, the partitioned displacement vector \mathbf{u}_s is decomposed by deformable and rigid body modes

$$\mathbf{u}_{s} = \begin{bmatrix} \Phi_{s} & \mathbf{R}_{s} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{s} \\ \alpha_{s} \end{bmatrix}, \qquad (3.21)$$

where Φ_s and \mathbf{R}_s are the eigenvector matrices of the deformable and rigid body modes, respectively. \mathbf{q}_s and $\boldsymbol{\alpha}_s$ are the corresponding generalized coordinate vectors. Note that, in the CB method, \mathbf{u}_s has only the deformable modes Φ_s because of the fixed interface boundary between substructures. However, in the F-CMS method using the free interface boundary, \mathbf{u}_s has not only the deformable modes Φ_s but also rigid body modes \mathbf{R}_s . In Equation 3.21, Φ_s and \mathbf{R}_s are calculated by the following substructural eigenvalue problems

$$[\mathbf{K}_{s}^{(k)} - \lambda_{j}^{(k)}\mathbf{M}_{s}^{(k)}](\boldsymbol{\phi}^{(k)})_{j} = \mathbf{0}, \quad j = 1, 2, ..., N_{q}^{(k)} + N_{\alpha}^{(k)}, \text{ for } k = 1, 2, ..., N_{s},$$
(3.22)

where $N_{\alpha}^{(k)}$ is the number of rigid body modes of the k^{th} substructure.

Then, $\Pi(\mathbf{u}_s, \boldsymbol{\lambda}_{\ell}, \mathbf{u}_b)$ can be expressed as the energy functional of four variables $\Pi(\mathbf{q}_s, \boldsymbol{\alpha}_s, \boldsymbol{\lambda}_{\ell}, \mathbf{u}_b)$, and $\delta \Pi(\mathbf{q}_s, \boldsymbol{\alpha}_s, \boldsymbol{\lambda}_{\ell}, \mathbf{u}_b)$ yields the following equations as

$$\begin{bmatrix} \mathbf{M}_{p} \frac{d^{2}}{dt^{2}} + \mathbf{K}_{p} \end{bmatrix} \mathbf{u}_{p} = \mathbf{f}_{p},$$

$$\mathbf{M}_{p} \frac{d^{2}}{dt^{2}} + \mathbf{K}_{p} = \begin{bmatrix} \mathbf{\Lambda}_{s} + \mathbf{I}_{q} \frac{d^{2}}{dt^{2}} & \mathbf{0} & \mathbf{\Phi}_{s}^{T} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\alpha} \frac{d^{2}}{dt^{2}} & \mathbf{R}_{b}^{T} & \mathbf{0} \\ \mathbf{B}^{T} \mathbf{\Phi}_{s} & \mathbf{R}_{b} & \mathbf{0} & -\mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u}_{p} = \begin{bmatrix} \mathbf{q}_{s} \\ \mathbf{\alpha}_{s} \\ \mathbf{\lambda}_{\ell} \\ \mathbf{u}_{b} \end{bmatrix}, \quad \mathbf{f}_{p} = \begin{bmatrix} \mathbf{\Phi}_{s}^{T} \mathbf{f}_{s} \\ \mathbf{R}_{s}^{T} \mathbf{f}_{s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{R}_{b} = \mathbf{B}^{T} \mathbf{R}_{s}, \quad \mathbf{I}_{q} = \mathbf{\Phi}_{s}^{T} \mathbf{M}_{s} \mathbf{\Phi}_{s}, \quad \mathbf{I}_{\alpha} = \mathbf{R}_{s}^{T} \mathbf{M}_{s} \mathbf{R}_{s}, \quad (3.23)$$

where t denotes the time variable. Note that Equation 3.23 contains all the rigid body modes \mathbf{R}_s and deformable modes $\mathbf{\Phi}_s$ as in Equation 3.10 in the CB method.

To construct an effective reduced model, only dominant substructural modes are retained from Equation 3.23. We here truncate the residual deformable modes from \mathbf{q}_s only, and all the other modes related with the rigid body motions $\boldsymbol{\alpha}_s$, the localized Lagrange multiplier vector $\boldsymbol{\lambda}_{\ell}$, and the interface DOFs \mathbf{u}_b are retained. Then, the eigenvector matrix $\boldsymbol{\Phi}_s$ of the deformable substructural modes can be divided into the dominant part $\boldsymbol{\Phi}_d$ and the residual part $\boldsymbol{\Phi}_r$ as in Equation 3.12 in the same way as the CB method.

Substituting Equation 3.12 into Equation 3.23 and performing the Gauss elimination on \mathbf{q}_r , the condensed equations are obtained

$$\begin{pmatrix}
\hat{\mathbf{\Lambda}}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{\alpha} \frac{d^{2}}{dt^{2}} & \mathbf{R}_{b}^{T} & \mathbf{0} \\
\mathbf{\Phi}_{db} & \mathbf{R}_{b} & -\hat{\mathbf{F}}_{rb} & -\mathbf{L}_{b} \\
\mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} & \mathbf{0}
\end{pmatrix}
\begin{vmatrix}
\mathbf{q}_{d} \\
\boldsymbol{\alpha}_{s} \\
\boldsymbol{\lambda}_{\ell} \\
\mathbf{u}_{b}
\end{vmatrix} = \begin{vmatrix}
\mathbf{\Phi}_{d}^{T} \mathbf{f}_{s} \\
\mathbf{R}_{s}^{T} \mathbf{f}_{s} \\
-\mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f}_{s} \\
\mathbf{0}
\end{vmatrix},$$

$$\hat{\mathbf{\Lambda}}_{d} = \mathbf{\Lambda}_{d} + \frac{d^{2}}{dt^{2}} \mathbf{I}_{d}, \, \hat{\mathbf{F}}_{r} = \mathbf{\Phi}_{r} \left[\mathbf{\Lambda}_{r} + \frac{d^{2}}{dt^{2}} \mathbf{I}_{r}\right]^{-1} \mathbf{\Phi}_{r}^{T}, \, \hat{\mathbf{F}}_{rb} = \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{B}, \, \mathbf{\Phi}_{db} = \mathbf{B}^{T} \mathbf{\Phi}_{d}, \quad (3.24)$$

in which we note that $\hat{\mathbf{F}}_r$ and $\hat{\mathbf{F}}_{rb}$ are the dynamic residual flexibilities of the full model and the interface boundary, respectively, and Φ_{db} is the eigenvector matrix of the dominant deformable modes in the interface boundary. We then invoke the harmonic response $(d^2/dt^2 = -\omega^2)$ with the natural frequency ω . Then, $\hat{\mathbf{F}}_r$ and $\hat{\mathbf{F}}_{rb}$ are expressed as functions of ω^2 , and these yield an intractable eigenvalue problem. To treat the transcendental nature of $\hat{\mathbf{F}}_{rb}$, we approximate $\hat{\mathbf{F}}_{rb}$ as

$$\hat{\mathbf{F}}_{rb} = \mathbf{\Phi}_{rb} \left[\mathbf{\Lambda}_r - \omega^2 \mathbf{I}_r \right]^{-1} \mathbf{\Phi}_{rb}^T$$

$$\approx \mathbf{\Phi}_{rb} \mathbf{\Lambda}_r^{-1} \mathbf{\Phi}_{rb}^T + \omega^2 \mathbf{\Phi}_{rb} \mathbf{\Lambda}_r^{-2} \mathbf{\Phi}_{rb}^T = \mathbf{F}_{rbs} + \omega^2 \mathbf{F}_{rbm}, \qquad (3.25)$$

where Φ_{rb} is defined as $\mathbf{B}^T \Phi_r$. Note that \mathbf{F}_{rbs} and \mathbf{F}_{rbm} are the static and dynamic parts of the interface residual flexibilities (see, e.g., [28]). Those residual terms are calculated using the dominant substructural modes as

$$\mathbf{F}_{rbs} = \mathbf{B}^T \mathbf{M}_s^{-1/2} \left[\mathbf{M}_s^{-1/2} \mathbf{K}_s \mathbf{M}_s^{-1/2} \right]^+ \mathbf{M}_s^{-1/2} \mathbf{B} - \mathbf{\Phi}_{db} \mathbf{\Lambda}_d^{-1} \mathbf{\Phi}_{db}^T,$$
(3.26a)

$$\mathbf{F}_{rbm} = \mathbf{B}^T \mathbf{M}_s^{-1/2} \left[\mathbf{M}_s^{-1/2} \mathbf{K}_s \mathbf{M}_s^{-1/2} \right]^{+2} \mathbf{M}_s^{-1/2} \mathbf{B} - \mathbf{\Phi}_{db} \mathbf{\Lambda}_d^{-2} \mathbf{\Phi}_{db}^T, \qquad (3.26b)$$

where the stiffness matrix \mathbf{K}_s is not invertible because of the substructural rigid body modes. Hence, we can use the pseudo-inverse denoted by the superscript +.

Substituting Equations 6.6 and 6.8 in Equation 3.24, we obtain the approximated equations of motion for the partitioned structure

$$\begin{bmatrix} -\bar{\mathbf{M}}_{p} \ \omega^{2} + \bar{\mathbf{K}}_{p} \end{bmatrix} \bar{\mathbf{u}}_{p} = \bar{\mathbf{f}}_{p},$$

$$-\bar{\mathbf{M}}_{p} \ \omega^{2} + \bar{\mathbf{K}}_{p} = \begin{bmatrix} \mathbf{\Lambda}_{d} - \omega^{2} \mathbf{I}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} & \mathbf{0} \\ \mathbf{0} & -\omega^{2} \mathbf{I}_{\alpha} & \mathbf{R}_{b}^{T} & \mathbf{0} \\ \mathbf{\Phi}_{db} & \mathbf{R}_{b} & -\mathbf{F}_{rbs} - \omega^{2} \mathbf{F}_{rbm} & -\mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} & \mathbf{0} \end{bmatrix},$$

$$\bar{\mathbf{u}}_{p} = \begin{bmatrix} \mathbf{q}_{d}^{T} \ \boldsymbol{\alpha}_{s}^{T} \ \boldsymbol{\lambda}_{\ell}^{T} \ \mathbf{u}_{b}^{T} \end{bmatrix}^{T}, \ \bar{\mathbf{f}}_{p} = \begin{bmatrix} [\mathbf{\Phi}_{d}^{T} \mathbf{f}_{s}]^{T} & [\mathbf{R}_{s}^{T} \mathbf{f}_{s}]^{T} & -[\mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f}_{s}]^{T} & \mathbf{0}^{T} \end{bmatrix}^{T}.$$
(3.27)

To construct the reduced eigenvalue problem from Equation 3.27, we consider the free vibration $(\mathbf{f} = \mathbf{0})$ and rearrange the matrices. Then, the final reduced eigenvalue problem of the F-CMS method becomes

$$\bar{\mathbf{K}}_{p} \bar{\mathbf{\Phi}}_{p} = \bar{\lambda} \bar{\mathbf{M}}_{p} \bar{\mathbf{\Phi}}_{p} \quad \text{with } \bar{\mathbf{u}}_{p} = \bar{\mathbf{\Phi}}_{p} \bar{\mathbf{q}}_{p},$$

$$\bar{\mathbf{K}}_{p} = \begin{bmatrix} \mathbf{\Lambda}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{b}^{T} & \mathbf{0} \\ \mathbf{\Phi}_{db} & \mathbf{R}_{b} & -\mathbf{F}_{rbs} & -\mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} & \mathbf{0} \end{bmatrix}, \quad \bar{\mathbf{M}}_{p} = \begin{bmatrix} \mathbf{I}_{d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{rbm} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \bar{\mathbf{\Phi}}_{p} = \begin{bmatrix} \bar{\mathbf{\Phi}}_{qd} \\ \bar{\mathbf{\Phi}}_{\alpha_{s}} \\ \bar{\mathbf{\Phi}}_{\lambda \ell} \\ \bar{\mathbf{\Phi}}_{u_{b}} \end{bmatrix}. \quad (3.28)$$

From the $3^{\rm rd}$ row in Equation 3.27, the global displacement vector \mathbf{u}_g can be approximated as

$$\mathbf{u}_g \approx \bar{\mathbf{u}}_g = \mathbf{L}\mathbf{u}_b = \mathbf{\Phi}_d \mathbf{q}_d + \mathbf{R}_s \boldsymbol{\alpha}_s - \hat{\mathbf{F}}_r \mathbf{B} \boldsymbol{\lambda}_\ell.$$
(3.29)

Similarly, the global eigenvector matrix Φ_g can be also approximated by $\bar{\Phi}_g$ from the 3rd row in the final reduced eigenvalue problem in Equation 3.28

$$\Phi_g \approx \bar{\Phi}_g = \mathbf{L}\bar{\Phi}_{u_b} = \Phi_d\bar{\Phi}_{q_d} + \mathbf{R}_s\bar{\Phi}_{\alpha_s} - \hat{\mathbf{F}}_r\mathbf{B}\bar{\Phi}_{\lambda_\ell}.$$
(3.30)

The F-CMS method is a flexibility-based method with residual flexibility compensation. The F-CMS method employs the free substructural modes, and then the residual modes that are not retained on a

substructural level are transformed into the dynamic residual flexibility $\hat{\mathbf{F}}_r$. This feature can be seen from the fact that $\hat{\mathbf{F}}_r$ in Equation 3.24 is not truncated but approximated as shown in Equation 6.6. In contrast, in the CB method, the substructural residual modes are truncated without any compensation. Therefore, the F-CMS method can give more accurate eigensolutions than the CB method in general.



Chapter 4. Enhanced CB method

Recently, the concept of residual flexibility has been widely used in the partitioned method [34] with free interface boundary condition, and then several CMS methods also developed in the 2000s [28, 29]. In our previous study [36], the conceptual idea to consider the effect of residual substructural modes was also applied to construct the enhanced transformation matrix used for the error estimation of the CB method.

At this point, it is natural to use the enhanced transformation matrix to improve the accuracy of the original CB method. However, the enhanced transformation matrix contains an unknown eigenvalue and thus it is not possible to use the transformation matrix in its present form for the improvement of the original CB method.

In order to overcome this difficulty, we borrowed O'callahan's idea, which was originally proposed to improve Guyan reduction [7]. That is, the unknown eigenvalue is approximated using O'callahan's approach. As a result, a new enhanced transformation matrix is obtained and, using it, an enhanced CB method are developed. Compared to the original CB method, the enhanced CB method can provide significantly improved reduced-order models with a low additional computational cost.



4.1 Formulation details

In the original CB method, to construct the reduced transformation matrix $\mathbf{\bar{T}}_{CB}$ in Equation 3.13, the residual substructural modes are simply truncated without any consideration. However, when the residual substructural modes are properly considered, the transformation matrix can be constructed more accurately.

Using Equation 3.12 in Equation C.3, \mathbf{u}_g can be rewritten

г

٦

$$\mathbf{u}_{g} = \begin{bmatrix} \mathbf{u}_{s} \\ \mathbf{u}_{b} \end{bmatrix} = \mathbf{T}_{CB} \begin{bmatrix} \mathbf{q}_{d} \\ \mathbf{q}_{r} \\ \mathbf{u}_{b} \end{bmatrix}, \quad \mathbf{T}_{CB} = \begin{bmatrix} \mathbf{\Phi}_{d} & \mathbf{\Phi}_{r} & -\mathbf{K}_{s}^{-1}\mathbf{K}_{c} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{b} \end{bmatrix}.$$
(4.1)

Using Equation 4.1 in Equation C.2, we obtain the equations of motion for the partitioned structure

$$\left[\frac{d^2}{dt^2}\mathbf{M}_p + \mathbf{K}_p\right]\mathbf{u}_p = \mathbf{f}_p,\tag{4.2a}$$

$$\mathbf{M}_{p} = \mathbf{T}_{CB}^{T} \mathbf{M}_{g} \mathbf{T}_{CB}, \ \mathbf{K}_{p} = \mathbf{T}_{CB}^{T} \mathbf{K}_{g} \mathbf{T}_{CB},$$
(4.2b)

$$\frac{d^2}{dt^2}\mathbf{M}_p + \mathbf{K}_p = \begin{bmatrix} \mathbf{\Lambda}_d & \mathbf{0} & \frac{a}{dt^2}\mathbf{M}_c \\ \mathbf{0}^T & \hat{\mathbf{\Lambda}}_r & \frac{d^2}{dt^2}\mathbf{D} \\ \frac{d^2}{dt^2}\bar{\mathbf{M}}_c^T & \frac{d^2}{dt^2}\mathbf{D}^T & \hat{\mathbf{K}}_b + \frac{d^2}{dt^2}\hat{\mathbf{M}}_b \end{bmatrix},$$
(4.2c)

$$\mathbf{u}_{p} = \begin{bmatrix} \mathbf{q}_{d} \\ \mathbf{q}_{r} \\ \mathbf{u}_{b} \end{bmatrix}, \ \mathbf{f}_{p} = \mathbf{T}_{CB}^{T} \begin{bmatrix} \mathbf{f}_{s} \\ \mathbf{f}_{b} \end{bmatrix},$$
(4.2d)

where the component matrices are defined by

$$\hat{\mathbf{\Lambda}}_d = \mathbf{\Lambda}_d + \frac{d^2}{dt^2} \mathbf{I}_d, \ \mathbf{\Lambda}_d = \mathbf{\Phi}_d^T \mathbf{K}_s \mathbf{\Phi}_d, \ \mathbf{I}_d = \mathbf{\Phi}_d^T \mathbf{M}_s \mathbf{\Phi}_d,$$
(4.3a)

$$\mathbf{M}_{c} = \mathbf{\Phi}_{d}^{T} [\mathbf{M}_{c} - \mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c}], \qquad (4.3b)$$

$$\hat{\mathbf{\Lambda}}_{r} = \mathbf{\Lambda}_{r} + \frac{d^{2}}{dt^{2}} \mathbf{I}_{r}, \ \mathbf{\Lambda}_{r} = \mathbf{\Phi}_{r}^{T} \mathbf{K}_{s} \mathbf{\Phi}_{r}, \ \mathbf{I}_{r} = \mathbf{\Phi}_{r}^{T} \mathbf{M}_{s} \mathbf{\Phi}_{r},$$
(4.3c)

$$\mathbf{D} = \mathbf{\Phi}_r^T \left[-\mathbf{M}_s \mathbf{K}_s^{-1} \mathbf{K}_c + \mathbf{M}_c \right], \qquad (4.3d)$$

$$\hat{\mathbf{M}}_{b} = \mathbf{M}_{b} + \mathbf{K}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} - \mathbf{M}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} - \mathbf{K}_{c}^{T} \mathbf{K}_{s}^{-1} \mathbf{M}_{c},$$
(4.3e)

$$\hat{\mathbf{K}}_b = \mathbf{K}_b - \mathbf{K}_c^T \mathbf{K}_s^{-1} \mathbf{K}_c.$$
(4.3f)

Note that Equation 4.2 is the original equations of motion that contain all the substructural modes. Using the second row in Equation 4.2a with $\mathbf{f}_p = \mathbf{0}$, we obtain

$$\mathbf{q}_r = -\hat{\mathbf{\Lambda}}_r^{-1} \left[\frac{d^2}{dt^2} \mathbf{D} \mathbf{u}_b \right].$$
(4.4)

Substituting Equation 4.4 into Equation 4.1, \mathbf{u}_s can be represented by

$$\mathbf{u}_{s} = \mathbf{\Phi}_{d}\mathbf{q}_{d} - \mathbf{K}_{s}^{-1}\mathbf{K}_{c}\mathbf{u}_{b} - \frac{d^{2}}{dt^{2}}\hat{\mathbf{F}}_{r}\left[-\mathbf{M}_{s}\mathbf{K}_{s}^{-1}\mathbf{K}_{c} + \mathbf{M}_{c}\right]\mathbf{u}_{b},\tag{4.5}$$

with

$$\hat{\mathbf{F}}_{r} = \mathbf{\Phi}_{r} \hat{\mathbf{\Lambda}}_{r}^{-1} \mathbf{\Phi}_{r}^{T} = \mathbf{\Phi}_{r} \left[\mathbf{\Lambda}_{r} + \frac{d^{2}}{dt^{2}} \mathbf{I}_{r} \right]^{-1} \mathbf{\Phi}_{r}^{T},$$
(4.6)

where $\hat{\mathbf{F}}_r$ represents the residual flexibility of the substructures.

We here invoke harmonic response $(d^2/dt^2 = -\lambda)$ and then $\hat{\mathbf{F}}_r$ can be approximated as

$$\hat{\mathbf{F}}_{r} = \boldsymbol{\Phi}_{r} \left[\boldsymbol{\Lambda}_{r} - \lambda \mathbf{I}_{r} \right]^{-1} \boldsymbol{\Phi}_{r}^{T} \approx \boldsymbol{\Phi}_{r} \boldsymbol{\Lambda}_{r}^{-1} \boldsymbol{\Phi}_{r}^{T} + \lambda \boldsymbol{\Phi}_{r} \boldsymbol{\Lambda}_{r}^{-2} \boldsymbol{\Phi}_{r}^{T} = \mathbf{F}_{rs} + \lambda \mathbf{F}_{rm},$$
(4.7)

where \mathbf{F}_{rs} and \mathbf{F}_{rm} are the static and dynamic parts of the residual flexibility, respectively.

Using Equation 4.7 in Equation 4.5 and truncating terms higher than order of λ , \mathbf{u}_s can be approximated

$$\mathbf{u}_{s} \approx \bar{\mathbf{u}}_{s} = \mathbf{\Phi}_{d}\mathbf{q}_{d} - \mathbf{K}_{s}^{-1}\mathbf{K}_{c}\mathbf{u}_{b} + \lambda\mathbf{F}_{rs}\left[-\mathbf{M}_{s}\mathbf{K}_{s}^{-1}\mathbf{K}_{c} + \mathbf{M}_{c}\right]\mathbf{u}_{b},\tag{4.8}$$

in which \mathbf{F}_{rs} is indirectly calculated by subtracting the dominant flexibility matrix from the full flexibility matrix as

$$\mathbf{F}_{rs} = \mathbf{K}_s^{-1} - \mathbf{\Phi}_d \mathbf{\Lambda}_d^{-1} \mathbf{\Phi}_d^T.$$
(4.9)

Using $\bar{\mathbf{u}}_s$ defined in Equation 4.8 instead of \mathbf{u}_s in Equation 3.13, we finally obtain

$$\mathbf{u}_g \approx \bar{\mathbf{u}}_g = \bar{\mathbf{T}}_{ECB} \bar{\mathbf{u}}_p, \quad \bar{\mathbf{T}}_{ECB} = \bar{\mathbf{T}}_{CB} + \bar{\mathbf{T}}_r, \tag{4.10}$$

with

$$\bar{\mathbf{T}}_{r} = \begin{bmatrix} \mathbf{0} & \lambda \mathbf{F}_{rs} \left[-\mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} + \mathbf{M}_{c} \right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
(4.11)

where $\bar{\mathbf{T}}_{ECB}$ is the transformation matrix enhanced by $\bar{\mathbf{T}}_r$. Note that, since $\bar{\mathbf{T}}_r$ contains the eigenvalue λ , it is regarded as a transformation matrix related with dynamic effect.

Here, a difficulty arises. Since the eigenvalue λ in $\bar{\mathbf{T}}_r$ is unknown, the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ cannot be used to improve the original CB method in its present form. To handle the unknown eigenvalue λ in $\bar{\mathbf{T}}_r$, we employ O'callahan's approach, which was proposed to improve Guyan reduction [7]. From Equation 3.14 with $\bar{\mathbf{f}}_p = \mathbf{0}$, the following relation is obtained

$$\lambda \bar{\mathbf{u}}_p = \bar{\mathbf{M}}_p^{-1} \bar{\mathbf{K}}_p \bar{\mathbf{u}}_p, \tag{4.12}$$

and, using Equation 4.12 in Equation 4.10, $\bar{\mathbf{T}}_r$ is newly redefined by

$$\bar{\mathbf{\Gamma}}_{r} = \begin{bmatrix} \mathbf{0} & \mathbf{F}_{rs} \left[-\mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} + \mathbf{M}_{c} \right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \bar{\mathbf{M}}_{p}^{-1} \bar{\mathbf{K}}_{p}.$$
(4.13)

Using the redefined $\bar{\mathbf{T}}_r$ in Equation 4.10, $\bar{\mathbf{T}}_{ECB}$ can be expressed without the unknown eigenvalue λ . Using the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ redefined by $\bar{\mathbf{T}}_r$ in Equation 4.10, the new reduced mass and stiffness matrices denoted by tilde (~) are defined

$$\tilde{\mathbf{M}}_{p} = \bar{\mathbf{T}}_{ECB}^{T} \mathbf{M}_{g} \bar{\mathbf{T}}_{ECB} = \bar{\mathbf{M}}_{p} + \bar{\mathbf{T}}_{r}^{T} \mathbf{M}_{g} \bar{\mathbf{T}}_{CB} + \bar{\mathbf{T}}_{CB}^{T} \mathbf{M}_{g} \bar{\mathbf{T}}_{r} + \bar{\mathbf{T}}_{r}^{T} \mathbf{M}_{g} \bar{\mathbf{T}}_{r},$$
(4.14a)

$$\tilde{\mathbf{K}}_{p} = \bar{\mathbf{T}}_{ECB}^{T} \mathbf{K}_{g} \bar{\mathbf{T}}_{ECB} = \bar{\mathbf{K}}_{p} + \bar{\mathbf{T}}_{r}^{T} \mathbf{K}_{g} \bar{\mathbf{T}}_{CB} + \bar{\mathbf{T}}_{CB}^{T} \mathbf{K}_{g} \bar{\mathbf{T}}_{r} + \bar{\mathbf{T}}_{r}^{T} \mathbf{K}_{g} \bar{\mathbf{T}}_{r}.$$
(4.14b)

Due to the compensation of the residual substructural modes in $\mathbf{\bar{T}}_{ECB}$, the reduced mass and stiffness matrices in Equation 4.14 are more precisely constructed than the original reduced matrices in Equation C.4. For this reason, using the newly defined $\mathbf{\tilde{M}}_p$ and $\mathbf{\tilde{K}}_p$, the solution accuracy of the reduced eigenvalue problem can be improved.

Table 4.1 shows the comparison of the original and enhanced CB methods. It is important to note that the original and enhanced CB methods produces the same size of reduced models. Compared to the original CB method, the residual flexibility \mathbf{F}_{rs} and the inverse of the reduced mass matrix $\bar{\mathbf{M}}_p^{-1}$ are additionally computed to construct the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ in the enhanced CB method.

However, \mathbf{F}_{rs} can simply calculated by reusing \mathbf{K}_s^{-1} and the dominant substructural eigensolutions, see Equation 4.9. Also, the size of the reduced mass matrix $\bar{\mathbf{M}}_p$ is small because it consists of a small number of dominant substructural modes and interface DOFs. For these reasons, we can easily identify the fact that the additional computational cost of the enhanced CB method is not high.

| | CB | Enhanced CB |
|------------------------------|------------------------|--|
| Transformation matrix | $ar{\mathbf{T}}_{CB}$ | $ar{\mathbf{T}}_{CB}+ar{\mathbf{T}}_r$ |
| Reduced mass matrix | $\bar{\mathbf{M}}_{m}$ | $ar{\mathbf{M}}_p + ar{\mathbf{T}}_r^T \mathbf{M}_g ar{\mathbf{T}}_{CB}$ |
| | 111 <i>p</i> | $+ ar{\mathbf{T}}_{CB}^T \mathbf{M}_g ar{\mathbf{T}}_r + ar{\mathbf{T}}_r^T \mathbf{M}_g ar{\mathbf{T}}_r$ |
| Roduced stiffness matrix | $\bar{\mathbf{k}}$ | $ar{\mathbf{K}}_p + ar{\mathbf{T}}_r^T \mathbf{K}_g ar{\mathbf{T}}_{CB}$ |
| | \mathbf{n}_p | $+ar{\mathbf{T}}_{CB}^T\mathbf{K}_gar{\mathbf{T}}_r+ar{\mathbf{T}}_r^T\mathbf{K}_gar{\mathbf{T}}_r$ |
| Size of the reduced matrices | \bar{N}_p | $ar{N}_p$ |

Table 4.1: Comparison between the CB and enhanced CB methods.

4.2 Numerical examples

In this section, we test the performance of the enhanced CB method compared to the original CB and F-CMS methods. It should be noted that, due to the use of localized Lagrange multipliers [28, 35], the F-CMS method requires more DOFs in reduced models (larger size of reduced matrices) than the original and enhanced CB methods for the same number of retained dominant substructural modes in general.

Four different structural problems are considered: simple plate, hyperboloid shell, stiffened plate and ring solid problems. These are modeled by 4-node MITC shell (see, e.g., Refs. [37, 38, 39, 40]) and 8-node brick elements. We here use the frequency cut-off mode selection method [41] to select the dominant substructural modes.

The following relative eigenvalue error is used to evaluate the performance of the enhanced CB method

$$\xi_i = \frac{\bar{\lambda}_i - \lambda_i}{\lambda_i},\tag{4.15}$$

in which ξ_i denotes the relative eigenvalue error for the i^{th} mode, and λ_i and $\bar{\lambda}_i$ are the exact and approximated eigenvalues, respectively. These eigenvalues are calculated from the global (original) and reduced eigenvalue problems.

4.2.1 Simple plate problem



We consider two numerical cases with 10 and 20 dominant substructural modes selected ($N_d = 10$ and $N_d = 20$). The numbers of retained substructural modes $N_d^{(k)}$ are listed in Table 4.2. Figure 4.2 presents the relative eigenvalue errors obtained by the original CB, F-CMS and enhanced CB methods. The results show that the enhanced CB method outperforms other two methods.



Figure 4.1: Simple plate problem.

| CMS | Case | $N_{d}^{(1)}$ | $N_d^{(2)}$ | N_d |
|--------------------|------|---------------|-------------|-------|
| CP and Enhanced CP | 1 | 7 | 3 | 10 |
| CD and Enhanced CD | 2 | 13 | 7 | 20 |
| ECMC | 1 | 7 | 3 | 10 |
| г-ОМБ | 2 | 13 | 7 | 20 |

Table 4.2: Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem.



Figure 4.2: Relative eigenvalue errors in the simple plate problem. (a) $N_d = 10$, (b) $N_d = 20$.

4.2.2 Hyperboloid shell problem

We here consider a hyperboloid shell structure of height H = 4.0m and thickness h = 0.05m. Young's modulus E is 69GPa, Poisson's ratio ν is 0.35, and density ρ_s is 2700kg/m³. The mid-surface of this shell structure is described by

$$x^{2} + y^{2} = 2 + z^{2}; \ z \in [-2, 2].$$
 (4.16)

We use a mesh of 20 (axial) × 40 (circumferential) MITC4 shell elements, see Figure 4.3. The finite element model of the hyperboloid shell is partitioned into four substructures $(N_s = 4)$.

We use 40 and 80 dominant substructural modes selected for two numerical cases ($N_d = 40$ and $N_d = 80$). The numbers of dominant substructural modes $N_d^{(k)}$ are listed in Table 4.3. Figure 4.4 shows that the enhanced CB method gives much better solution accuracy than other two methods.

| \mathbf{CMS} | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | $N_d^{(3)}$ | $N_d^{(4)}$ | N_d |
|--------------------|------|-------------|-------------|-------------|-------------|-------|
| CP and Enhanced CP | 1 | 17 | 3 | 17 | 3 | 40 |
| OD and Enhanced OD | 2 | 33 | 7 | 33 | 7 | 80 |
| E CMS | 1 | 15 | 5 | 15 | 5 | 40 |
| F-UMS | 2 | 29 | 11 | 29 | 11 | 80 |
| | | 15 | | | | |
| | | | | | | |

Table 4.3: Retained substructural mode numbers $N_d^{(k)}$ in the hyperboloid shell problem.





Figure 4.3: Hyperboloid shell problem.



Figure 4.4: Relative eigenvalue errors in the hyperboloid shell problem. (a) $N_d = 40$, (b) $N_d = 80$.
4.2.3 Stiffened plate problem

We here consider a stiffened plate with free boundary, see Figure 4.5. Length L is 4.8m, width B is 3.2m, and thickness h is 0.03m. The flat plate has two longitudinal and four transverse stiffeners, and height H is 0.5m. Young's modulus E is 210GPa, Poisson's ratio ν is 0.3, and density ρ_s is 7850kg/m³. The bottom plate is modeled by a mesh of 24×16 shell finite elements, and the longitudinal and transverse stiffeners are modeled by meshes of 24 × 2 and 16 × 2 shell finite elements, respectively. This stiffened plate is partitioned into six substructures ($N_s = 6$).

We use 50 and 80 dominant substructural modes selected in two numerical cases ($N_d = 50$ and $N_d = 80$), and the numbers of dominant substructural modes $N_d^{(k)}$ are listed in Table 4.4. The relative eigenvalue errors are plotted in Figure 4.6. The results show the robustness of the enhanced CB method.

| abie 111 1000amea sabbu | acturat | inouo i | iumo on | ' d | 111 0110 50 | | proce i | 100101 |
|-------------------------|---------|-------------|-------------|-------------|-------------|-------------|-------------|--------|
| CMS | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | $N_d^{(3)}$ | $N_d^{(4)}$ | $N_d^{(5)}$ | $N_d^{(6)}$ | N_d |
| CD and Enhanced CD | 1 | 20 | 8 | 3 | 3 | 8 | 8 | 50 |
| CD and Enhanced CD | 2 | 29 | 17 | 6 | 6 | 11 | 11 | 80 |
| E CMS | 1 | 11 | 11 | 7 | 7 | 7 | 7 | 50 |
| F-0M5 | 2 | 18 | 18 | 11 | 11 | 11 | 11 | 80 |

Table 4.4: Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem.







Figure 4.5: Stiffened plate problem.



Figure 4.6: Relative eigenvalue errors in the stiffened plate problem. (a) $N_d = 50$, (b) $N_d = 80$.

4.2.4 Ring solid problem

Let us consider a ring solid problem, see Figure 4.7. Height H is 0.05m, and the radii R_1 and R_2 are 0.13m and 0.1m, respectively. Young's modulus E is 72GPa, Poisson's ratio ν is 0.33, and density ρ_s is 2796kg/m³. The ring solid structure is modeled by a mesh of 40 (circumferential) × 3 (radial) × 5 (axial) brick elements and is partitioned into four identical substructures ($N_s = 4$).

From each substructure, 10 and 20 dominant substructural modes are retained for two numerical cases ($N_d = 10$ and $N_d = 20$). Figure 4.8 consistently demonstrates the excellent performance of the enhanced CB method.

We finally note that, when we obtain Equation 4.8, it is possible to contain terms higher than order of λ and thus the enhanced transformation matrix can be more accurate. However, we could not get meaningful improvement in the solution accuracy with the higher order enhanced transformation matrix.







Figure 4.7: Solid ring problem.



Figure 4.8: Relative eigenvalue errors in the solid ring problem. (a) $N_d = 40$, (b) $N_d = 80$.

4.3 Closure

In this study, we developed a new component mode synthesis (CMS) method by improving the well-known Craig-Bampton (CB) method. Unlike the original CB method, the residual substructural modes are considered to construct the transformation matrix. As a result, the original CB transformation matrix is enhanced by the additional dynamic term, in which the unknown eigenvalue is approximated using O'callahan's approach.

Using the enhanced transformation matrix, global (original) structural models can be more precisely reduced and then the accuracy of reduced models is dramatically improved with a low additional computational cost. The excellent performance of the enhanced CB method was demonstrated through various numerical examples.

This concept of the enhanced CB method also has been employed for enhancing the automated multi-level substructuring (AMLS) method [42]. A challenge for future work is to develop an error estimation method for the enhanced CB method for measuring the reliability of reduced models.



Chapter 5. F-CMS method with interface reduction

In the original formulation of the F-CMS method, internal DOFs are only reduced by dominant substructural modes, see Section 3.3. However, when the solid structure is used in the model reduction, the size of interface DOFs are significant sometimes, and then the size of interface DOFs is a quite big proportion of total size in the reduced models. To solve this problem, Markovic et al. [30] proposed an interface reduction technique in the F-CMS method. In this approach, the interface boundary DOFs can be represented by the dominant modes of adjacent field, and then the number of interface boundary DOFs can be efficiently reduced without loss of precision. Although the proposed method shows good performance, it could be more precisely formulated and has possibilities to improve the solution accuracy. Especially, due to the complexity of the formulations, the approximated global eigenvectors were not defined in the interface boundary reduction level. These are motivations of the present research.

Here, we propose a well formulated three level reduction technique in the F-CMS method (see Figure 5.1). In this formulation, the reduction procedures are sequentially performed internal, localized Lagrange multipliers (LLM), and interface DOFs. In particular, to improve the solution accuracy, we use the static correction to construct the dominant filed of the interface boundary displacement vector, and then the residual dynamic term is reactivated in the final formulation. Due to the modification of the original formulation, the approximated global eigenvector can be defined in every reduction level.

Since reduction procedures of the internal DOFs (Reduction level 1) is already defined in Section 3.3, we here start from the reduction level 2 for the LLM.



Figure 5.1: Reduction procedures. (a) Fully retained model. (b) Level 1: reduction of internal DOFs. (c) Level 2: reduction of localized Lagrange multipliers. (d) Level 3: reduction of interface DOFs

5.1 Reduction of localized Lagrange multiplier

In the reduction level 2, the LLM λ_{ℓ} is reduced. At first, λ_{ℓ} can be decomposed as the dominant term λ_d and the residual term λ_r , and it can be represented as

$$\boldsymbol{\lambda}_{\ell} = \begin{bmatrix} \boldsymbol{\Psi}_{d} & \boldsymbol{\Psi}_{r} \end{bmatrix} \begin{bmatrix} \boldsymbol{\chi}_{d} \\ \boldsymbol{\chi}_{r} \end{bmatrix},$$
$$\boldsymbol{\lambda}_{d} = \boldsymbol{\Psi}_{d} \boldsymbol{\chi}_{d}, \ \boldsymbol{\lambda}_{r} = \boldsymbol{\Psi}_{r} \boldsymbol{\chi}_{r},$$
(5.1)

where Ψ_d and Ψ_r are the mode shapes, and χ_d and χ_r are the generalized coordinates of the dominant and residual terms of the LLM, respectively.

The LLM field is directly coupled with the retained internal mode shapes which are Φ_d and \mathbf{R} at the boundary Γ , see Figure 5.1(d). Therefore, Ψ_d can be constructed as the spaces spanned by Φ_{db} and \mathbf{R}_b , and Ψ_r also can be constructed as the corresponding null space of Ψ_d using the matrix orthogonality as

$$\Psi_{d} = \operatorname{Span}(\Phi_{db}) \cup \operatorname{Span}(\mathbf{R}_{b}), \ \Psi_{r} \perp \Psi_{d}$$
$$\Longrightarrow \Phi_{db}^{T} \Psi_{r} = \mathbf{R}_{b}^{T} \Psi_{r} = \mathbf{0}.$$
(5.2)

Note that Ψ_r is not the physical null space, therefore it should be carefully considered to the convergence problems using F-CMS method with the boundary reduction.

Substituting Equations 5.1 and 5.2 in Equation 3.23, the following equation are obtained

$$\begin{bmatrix} \mathbf{\Lambda}_{d} - \omega^{2} \mathbf{I}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} \mathbf{\Psi}_{d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\omega^{2} \mathbf{I}_{\alpha} & \mathbf{R}_{b}^{T} \mathbf{\Psi}_{d} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Psi}_{d}^{T} \mathbf{\Phi}_{db} & \mathbf{\Psi}_{d}^{T} \mathbf{R}_{b} & -\hat{\mathbf{F}}_{dd} & -\hat{\mathbf{F}}_{dr} & -\mathbf{\Psi}_{d}^{T} \mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{F}}_{rd} & -\hat{\mathbf{F}}_{rr} & -\mathbf{\Psi}_{r}^{T} \mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} \mathbf{\Psi}_{d} & -\mathbf{L}_{b}^{T} \mathbf{\Psi}_{r} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{d} \\ \boldsymbol{\alpha} \\ \boldsymbol{\chi}_{d} \\ \boldsymbol{\chi}_{d} \\ \mathbf{\chi}_{r} \\ \mathbf{u}_{b} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{d}^{T} \mathbf{f} \\ -\mathbf{\Psi}_{d}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ -\mathbf{\Psi}_{d}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ -\mathbf{\Psi}_{r}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ \mathbf{0} \end{bmatrix} ,$$

$$\hat{\mathbf{F}}_{ij} = \mathbf{\Psi}_{i}^{T} \hat{\mathbf{F}}_{rb} \mathbf{\Psi}_{j}, \quad i, j \in [d, r].$$

$$(5.3)$$

To construct the reduced structural system from Equation 5.3, the residual term χ_r might be condensed out. From the 4th row in Equation 5.3, χ_r is represented as

$$\boldsymbol{\chi}_{r} = \hat{\mathbf{F}}_{rr}^{-1} (-\hat{\mathbf{F}}_{rd} \boldsymbol{\chi}_{d} - \boldsymbol{\Psi}_{r}^{T} \mathbf{L}_{b} \mathbf{u}_{b} + \boldsymbol{\Psi}_{r}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f}),$$
(5.4)

where $\hat{\mathbf{F}}_{rr}^{-1}$ are not directly calculated. Therefore, using Woodbury matrix identity [43], the low order approximation of $\hat{\mathbf{F}}_{rr}^{-1}$ can be performed as

$$\hat{\mathbf{F}}_{rr}^{-1} = (\boldsymbol{\Psi}_{r}^{T}\hat{\mathbf{F}}_{rb}\boldsymbol{\Psi}_{r})^{-1} \approx (\boldsymbol{\Psi}_{r}^{T}\hat{\mathbf{F}}_{rbs}\boldsymbol{\Psi}_{r} + \omega^{2}\boldsymbol{\Psi}_{r}^{T}\hat{\mathbf{F}}_{rbm}\boldsymbol{\Psi}_{r})^{-1} \approx \mathbf{F}_{s}^{rr} - \omega^{2}\mathbf{F}_{m}^{rr},$$
$$\mathbf{F}_{s}^{rr} = (\boldsymbol{\Psi}_{r}^{T}\hat{\mathbf{F}}_{rbs}\boldsymbol{\Psi}_{r})^{-1}, \quad \mathbf{F}_{m}^{rr} = \mathbf{F}_{s}^{rr}(\boldsymbol{\Psi}_{r}^{T}\hat{\mathbf{F}}_{rbm}\boldsymbol{\Psi}_{r})\mathbf{F}_{s}^{rr}.$$
(5.5)

The details of Woodbury matrix identity are explained in Appendix.

Substituting Equations 5.4 and 5.5 in Equation 5.3, following approximated equation of motion in reduction level 2 is obtained as

$$\begin{bmatrix} \mathbf{\Lambda}_{d} - \omega^{2} \mathbf{I}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{0} & -\omega^{2} \mathbf{I}_{\alpha} & \mathbf{R}_{b}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{\Psi}_{d}^{T} \mathbf{\Phi}_{db} & \mathbf{\Psi}_{d}^{T} \mathbf{R}_{b} & \hat{\mathbf{G}} & -\hat{\mathbf{\Psi}}_{d}^{T} \mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} \hat{\mathbf{\Psi}}_{d} & \hat{\mathbf{H}}_{rb} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{d} \\ \boldsymbol{\alpha} \\ \boldsymbol{\chi}_{d} \\ \mathbf{u}_{b} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{d}^{T} \mathbf{f} \\ -\hat{\mathbf{\Psi}}_{d}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ -\hat{\mathbf{\Psi}}_{d}^{T} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ \mathbf{L}_{b}^{T} \hat{\mathbf{H}}_{r} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \end{bmatrix},$$
$$\hat{\mathbf{G}} = -\hat{\mathbf{F}}_{dd} + \hat{\mathbf{F}}_{dr} \hat{\mathbf{F}}_{rr}^{-1} \hat{\mathbf{F}}_{rd}, \quad \hat{\mathbf{\Psi}}_{d} = \mathbf{\Psi}_{d} - \mathbf{\Psi}_{r} \hat{\mathbf{F}}_{rr}^{-1} \hat{\mathbf{F}}_{rd},$$
$$\hat{\mathbf{H}}_{r} = \mathbf{\Psi}_{r} \hat{\mathbf{F}}_{rr}^{-1} \mathbf{\Psi}_{r}^{T}, \quad \hat{\mathbf{H}}_{rb} = \mathbf{L}_{b}^{T} \hat{\mathbf{H}}_{r} \mathbf{L}_{b}. \tag{5.6}$$

To obtain the reduced eigenvalue problem from Equation 5.6, all the transcendental quantities, superposed by hat, are approximated as

$$\hat{\mathbf{G}} \approx \mathbf{G}_s + \omega^2 \mathbf{G}_m, \ \hat{\mathbf{H}}_{rb} \approx \mathbf{H}_{rbs} + \omega^2 \mathbf{H}_{rbm}, \ \hat{\boldsymbol{\Psi}}_d \approx \mathbf{P}_{ds} + \omega^2 \mathbf{P}_{dm}.$$
(5.7)

Then, the reduced eigenvalue problem of the reduction level 2 can be obtained as

$$\bar{\mathbf{K}}_{p}^{\text{lv2}} \bar{\Phi}_{p}^{\text{lv2}} = \bar{\lambda} \bar{\mathbf{M}}_{p}^{\text{lv2}} \bar{\Phi}_{p}^{\text{lv2}} \quad \text{with } \bar{\mathbf{u}}_{p}^{\text{lv2}} = \bar{\Phi}_{p}^{\text{lv2}} \bar{\mathbf{q}}_{p}^{\text{lv2}},$$

$$\bar{\mathbf{K}}_{p}^{\text{lv2}} = \begin{bmatrix} \mathbf{\Lambda}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{b}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{\Psi}_{d}^{T} \Phi_{db} & \mathbf{\Psi}_{d}^{T} \mathbf{R}_{b} & \mathbf{G}_{s} & -\mathbf{P}_{ds}^{T} \mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b}^{T} \mathbf{P}_{ds} & \mathbf{H}_{rbs} \end{bmatrix}, \quad \bar{\mathbf{M}}_{p}^{\text{lv2}} = \begin{bmatrix} \mathbf{I}_{d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{G}_{m} & \mathbf{P}_{dm}^{T} \mathbf{L}_{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{b}^{T} \mathbf{P}_{dm} & -\mathbf{H}_{rbm} \end{bmatrix},$$

$$\bar{\mathbf{\Phi}}_{p}^{\text{lv2}} = \begin{bmatrix} \bar{\mathbf{\Phi}}_{q_{d}}^{T} & \bar{\mathbf{\Phi}}_{\alpha}^{T} & \bar{\mathbf{\Phi}}_{\chi_{d}}^{T} & \bar{\mathbf{\Phi}}_{u_{b}}^{T} \end{bmatrix}^{T}. \quad (5.8)$$

From the 3^{rd} row in Equation 5.8, $\bar{\Phi}_g$ can be calculated

$$\bar{\mathbf{\Phi}}_{g} \approx \mathbf{L}\bar{\mathbf{\Phi}}_{u_{b}} = \mathbf{\Phi}_{d}\bar{\mathbf{\Phi}}_{q_{d}} + \mathbf{R}\bar{\mathbf{\Phi}}_{\alpha} - \hat{\mathbf{F}}_{r}\mathbf{B}\mathbf{\Psi}_{d}\bar{\mathbf{\Phi}}_{\chi_{d}}, \text{ if } \hat{\mathbf{\Psi}}_{d} \approx \mathbf{\Psi}_{d}, \text{ in Level 2.}$$
(5.9)

5.2 Reduction of interface DOFs

In the reduction level 3, the interface DOF \mathbf{u}_b is reduced. Here, \mathbf{u}_b can be decomposed as the dominant term \mathbf{u}_d and the residual term \mathbf{u}_r , and it can be represented as

$$\mathbf{u}_{b} = \begin{bmatrix} \Xi_{d} & \Xi_{r} \end{bmatrix} \begin{bmatrix} \varphi_{d} \\ \varphi_{r} \end{bmatrix},$$
$$\mathbf{u}_{d} = \Xi_{d}\varphi_{d}, \quad \mathbf{u}_{r} = \Xi_{r}\varphi_{r},$$
(5.10)

where Ξ_d and Ξ_r are the mode shapes, and φ_d and φ_r are the generalized coordinates of the dominant and residual terms of the interface DOF, respectively.

In the analogous way to calculate the dominant LLM field, Ξ_d can be constructed using the λ_ℓ field which is the adjacent sub-domains, see Figure 5.1(d). Using the static terms of dominant mode shapes $\hat{\Psi}_d$, Ξ_d can be constructed, and Ξ_r also can be constructed as the corresponding null space of Ξ_d as

$$\Xi_{d} = \operatorname{Span}(\mathbf{L}_{b}^{T}\mathbf{P}_{ds}), \ \Xi_{r} \bot \Xi_{d}$$
$$\Longrightarrow \mathbf{P}_{ds}^{T}\mathbf{L}_{b}\Xi_{r} = \mathbf{0}.$$
(5.11)

In the previous research, Ξ_r is assumed as the null space of $\hat{\Psi}_d^T \mathbf{L}_b$ [30]. Then, the dynamic part \mathbf{P}_{dm} is also neglected with the static part \mathbf{P}_{ds} . In contrast, in the present formulation, Ξ_r is defined as the null space of $\mathbf{P}_{ds}^T \mathbf{L}_b$, see Equation 5.11. Therefore, \mathbf{P}_{dm} is retained in the reduction level 3, and it leads to the improvement of the accuracy performance in the eigensolutions. This feature are demonstrated in Section 5 with numerical examples. Note that Ξ_r is also arbitrary null space such as Ψ_r .

Substituting Equations 5.10 and 5.11 in Equation 5.6, the following equation are obtained

-

- -

From the 5th row in Equation 5.12, φ_r is represented as

$$\boldsymbol{\varphi}_{r} = \hat{\mathbf{H}}_{rr}^{-1} (\boldsymbol{\Xi}_{r}^{T} \mathbf{L}_{b}^{T} \hat{\mathbf{H}}_{r} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} + \omega^{2} \boldsymbol{\Xi}_{r}^{T} \mathbf{L}_{b}^{T} \mathbf{P}_{dm} \boldsymbol{\chi}_{d} - \hat{\mathbf{H}}_{rd} \boldsymbol{\varphi}_{d}).$$
(5.13)

We here substitute Equation 5.13 in Equation 5.12, and condense out the terms of φ_r . The higher order terms over ω^2 are neglected. Then, following approximated equation of motion in reduction level 3 is obtained as

$$\begin{bmatrix} \mathbf{\Lambda}_{d} - \omega^{2} \mathbf{I}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{0} & -\omega^{2} \mathbf{I}_{\alpha} & \mathbf{R}_{b}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{\Psi}_{d}^{T} \mathbf{\Phi}_{db} & \mathbf{\Psi}_{d}^{T} \mathbf{R}_{b} & \hat{\mathbf{G}} & \hat{\mathbf{Q}}_{d} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}_{d}^{T} & \hat{\mathbf{J}} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{d} \\ \boldsymbol{\alpha} \\ \boldsymbol{\chi}_{d} \\ \boldsymbol{\varphi}_{d} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{d}^{T} \mathbf{f} \\ \mathbf{R}^{T} \mathbf{f} \\ (-\mathbf{P}_{ds}^{T} - \omega^{2} \hat{\mathbf{P}}_{dm}^{T}) \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \\ \hat{\mathbf{Z}}_{d}^{T} \mathbf{L}_{b}^{T} \mathbf{h}_{r} \mathbf{B}^{T} \hat{\mathbf{F}}_{r} \mathbf{f} \end{bmatrix},$$
$$\hat{\mathbf{Q}}_{d} = (-\mathbf{P}_{ds}^{T} - \omega^{2} \hat{\mathbf{P}}_{dm}^{T}) \mathbf{L}_{b} \mathbf{\Xi}_{d}, \quad \hat{\mathbf{P}}_{dm} = \mathbf{P}_{dm} - \hat{\mathbf{H}}_{r} \hat{\mathbf{E}}_{rb} \mathbf{P}_{dm},$$
$$\hat{\mathbf{E}}_{rb} = \mathbf{L}_{b} \mathbf{\Xi}_{r} \hat{\mathbf{H}}_{rr}^{-1} \mathbf{\Xi}_{r}^{T} \mathbf{L}_{b}^{T}, \quad \hat{\mathbf{J}} = \hat{\mathbf{H}}_{dd} - \hat{\mathbf{H}}_{dr} \hat{\mathbf{H}}_{rr}^{-1} \hat{\mathbf{H}}_{rd}, \quad \hat{\mathbf{\Xi}}_{d} = \mathbf{\Xi}_{d} - \mathbf{\Xi}_{r} \hat{\mathbf{H}}_{rr}^{-1} \hat{\mathbf{H}}_{rd}, \quad (5.14)$$

To obtain the reduced eigenvalue problem from Equation 5.14, all the transcendental quantities, superposed by hat, are approximated as

$$\hat{\mathbf{Q}}_d \approx \mathbf{Q}_{ds} + \omega^2 \mathbf{Q}_{dm}, \ \hat{\mathbf{J}} \approx \mathbf{J}_s + \omega^2 \mathbf{J}_m.$$
(5.15)

Then, the reduced eigenvalue problem of the reduction level 3 with free vibration $(\mathbf{f} = \mathbf{0})$ can be obtained as

$$\bar{\mathbf{K}}_{p}^{\text{lv3}} \bar{\mathbf{\Phi}}_{p}^{\text{lv3}} = \bar{\lambda} \bar{\mathbf{M}}_{p}^{\text{lv3}} \bar{\mathbf{\Phi}}_{p}^{\text{lv3}} \quad \text{with} \quad \bar{\mathbf{u}}_{p}^{\text{lv3}} = \bar{\mathbf{\Phi}}_{p}^{\text{lv3}} \bar{\mathbf{q}}_{p}^{\text{lv3}},$$

$$\bar{\mathbf{K}}_{p}^{\text{lv3}} = \begin{bmatrix} \mathbf{\Lambda}_{d} & \mathbf{0} & \mathbf{\Phi}_{db}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{b}^{T} \mathbf{\Psi}_{d} & \mathbf{0} \\ \mathbf{\Psi}_{d}^{T} \mathbf{\Phi}_{db} & \mathbf{\Psi}_{d}^{T} \mathbf{R}_{b} & \mathbf{G}_{s} & \mathbf{Q}_{ds} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{ds}^{T} & \mathbf{J}_{s} \end{bmatrix}, \quad \bar{\mathbf{M}}_{p}^{\text{lv3}} = \begin{bmatrix} \mathbf{I}_{d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{G}_{m} & -\mathbf{Q}_{dm} \\ \mathbf{0} & \mathbf{0} & -\mathbf{Q}_{dm}^{T} & -\mathbf{J}_{m} \end{bmatrix},$$

$$\bar{\mathbf{\Phi}}_{p}^{\text{lv3}} = \begin{bmatrix} \bar{\mathbf{\Phi}}_{q_{d}}^{T} & \bar{\mathbf{\Phi}}_{\alpha}^{T} & \bar{\mathbf{\Phi}}_{\chi_{d}}^{T} & \bar{\mathbf{\Phi}}_{\varphi_{d}}^{T} \end{bmatrix}^{T}. \quad (5.16)$$

From the $3^{\rm rd}$ row in Equation 5.16, $\bar{\mathbf{\Phi}}_g$ can be calculated

$$\bar{\mathbf{\Phi}}_{g} \approx \mathbf{L} \mathbf{\Xi}_{d} \bar{\mathbf{\Phi}}_{\varphi_{d}} = \mathbf{\Phi}_{d} \bar{\mathbf{\Phi}}_{q_{d}} + \mathbf{R} \bar{\mathbf{\Phi}}_{\alpha} - \hat{\mathbf{F}}_{r} \mathbf{B} \mathbf{\Psi}_{d} \bar{\mathbf{\Phi}}_{\chi_{d}}, \text{ if } \hat{\mathbf{\Psi}}_{d} \approx \mathbf{\Psi}_{d}, \text{ in Level 3.}$$
(5.17)

Note that the calculation procedures of $\bar{\Phi}_g$ in levels 2 and 3 are not performed unless an assumption $(\hat{\Psi}_d \approx \Psi_d)$ satisfied, see Equations 5.9 and 5.17. Then, the solutions of the reduced model in the levels 2 and 3 have the convergence problems. Hence, this assumption might be carefully considered when the present formulation is used for the boundary reduction.

5.3 Numerical example

We here consider a rectangular plate problem with free boundary condition, see Figure 5.2. Length L and width B are 3m and 1.5m, respectively. Thickness h is 0.01m. Young's modulus E is 69GPa, Poisson's ratio ν is 0.35, and density ρ_s is 2700kg/m³. The plate is modeled by a 30 × 15 mesh of the four-node MITC shell finite elements, and the structural model is partitioned into two substructures $(N_s = 2)$.

We here consider two numerical cases which are differently selected numbers of the retained modes, and these are listed in Table 5.1. The frequency cut-off mode selection method is used to select the dominant substructural modes. Figure 5.3 shows the relative eigenvalue errors of the present method and the previous methods. The present formulation shows similar or better solution accuracy compared with the previous methods.



Figure 5.2: Rectangular plate problem.



Figure 5.3: Relative eigenvalue error in the rectangular plate problem. (a) Case 1 and (b) Case 2.

| | | \mathbf{q}_{d} | ${oldsymbol lpha}$ | $oldsymbol{\lambda}_\ell$ | \mathbf{u}_b | Total |
|--------|------------------------------------|------------------|--------------------|---------------------------|----------------|-------|
| | Global | 1485 | 3 | - | - | 1488 |
| | CB | 12 | - | - | 123 | 135 |
| | Damijan et al. $\left(2007\right)$ | 12 | 6 | 18 | 36 | 72 |
| Case 1 | Present (Level 1) | 12 | 6 | 246 | 123 | 387 |
| | Present (Level 2) | 12 | 6 | 18 | 123 | 159 |
| | Present (Level 3) | 12 | 6 | 18 | 18 | 54 |
| | CB | 22 | - | - | 123 | 145 |
| | Damijan et al. (2007) | 22 | 6 | 28 | 56 | 112 |
| Case 2 | Present (Level 1) | 22 | 6 | 246 | 123 | 397 |
| | Present (Level 2) | 22 | 6 | 28 | 123 | 179 |
| | Present (Level 3) | 22 | 6 | 28 | 28 | 84 |

Table 5.1: Numbers of retained modes and DOFs in the rectangular plate problem.

5.4 Closure

We extended the interface reduction technique of the F-CMS method performed by Markovic et al. [30]. To obtain the precise formulation, we propose three-level reduction procedures for the internal, localized Lagrange multipliers and interface DOFs. Especially, the static correction is used in the reduction level 3 for the interface DOFs unlike the previous research, and then the additional residual dynamic terms are survived in the final eigenvalue problem. Consequently, the present research offers an enhanced eigenvalue problem compared with the previous eigenvalue problem. Numerical results showed the improvement of the solution accuracy with reducing the matrix size in the present formulation. In addition, the eigenvector relations between the global and reduced models were completely defined in the whole reduction levels.

Chapter 6. Mode selection method for CMS methods

In the CMS methods, a global structure is partitioned into multiple substructures, and substructural modes are obtained by solving individual substructural eigenvalue problems. A reduced model is constructed by retaining only the dominant substructural modes that are selected. Therefore, the accuracy of the reduced model strongly depends on the composition of the selected dominant substructural modes. A key challenge is then to develop a high-fidelity mode selection method to construct a reliable reduced model, in which the global model is accurately reflected and the size of the reduced model is as small as possible.

To date, the frequency cut-off mode selection method has been widely used for the CMS methods, which was initially proposed by Hurty [41, 44] and subsequently embellished by others [45, 46, 47]. The frequency cut-off mode selection method uses the natural frequencies calculated from substructural eigenvalue problems. The basic idea relies on the assumption which states, in general, the lower substructural modes contribute more substantially to the lower global modes than the higher substructural modes. Therefore, the dominant substructural modes are sequentially selected from lowest natural frequencies. While the rule is simple, the lower substructural modes do not necessarily contribute adequately to the lower global modes while leaving out some critical higher substructural modes. This fact leads to the limitation in the solution accuracy of the frequency cut-off mode selection method.

During the 1990s and 2000s, several mode selection methods [48, 49, 50, 51, 52, 53] were developed for the stiffness-based CMS method, in particular, Craig-Bampton (CB) method. A key idea for their mode selection methods is based on the observation that the components of a reduced interior mass matrix that is constructed with the fixed interface modal participation factor is ranked, which is then subsequently used for mode selection. There was also a different mode selection method for a flexibilitybased CMS (F-CMS) method [28]. The method relies on the observation that the relative contribution of each substructural mode to the substructural interface flexibility can be used as a guide to select dominant substructural modes in the F-CMS method. These mode selection methods may be collectively viewed as *a priori* mode selection methods.

In practice, reduced-order structural models find their applications in the design of vibration and noise reduction, alleviation of vibration-induced fatigue, active control synthesis, and sustained performance evaluations of vibrating structures. In other words, reduced-order models are usually developed after a series of model updating to arrive at tailored reduced-order models intended for specific applications. The task of model updating, thus, can be greatly facilitated if there are *a posteriori* mode selection strategies. To this end, the objective of this study is to develop an *a posteriori* mode selection method. That is, we carry out a preliminary CMS employing the frequency cutoff concept or an *a priori* mode selection method (although we have not explored this option in the present study). Then, we rank the contributions of substructural modes to target global modes that we want to approximate more accurately. We repeat the procedure by injecting additional substructural modes if necessary until the specified accuracy of the target global modes are realized, and ejecting the relative superfluous modes is performed in the iteration procedure at once. The rest of this chapter is organized as follows.

Section 6.1 presents a review of the three available mode selection methods for stiffness-based CMS techniques, specialized to the CB method. In their 1996 paper, Kammer and Triller[50] applied a balanced systems theory-based realization procedure[54] to derive three *effective interface mass (EIM)* matrices, corresponding to three output measures, viz., the acceleration, the velocity and the displacement output. It so turns out that the *coupling matrices* developed by Givoli et al.[52] employing the Dirichlet-to-Neumann method, and the moment-matching principle adopted by Liao et al.[53] correspond to the velocity-output EIM and the displacement EIM, respectively. The proposed *a posteriori* mode selection method is presented in Section 6.2. An error estimation and an accuracy control strategy are also presented in Section 6.3. As the proposed mode selection method is also applicable to a flexibility-based CMS method[28], we summarize its formulation in Section 6.4. Numerical experiments with the proposed mode selection method are offered as applied to a plate, cylindrical and hemispherical shells in Section 6.5.

6.1 Review of Existing Mode Selection Methods

6.1.1 Kammer's method

In 1996, Kammer and Triller[50] presented their mode select methods for three different MIMO (multiple-input and multiple-output) scenarios for the CB method. Apparently, it appears that neither Givoli et al.[52] nor Liao et al.[53] was aware of Kammer and Triller's 1996 work as the work of Kammer and Triller[50] was not referenced in their papers. It turns out that the methods proposed by Givoli et al.[52] and Liao et al.[53] can be viewed as a special case of Kammer and Triller's method by adopting different weighting norms. Employing the systems theory-based input-output relations, Kammer and Triller arrived at the following expression for the interface equation subject to modal acceleration output:

$$\mathbf{M}_{i}\ddot{\mathbf{u}}_{i} + \mathbf{K}_{i}\mathbf{u}_{i} = \mathbf{f}_{i} + \mathbf{f}_{q}^{a},$$
$$\mathbf{f}_{q}^{a} = \sum_{j=1}^{n_{s}} \omega_{j}^{-1}S_{j}^{a}\ddot{v}_{j}, \ \mathbf{S}_{j}^{a} = \hat{\mathbf{M}}_{c}\mathbf{\Phi}_{\mathbf{d}_{j}}\mathbf{\Phi}_{\mathbf{d}_{j}}^{T}\hat{\mathbf{M}}_{c}^{T}, \ \ddot{\mathbf{v}}_{j}(t) = \int_{0}^{t} \ddot{\mathbf{u}}_{b}\sin[\omega_{j}(t-\tau)] \ d\tau, \tag{6.1}$$

where \mathbf{M}_i and \mathbf{K}_i are the *i*th substructure mass and stiffness matrices statically reduced to the interface degrees of freedom, respectively; and superscript *a* denotes acceleration input; and \mathbf{S}^a is labeled as *the effective interface mass matrix (EIM)* for acceleration output by Kammer and Triller[50] and *the coupling matrix* by Givoli et al. and Liao et al.[52, 53].

When the modal velocity and modal displacement output are specified, Kammer and Triller resorted to the so-called balanced realization procedure proposed by Moore[54] to obtain the following effective interface mass matrix for \mathbf{S}^{v} for the modal velocity output and that for the modal displacement output \mathbf{S}^{d} as:

$$\mathbf{S}_{j}^{v} = \hat{\mathbf{M}}_{c} \boldsymbol{\Phi}_{\mathbf{d}_{j}} \boldsymbol{\omega}_{j}^{-2} \boldsymbol{\Phi}_{\mathbf{d}_{j}}^{T} \hat{\mathbf{M}}_{c}^{T}, \tag{6.2}$$

$$\mathbf{S}_{j}^{d} = \hat{\mathbf{M}}_{c} \mathbf{\Phi}_{\mathbf{d}_{j}} \boldsymbol{\omega}_{j}^{-4} \mathbf{\Phi}_{\mathbf{d}_{j}}^{T} \hat{\mathbf{M}}_{c}^{T}.$$
(6.3)

6.1.2 The OMR method

While the derivational process for deriving the optimal modal reduction (OMR) method[52] is independent of the procedure employed by Kammer and Triller, the effective interface mass matrix derived by invoking an optimal modal reduction method by Givoli et al. turns out to be the same as the modal acceleration EIM derived by Kammer and Triller:

$$\underbrace{\mathbf{S}_{j}^{OMR}}_{\text{Givoli et al. [52]}} = \underbrace{\mathbf{S}_{j}^{a}}_{\text{Kammer and Triller [50]}}$$
(6.4)

6.1.3 The CMS^P_{χ} method

Similarly, the effective interface mass matrix derived by Liao et al [53]. by utilizing a momentmatching principle is the same as the modal velocity EIM derived by Kammer and Triller:

$$\underbrace{\mathbf{S}_{j}^{CMS_{\chi}^{P}}}_{\text{iao et al. [53]}} = \underbrace{\mathbf{S}_{j}^{v}}_{\text{Kammer and Triller [50]}}$$
(6.5)

Remark 1: It should be noted that Kammer and Triller used *trace* of the effective interface mass matrix in their applications whereas Givoli et al. and Liao et al. appear to have used L_2 -norm.

The three foregoing EIM matrices have been used to select the substructural modes *a priori*. While they offer a ranking of substructural modes, which clearly constitutes an improvement over the standard frequency cut-off method, it is not clear how one may utilize EIM matrices in improving adaptively as well as interactively updating the reduced model further once a reduced-order model is constructed. This is because in practice several iterative model improvements are often required for various intended applications with the reduced-order models, such as control synthesis, design optimization, performance evaluation, etc.

In the following we present a *a posteriori* mode selection method which can make use of intermediate reduced-order models, once a preliminary reduced-order model is at hand, thus allowing subsequent model updating depending on specific applications.

6.2 Proposed Mode selection method

I

This section presents an *a posteriori* mode selection method that utilizes a quantitative measure of how substructural interior modes contribute to the global modes. To this end, we begin with a set of dominant substructural modes chosen by the frequency cut-off mode selection method (see, e.g., references [41, 44, 45, 46]). The basic assumption is that, in general, lower substructural modes contribute more substantially to lower global modes than higher substructural modes. Therefore, dominant substructural modes are sequentially selected from the lowest substructural frequency.

However, as described below, the reduced model rendered by the frequency cut-off approach may result in inadequate models. We begin with a free-free cylindrical shell partitioned into two substructures as shown in Figure 6.1, with relatively long (Ω_1) and short (Ω_2) cylinders. Figure 6.1 also shows the relation between the global and substructural mode shapes. It is observed that, for the relatively higher global modes in Figure 6.1(b), the basic assumption of the frequency cut-off mode selection method does not hold. For example, global mode 10 is more strongly related to mode 39 of the substructure Ω_2 than other lower substructural modes. This example demonstrates the inherent limitation of the frequency cut-off mode selection method. We now present in a step-by-step manner the proposed mode selection method.



Figure 6.1: Relation between global and substructural mode shapes in a cylindrical shell. (a) Global modes 1 to 9, (b) Global modes 10 to 39.

6.2.1 Modal relation from substructural to global eigenvector

In the original CB method, we compute the approximate global eigenvector matrices $\bar{\Phi}_g$ in terms of the dominant substructural eigenvector matrix Φ_d and the eigenvector matrix $\bar{\Phi}_p$ obtained from the reduced-order model as restated below:

$$\bar{\mathbf{\Phi}}_{g} = \begin{bmatrix} \mathbf{\Phi}_{d} \bar{\mathbf{\Phi}}_{q_{d}} - \mathbf{K}_{s}^{-1} \mathbf{K}_{c} \bar{\mathbf{\Phi}}_{u_{b}} \\ \bar{\mathbf{\Phi}}_{u_{b}} \end{bmatrix}.$$
(6.6)

The preceding equation reveals that the approximate global eigenvector matrix $\bar{\Phi}_g$ consists of several components among which a direct link between the fixed interface substructural mode shapes in Φ_d and the global mode shapes in $\bar{\Phi}_g$ is manifested in the term, $\Phi_d \bar{\Phi}_{qd}$. Observe that Φ_d is obtained from the individual substructural eigenvalue problems while the generalized coordinate eigenvector matrix $\bar{\Phi}_{qd}$ corresponding to the generalized coordinates \mathbf{q}_d is calculated from the eigenvalue problem of the reduced model. From a physical point of view, $\bar{\Phi}_{qd}$ acts as the post-multiplication weighting vector set for each of the substructural mode shapes. For example, the j^{th} global mode shape $\bar{\Phi}_g(1:N_g, j)$ is obtained by

$$\bar{\mathbf{\Phi}}_{g}(1:N_{d},j) = \sum_{i=1}^{\bar{N}_{p}} \mathbf{\Phi}_{d}(1:N_{d},i) \cdot \bar{\mathbf{\Phi}}_{q_{d}}(i,j),$$
(6.7)

where N_g , \bar{N}_p , and N_d denote the numbers of DOFs in the global and reduced models, and the number of dominant substructural modes, respectively. Hence, the magnitude of $|\bar{\Phi}_{q_d}(i,j)|$ can be used as an indicative of the contribution of the i^{th} substructural mode shape to the j^{th} global mode shape. Interpreted in terms of the substructural mode shapes, the magnitude of the i^{th} row of $|\sum_{j=1}^{\bar{N}_p} \bar{\Phi}_{q_d}(i,j)|$ offers an indication of the contribution of the i^{th} substructural mode shape to the overall global mode shapes.

In other words, $\bar{\mathbf{\Phi}}_{q_d}$ in Equation 6.6 is simply a relation matrix between the approximate global eigenvector matrix $\bar{\mathbf{\Phi}}_g$ and the dominant substructural interior eigenvector matrix $\mathbf{\Phi}_d$, viz.,

$$\bar{\mathbf{\Phi}}_g \xleftarrow{\bar{\mathbf{\Phi}}_{q_d}}{\longrightarrow} \mathbf{\Phi}_d . \tag{6.8}$$

Consequently, the generalized coordinate eigenvector matrix $\bar{\Phi}_{q_d}$ can be utilized as an indicator of the contributions of the dominant substructural modes to the global modes. To succinctly see how $\bar{\Phi}_{q_d}$ links the substructural mode shapes to the global mode shapes, we expand $\bar{\Phi}_{q_d}$ to read:

$$\bar{\boldsymbol{\Phi}}_{q_{d}} = \begin{bmatrix} \bar{\boldsymbol{\Phi}}_{q_{d}}^{(1)} \\ \bar{\boldsymbol{\Phi}}_{q_{d}}^{(2)} \\ \vdots \\ \bar{\boldsymbol{\Phi}}_{q_{d}}^{(k)} \\ \vdots \\ \bar{\boldsymbol{\Phi}}_{q_{d}}^{(N_{s})} \end{bmatrix}, \quad \bar{\boldsymbol{\Phi}}_{q_{d}}^{(k)} = \begin{bmatrix} (\bar{\phi}_{q_{d}}^{(k)})_{11} & (\bar{\phi}_{q_{d}}^{(k)})_{12} & \cdots & (\bar{\phi}_{q_{d}}^{(k)})_{1\bar{N}_{p}} \\ (\bar{\phi}_{q_{d}}^{(k)})_{21} & (\bar{\phi}_{q_{d}}^{(k)})_{22} & \cdots & (\bar{\phi}_{q_{d}}^{(k)})_{2\bar{N}_{p}} \\ \vdots & \vdots & (\bar{\phi}_{q_{d}}^{(k)})_{ij} & \vdots \\ (\bar{\phi}_{q_{d}}^{(k)})_{N_{d}^{(k)}1} & (\bar{\phi}_{q_{d}}^{(k)})_{N_{d}^{(k)}2} & \cdots & (\bar{\phi}_{q_{d}}^{(k)})_{N_{d}^{(k)}\bar{N}_{p}} \end{bmatrix}, \quad (6.9)$$

where $\bar{\Phi}_{q_d}$ is the $N_d \times \bar{N}_p$ matrix $(N_d = N_d^{(1)} + N_d^{(2)} + ... + N_d^{(k)})$. The component matrix $\bar{\Phi}_{q_d}^{(k)}$ $(N_d^{(k)} \times \bar{N}_p$ matrix) gives the contribution of the substructural modes of the k^{th} substructure to the entire global modes. $(\bar{\phi}_{q_d}^{(k)})_{ij}$ is a component of $\bar{\Phi}_{q_d}^{(k)}$ corresponding to the contribution of the i^{th} substructural mode in the k^{th} substructure to the j^{th} global mode. Here, N_s denotes the number of substructures.

We therefore propose that the magnitude of each row of $\bar{\Phi}_{q_d}$ for a substructure be used as an indicator of the contribution level of the corresponding substructural mode to the global modes in $\bar{\Phi}_{g}$. In so doing, of several possible measures of utilizing the row vectors of $\bar{\Phi}_{q_d}$, the following simple norm is adopted

$$C_i^{(k)} = \sqrt{\sum_{j=1}^{\bar{N}_p} \left[(\bar{\phi}_{q_d}^{(k)})_{ij} \right]^2},$$

 $i = 1, 2, ..., N_d^{(k)}, \ j = 1, 2, ..., \bar{N}_p \text{ and } k = 1, 2, ..., N_s,$
(6.10)

where $C_i^{(k)}$ is an indicator of the contribution of the i^{th} substructural mode of the k^{th} substructure to the global modes.

It is noted that each substructural mode makes a different contribution to the global modes in $\bar{\Phi}_g$. That is, the magnitude $C_i^{(k)}$ has a different scalar value for each substructural mode. This observation enables us to rank the substructural modes based on their contributions to the global modes.

Remark 2: It is noted that the second term in the first row of Equation 6.6, viz., $\mathbf{K}_s^{-1}\mathbf{K}_c \bar{\mathbf{\Phi}}_{u_b}$, represents the contribution of the interface mode shapes to the interior global mode shapes. This term would play an important role when one carries out the reduction of the interface degrees of freedom. We plan to report in a separate paper a strategy of reducing the interface DOFs.

Mode selection procedure

Suppose we have carried out a preliminary CMS synthesis employing the frequency cut-off procedure (or alternatively some other *a priori* mode selection method) and obtained both the global mode shapes and the interior substructural mode shapes. We label the preliminary reduced model (intermediate reduced model) by tilde $(\tilde{\cdot})$ and the final reduced model by an overbar $(\bar{\cdot})$.

The numbers of the selected substructural modes for the intermediate and final reduced models are also defined by \tilde{N}_d ($\tilde{N}_d = \tilde{N}_d^{(1)} + \tilde{N}_d^{(2)} + \ldots + \tilde{N}_d^{(k)}$) and \bar{N}_d ($\bar{N}_d = \bar{N}_d^{(1)} + \bar{N}_d^{(2)} + \ldots + \bar{N}_d^{(k)}$), respectively. Then, the total numbers of modes in the intermediate and final reduced models, \tilde{N}_p and \bar{N}_p , are given by

$$\tilde{N}_p = \tilde{N}_d + N_{u_b}, \ \bar{N}_p = \bar{N}_d + N_{u_b}.$$
 (6.11)

where N_{u_b} denotes the number of interface boundary DOFs.

6.2.2

Figure 6.2 shows the three eigenvalue problems considered in the global, intermediate reduced and final reduced models. The computational cost of the present mode selection method depends on the size of the intermediate reduced eigenvalue problems. The numbers of DOFs in the three eigenvalue problems are

$$\bar{N}_p < \bar{N}_p \ll N_g. \tag{6.12}$$

Since \tilde{N}_p for the intermediate reduced model is much smaller than N_g for the global model, the computational cost to solve the eigenvalue problem of the intermediate reduced model is not high. Note that \bar{N}_d should be sufficiently smaller than \tilde{N}_d . Determining the size of \tilde{N}_p will be discussed in the numerical experiments.

(a) (b) (c) (c)
$$\overline{K}_{p}\overline{\Phi}_{p} = \lambda M_{g}\Phi_{g}$$
 $\overline{\widetilde{N}}_{d}$ $\overline{K}_{p}\overline{\Phi}_{p} = \widetilde{\lambda} \widetilde{M}_{p}\overline{\Phi}_{p}$ $\overline{\overline{N}}_{d}$ $\overline{K}_{p}\overline{\Phi}_{p} = \overline{\lambda} \overline{M}_{g}\overline{\Phi}_{p}$

Figure 6.2: Three eigenvalue problems for (a) the global eigenvalue problem, (b) the intermediate reduced eigenvalue problem, and (c) the final reduced eigenvalue problem. \tilde{N}_d and \bar{N}_d are the numbers of the substructural modes retained in the intermediate and final reduced models, respectively.

Here, the substructural modal contribution $C_i^{(k)}$ can be obtained from the intermediate reduced model given by $\tilde{\mathbf{M}}_p$ and $\tilde{\mathbf{K}}_p$. Therefore, Equation 6.10 becomes

$$C_i^{(k)} = \sqrt{\sum_{j=1}^{\tilde{N}_p} \left[(\tilde{\phi}_{q_d}^{(k)})_{ij} \right]^2},$$

= 1, 2, ..., $\tilde{N}_d^{(k)}$, $j = 1, 2, ..., \tilde{N}_p$ and $k = 1, 2, ..., N_s$. (6.13)

Let us explain a very important point of the present mode selection method. In Equation 6.13, the substructural contribution $C_i^{(k)}$ is calculated for the entire global modes of the intermediate reduced model. However, in engineering practice we are interested in only several lower global modes. Using weighting factors κ_j , it is possible to differently weight each global mode. Lager weighting factors should be used for more important global modes or global modes that we are more interested in. Then, Equation 6.13 is redefined as

i

$$C_i^{(k)} = \sqrt{\sum_{j=1}^{\tilde{N}_p} \left[\kappa_j(\tilde{\phi}_{q_d}^{(k)})_{ij}\right]^2},$$

 $i = 1, 2, ..., \tilde{N}_d^{(k)}, \ j = 1, 2, ..., \tilde{N}_p \text{ and } k = 1, 2, ..., N_s.$
(6.14)

For simplicity, in this study we determine the weighting factors with the following rule

$$\kappa_j = 1 \text{ for } N_t^{\mathrm{L}} \le j \le N_t^{\mathrm{U}}, \text{ otherwise } \kappa_j = 0,$$
(6.15)

in which $N_t^{\rm L}$ and $N_t^{\rm U}$ denote the lower and upper limits of the target global modes, respectively. Both mode numbers are used to define the range of the target global modes. N_t is the target range size $(N_t = N_t^{\rm U} - N_t^{\rm L})$. Using the weighting factors κ_j in Equation 6.15, the substructural contributions to the target global modes from $N_t^{\rm L}$ to $N_t^{\rm U}$ are calculated using Equation 6.14 and the contributions $C_i^{(k)}$ are ranked. Then, the substructural modes which contribute more to the target global modes are selected in order of $C_i^{(k)}$.

As mentioned, there exist higher substructural modes that contribute more to the target global modes than lower substructural modes as shown in Figure 6.1. The present mode selection method can pick up the non-sequential modal relation in frequency. This feature enables the improved performance of the present mode selection method in capturing the important modal characteristics of the global models. Note that the present mode selection method is independent of frequency spectra of applied forces, and can be applied to various CMS methods if the relation between Φ_d and $\bar{\Phi}_g$ can be defined. The present mode selection procedure is summarized in Table 6.1:

• (Step 1) Initial preparation. The global matrices \mathbf{M}_g and \mathbf{K}_g are given at first. The lower limit N_t^{L} and the upper limit N_t^{U} are determined for the range of the target global modes considering the interesting natural frequency range ($\omega_t^{\mathrm{L}} \sim \omega_t^{\mathrm{U}}$) of the global structure. We then determine the numbers of the dominant substructural modes \tilde{N}_d and \bar{N}_d retained in the intermediate and final reduced model, respectively. \tilde{N}_d and \bar{N}_d are defined by $\tilde{\omega}_t$ and $\bar{\omega}_t$ considering the upper limit of the interesting natural frequency ω_t^{U} by

$$\tilde{\omega}_t = \gamma \beta \omega_t^{\mathrm{U}}, \ \bar{\omega}_t = \beta \omega_t^{\mathrm{U}}. \tag{6.16}$$

In order to use the new mode selection method, we need to determine how much larger than \bar{N}_d is \tilde{N}_d by γ . We will study on the factor γ in Section 5.1. Another parameter β is recommended as $1.3 \sim 5$ in commercial FE software like ABAQUS [55].

• (Step 2) Construction of the intermediate reduced model. The frequency cut-off mode selection method is applied to select the dominant substructural modes for the intermediate reduced model. That is, using the substructural frequencies obtained by solving the substructural eigenvalue problems, the dominant substructural modes are selected in order from the lowest substructural frequency regardless of substructures. The selected substructural mode numbers are arranged in a vector $\tilde{\mathbf{N}}_d$ as

$$\tilde{\mathbf{N}}_d = \begin{bmatrix} (\tilde{\mathbf{N}}_d^{(1)})^T & (\tilde{\mathbf{N}}_d^{(2)})^T & \cdots & (\tilde{\mathbf{N}}_d^{(k)})^T & \cdots & (\tilde{\mathbf{N}}_d^{(N_s)})^T \end{bmatrix}^T,$$
(6.17)

where $\tilde{\mathbf{N}}_{d}^{(k)}$ is the vector which contains the selected mode numbers of the k^{th} substructure in the intermediate reduced model. The intermediate reduced matrices $\tilde{\mathbf{M}}_{p}$ and $\tilde{\mathbf{K}}_{p}$ are constructed by retaining intermediately selected substructural modes.

• (Step 3) Construction of the final reduced model. The eigenvalue problem of the intermediate reduced model is solved with $\tilde{\mathbf{M}}_p$ and $\tilde{\mathbf{K}}_p$, and the generalized coordinate eigenvector matrix $\tilde{\mathbf{\Phi}}_{q_d}$ is obtained. The substructural contribution $C_i^{(k)}$ is calculated using Equation 6.14 and the dominant substructural modes are selected in order of the contribution from the substructural mode that contributes most largely to the target global modes. The selected substructural mode numbers are arranged in $\bar{\mathbf{N}}_d$ as

$$\bar{\mathbf{N}}_{d} = \begin{bmatrix} (\bar{\mathbf{N}}_{d}^{(1)})^{T} & (\bar{\mathbf{N}}_{d}^{(2)})^{T} & \cdots & (\bar{\mathbf{N}}_{d}^{(k)})^{T} & \cdots & (\bar{\mathbf{N}}_{d}^{(N_{s})})^{T} \end{bmatrix}^{T},$$
(6.18)

where the vector $\bar{\mathbf{N}}_{d}^{(k)}$ contains the selected substructural mode numbers of the k^{th} substructure in the final reduced model. Using the finally selected substructural modes, the final reduced matrices $\bar{\mathbf{M}}_{p}$ and $\bar{\mathbf{K}}_{p}$ are constructed.

- Step 1. Initial preparation
 - (a) \mathbf{M}_g and \mathbf{K}_g are given.
 - (b) The range of the target global modes is determined by $\omega_t^{\rm L} \sim \omega_t^{\rm U}$: $N_t^{\rm L} \sim N_t^{\rm U}$.
 - (c) \tilde{N}_d and \bar{N}_d are determined by $\tilde{\omega}_d$ and $\bar{\omega}_d$, respectively: $\tilde{\omega}_d = \gamma \beta \omega_t^{\mathrm{U}}, \ \bar{\omega}_d = \beta \omega_t^{\mathrm{U}},$

Step 2. Construction of the intermediate reduced model

- (a) The substructural eigenvalue problems are solved $[\mathbf{K}_{s}^{(k)} - \lambda_{i}^{(k)}\mathbf{M}_{s}^{(k)}](\boldsymbol{\phi}^{(k)})_{j} = \mathbf{0}, \quad \text{for } k = 1, 2, ..., N_{s}.$
- (b) The dominant substructural modes are selected by the frequency cut-off mode selection method.
- (c) The intermediate reduced model is constructed with $\tilde{\mathbf{M}}_p$ and $\tilde{\mathbf{K}}_p$.

Step 3. Construction of the final reduced model

- (a) The intermediate eigenvalue problem is solved and $\tilde{\Phi}_{q_d}$ is found:
 - $\tilde{\mathbf{K}}_p(\tilde{\boldsymbol{\phi}})_j = \tilde{\lambda}_j \tilde{\mathbf{M}}_p(\tilde{\boldsymbol{\phi}})_j, \quad j = 1, 2, ..., \tilde{N}_p.$
- (b) The contributions of the substructural modes to the target global modes are calculated: $C_j^{(k)}$ in Equation 6.14.
- (c) The dominant substructural modes are selected in order of $C_i^{(k)}.$
- (d) The final reduced model is constructed with $\bar{\mathbf{M}}_p$ and $\bar{\mathbf{K}}_p$.



6.3 Error estimation and accuracy control

The reduced-order model thus constructed employing the present mode selection method still is wanting of one ultimate fidelity goal: the accuracy of the global modes and mode shapes computed from the reduced-order model. We extend the present mode selection method to offer an error estimation method and a strategy to assess the accuracy of the reduced model.

In the present mode selection method, the norms of the row vectors in the generalized coordinate eigenvector matrix $\tilde{\Phi}_{q_d}$ are used to calculate the substructural contribution $C_i^{(k)}$ to the target global modes, see Equation 6.14. $\tilde{\Phi}_{q_d}$ can also be represented by the column vectors $(\tilde{\phi}_{q_d})_j$

$$\tilde{\boldsymbol{\Phi}}_{q_d} = \left[\begin{array}{cccc} (\tilde{\boldsymbol{\phi}}_{q_d})_1 & (\tilde{\boldsymbol{\phi}}_{q_d})_2 & \cdots & (\tilde{\boldsymbol{\phi}}_{q_d})_j & \cdots & (\tilde{\boldsymbol{\phi}}_{q_d})_{\tilde{N}_p} \end{array} \right].$$
(6.19)

When we only retain the several rows corresponding to the selected dominant substructural modes in $\bar{\mathbf{N}}_d$ from $\tilde{\mathbf{\Phi}}_{q_d}$, the components of $\tilde{\mathbf{\Phi}}_{q_d}^{\bar{\mathbf{N}}_d}$ are defined by

$$(\tilde{\phi}_{q_d})_{ij}$$
 for $i \in \bar{\mathbf{N}}_d$, $j = 1, 2, ..., \tilde{N}_p$. (6.20)

Whence, $\tilde{\Phi}_{q_d}^{\bar{\mathbf{N}}_d}$ can be represented by the reduced column vectors $\tilde{\phi}_{q_d}^{\bar{\mathbf{N}}_d}$

$$\tilde{\Phi}_{q_d}^{\bar{\mathbf{N}}_d} = \left[\begin{array}{ccc} (\tilde{\phi}_{q_d}^{\bar{\mathbf{N}}_d})_1 & (\tilde{\phi}_{q_d}^{\bar{\mathbf{N}}_d})_2 & \cdots & (\tilde{\phi}_{q_d}^{\bar{\mathbf{N}}_d})_j & \cdots & (\tilde{\phi}_{q_d}^{\bar{\mathbf{N}}_d})_{\tilde{N}_p} \end{array} \right].$$
(6.21)

From Equations 6.19 and 6.21, the difference between $\tilde{\phi}_{q_d}$ and $\tilde{\phi}_{q_d}^{\mathbf{N}_d}$ can be measured by

$$\varepsilon_j = 1 - \frac{||(\tilde{\boldsymbol{\phi}}_{q_d}^{\mathbf{N}_d})_j||}{||(\tilde{\boldsymbol{\phi}}_{q_d})_j||}, \quad N_t^{\mathbf{L}} \le j \le N_t^{\mathbf{U}}.$$
(6.22)

Using ε_j , it is possible to estimate the natural frequency error. Note that ε_j is the sum of the contributions of the dominant substructural modes in $\bar{\mathbf{N}}_d$. Since ε_j is based on the present mode selection method, the performance of ε_j also depends on the solution accuracy of the intermediate eigenvalue problem.

In Equation 6.15, we used weighting factors $\kappa_j = 1$ for the range of the target global modes defined by $N_t^{\rm L}$ and $N_t^{\rm U}$ and weighting factors $\kappa_j = 0$ for other global modes. However, it is also possible to use different κ_j for each global mode or for the other ranges of global modes.

To improve the accuracy of the reduced model that has relatively large error in several global modes, first, the inaccurately approximated global modes are detected by using the proposed error estimation method in Equation 6.22. We then redefine κ_j or the target range considering the detected global modes, and additionally select substructural modes that largely contribute to only the inaccurately approximated global modes by using Equation 6.14. Finally, the selected modes are added in the final reduced model. The feasibility of the proposed strategy is demonstrated in Section 6.

6.4 Application in the F-CMS method

It so turns out that the mode selection method developed in the preceding sections is applicable to flexibility-based CMS (F-CMS) method[28] as well. A major difference between the CB and the F-CMS methods is that each of the partitioned substructures is completely free except inheriting the global boundary conditions as shown in Figure 3.1(d). Hence, the substructures do not have common interfaces. This means that substructural modes and mode shapes of a substructure get coupled only through the common interface displacements and interface Lagrange multipliers. Formulation details of the F-CMS method is presented in Section 3.3.

6.4.1 A priori mode selection method

It was proposed in [28, 35] that the frequency-independent residual flexibility (\mathbf{F}_{rbs}) can be used to select the substructural modes by ranking the contributions of each substructural mode. For example, if a substructural mode, say, the j^{th} mode (Λ_d)_j contributes significantly, then the following error norm

$$C_{j}^{ind} = \frac{||\mathbf{F}_{bb} - (\mathbf{\Phi}_{db}\mathbf{\Lambda}_{d}^{-1}\mathbf{\Phi}_{db}^{T})_{j}||}{||\mathbf{F}_{bb}||}, \quad j = 1, 2, ..., N_{d}$$
(6.23)

would provide an indication of the ranking of each substructural mode. Using summation of contribution of each substructural mode to the interface flexibility, the following cumulative error norm is also defined

$$C_{j}^{cum} = \frac{||\mathbf{F}_{bb} - \sum_{j=1}^{N_{d}} (\mathbf{\Phi}_{db} \mathbf{\Lambda}_{d}^{-1} \mathbf{\Phi}_{db}^{T})_{j}||}{||\mathbf{F}_{bb}||},$$
(6.24)

where C_j^{cum} denotes the cumulative error norm, and it can be used to a stop criterion of the mode selection method.

While a direct comparison is not possible, the preceding *a priori* mode selection method for the F-CMS method is qualitatively analogous to the effective interface mass matrix indicator (\mathbf{S}^v in Equation (6.2)) introduced by Kammer and Triller[50] and Liao et al.[53]. Although not explored in [28],

if one were to employ \mathbf{F}_{rbm} in a similar manner as \mathbf{F}_{rbs} , a F-CMS mode selection method akin to the displacement out case (see Equation (6.3)) result.

6.4.2 A posteriori mode selection method

In the F-CMS method, the reduced global eigenvector $\bar{\Phi}_g$ is defined by

$$\bar{\mathbf{\Phi}}_g = \mathbf{\Phi}_d \bar{\mathbf{\Phi}}_{q_d} + \mathbf{R}_s \bar{\mathbf{\Phi}}_{\alpha_s} - \hat{\mathbf{F}}_r \mathbf{B} \bar{\mathbf{\Phi}}_{\lambda_\ell}, \tag{6.25}$$

in which $\bar{\Phi}_g$ is also $N_g \times \bar{N}_p$ matrix. In the F-CMS method, \bar{N}_p is differently defined with the CB method as

$$\bar{N}_p = \bar{N}_d + N_\alpha + N_{\lambda\ell} + N_{u_b}, \tag{6.26}$$

where N_{α} and $N_{\lambda_{\ell}}$ denote the numbers of rigid body mode and interface boundary DOFs, respectively.

Note that the first term in the above equation has the same form that is exploited in the development of the mode selection method in the preceding section (see Equations 6.6 and 6.8). Therefore, the present *a posteriori* mode selection method in the previous section developed for the stiffness-based CMS method is equally applicable to the flexibility-based CMS method as well.

6.5 Numerical examples

The present mode selection method (see Equation 6.14) is evaluated by using the CB and F-CMS methods in comparison with the frequency cut-off mode selection method and the flexibility-based mode selection method in Equation 6.23. In so doing, the following relative global frequency error is used

$$\omega_{j(\text{relative error})} = |\omega_j - \bar{\omega}_j| / \omega_j, \qquad (6.27)$$

where the j^{th} natural frequency ω_j calculated from the global (original) eigenvalue problem is used as the reference solution, and the corresponding approximate natural frequency $\bar{\omega}_j$ is calculated in the CB and F-CMS methods, respectively. We also demonstrate the feasibility of the proposed error estimation and accuracy control strategy. In this section, we use $\beta = 3$ and $\beta = 2$ (see Equation 6.16) in the CB and F-CMS methods, respectively.

To evaluate the performance of the present mode selection method, we performed various numerical tests considering structural shapes, partition types and retained mode numbers. In most cases, the present mode selection method presents the best solution accuracy among the previous two mode selection methods considered in this study. However, in this work, we present the results of three structural problems: a square plate, a cylindrical shell and a hemisphere shell.

In all the numerical examples, the elastic modulus E is 69GPa, Poisson's ratio ν is 0.35, and the structural density ρ_s is 2700kg/m³.

6.5.1 Cylindrical shell problem

Let us consider a cylindrical shell with free boundary at both ends. Length L is 12m, radius R is 0.5m, and thickness h is 0.005m (h/R = 1/100 and L/R = 24). The cylindrical shell is modeled by a

 24×24 mesh of shell finite elements in the axial and circumferential directions, and is partitioned into two same substructures as shown in Figure 6.3. Here, we use the interesting natural frequency range from $\omega_t^{\rm L} = 0$ to $\omega_t^{\rm L} = 59$ rad/sec. Then, $N_t^{\rm L}$ and $N_t^{\rm U}$ are determined as 1 and 20, respectively.

To investigate the effect of γ on the performance of the present mode selection method, we calculate the relative natural frequency errors for 3 different γ (1.5, 2.0 and 2.5) and the results are presented in Figure 6.4. We observe that the same performance is obtained when γ is larger than 2.0, that is, there is a performance saturation for γ . The fact implies that $\gamma = 2.0$ is a proper choice for the intermediate reduced model considering both computational cost and accuracy. Although the performance of the present mode selection method depends on the choice of γ , we will use $\gamma = 2.0$ in the following numerical examples. The numbers of the selected substructural modes \tilde{N}_d for the intermediate reduced model are plotted in Figure 6.5. In the cylindrical shell problem, \tilde{N}_d is about twice larger than \bar{N}_d when $\gamma = 2.0$.

The performance of the present mode selection method is compared with the previous mode selection methods. Figure 6.6 shows that the present mode selection method gives the best solution accuracy among them. The numbers of the selected dominant substructural modes $\bar{N}_d^{(k)}$ are listed in Table 6.2. It should be noted that Equation 6.23 of the flexibility-based mode selection method [28] is only used for the numerical test. However, when the cumulative mode selection criterion defined in Equation 6.24 is used with Equation 6.23, the mode selection performance can be improved.



Figure 6.3: Cylindrical shell problem 1.

| | | d | | |
|-------|-----------------------|-------------------|-------------------|------------------------------------|
| CMS | Mode selection method | $\bar{N}_d^{(1)}$ | $\bar{N}_d^{(2)}$ | $\bar{N}_d = \sum \bar{N}_d^{(k)}$ |
| CD | Freq.cut-off | 23 | 23 | 46 |
| СВ | Present | 23 | 23 | 46 |
| | Freq.cut-off | 19 | 19 | 38 |
| F-CMS | Park and Park | 19 | 19 | 38 |
| | Present | 19 | 19 | 38 |
| | Present | 19 | 19 | 38 |

Table 6.2: Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the cylindrical shell problem 1.



Figure 6.4: Relative natural frequency error of the present mode selection method depending on γ in the cylindrical shell problem 1. (a) CB method, (b) F-CMS method.



Figure 6.5: Retained substructural mode numbers \tilde{N}_d in the cylindrical shell problem 1. (a) CB method, (b) F-CMS method.

The present mode selection method is also tested in different partition type of cylindrical shell problem in Figure 6.7. Figure 6.8 shows that the present mode selection method also gives better solution accuracy than the other mode selection methods. The numbers of the selected dominant substructural modes $\bar{N}_d^{(k)}$ are listed in Table 6.3.

For the partition type in Figure 6.3, when the F-CMS method is used, it is observed that the natural frequency error is relatively large in modes 5, 6, 15 and 16, see Figure 6.6(b). To handle the inaccurately approximated modes, we use the error estimation and accuracy control strategy proposed in Section 4. The key concept of the proposed accuracy control strategy is that the range of the target global modes can be freely changed. We capture the inaccurately approximated modes and improve the accuracy in the reduced model by additionally selecting several substructural modes using the present mode selection method. Figure 6.9(a) shows ε_j in Equation 6.22 for the F-CMS method. Applying the proposed error estimation method when $\bar{N}_d = 38$ that is initially selected, the inaccurately approximated global modes are detected. Then, we increase the number of the selected dominant modes $\bar{N}_d = 44$ with and without using the accuracy control strategy. The results are shown in Figure 6.9(b). When additional 6 substructural modes are selected using the redefined target range for global modes 5, 6, 15 and 16, the accuracy of the those approximated modes are improved as shown in Figures 6.9(a) and (b). However, the enough improvement is only achieved employing the proposed accuracy control strategy.



Figure 6.6: Relative natural frequency error in the cylindrical shell problem 1. (a) CB method, (b) F-CMS method.



Figure 6.7: Cylindrical shell problem 2.

Table 6.3: Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the cylindrical shell problem 2.

| CMS | Mode selection method | $\bar{N}_d^{(1)}$ | $\bar{N}_d^{(2)}$ | $\bar{N}_d = \sum \bar{N}_d^{(k)}$ |
|-------|-----------------------|-------------------|-------------------|------------------------------------|
| CB | Freq.cut-off | 14 | 29 | 43 |
| СВ | Present | 14 | 29 | 43 |
| | Freq.cut-off | 14 | 26 | 40 |
| F-CMS | Park and Park | 26 | 14 | 40 |
| | Present | 16 | 24 | 40 |



Figure 6.8: Relative natural frequency error in the cylindrical shell problem 2. (a) CB method, (b) F-CMS method.



Figure 6.9: Error estimation and accuracy control in the cylindrical shell problem 1 using the F-CMS method. (a) Error estimation using Equation 6.22, (b) Accuracy control for the global modes 5, 6, 15 and 16.

6.5.2 Square plate problem

Let us consider a square plate with free boundary. Length L is 1m and thickness h is 0.002m (h/L = 1/500). The plate is modeled by a 10 × 10 mesh of shell finite elements and is partitioned into two substructures Ω_1 and Ω_2 , see Figure 6.10.

We here use $\omega_t^{\rm L}=0$ and $\omega_t^{\rm U}=54$ rad/sec. Therefore, $N_t^{\rm L}$ and $N_t^{\rm U}$ are determined as 1 and 12, respectively. Using the mode selection procedures, we can select the dominant substructural modes for the square plate problem using the CB and F-CMS methods. The numbers of the selected dominant substructural modes $\bar{N}_d^{(k)}$ are listed in Table 6.4 and the good performance of the present mode selection method is demonstrated in Figure 6.11. The substructural mode numbers in the CB method are specifically listed in Table 6.5, which clearly shows differently selected substructural modes using the frequency cut-off and present mode selection methods.



Figure 6.10: Square plate problem.

| | | | a | 1 1 | |
|----------------|-----------------------|-------------------|-------------------|------------------------------------|---|
| \mathbf{CMS} | Mode selection method | $\bar{N}_d^{(1)}$ | $\bar{N}_d^{(2)}$ | $\bar{N}_d = \sum \bar{N}_d^{(k)}$ | - |
| CB | Freq.cut-off | 25 | 2 | 27 | - |
| СD | Present | 24 | 3 | 27 | |
| | Freq.cut-off | 19 | 1 | 20 | - |
| F-CMS | Park and Park | 11 | 9 | 20 | |
| | Present | 17 | 3 | 20 | |

Table 6.4: Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the square plate problem.



Figure 6.11: Relative natural frequency error in the square plate problem. (a) CB method, (b) F-CMS method.

| Substructural | Freq.cut-off (rad/sec) | | Preser | nt, $C_j^{(k)}$ |
|---------------|------------------------|------------------------|-----------------------------|------------------------|
| mode number | Ω_1 | Ω_2 | Ω_1 | Ω_2 |
| 1 | 2.268E+00 (°) | 3.860E+01 (°) | 8.589E+00 (°) | 5.871E-01 (°) |
| 2 | $3.583E{+}00~(\circ)$ | $1.598E{+}02~(\circ)$ | $2.988E + 00 (\circ)$ | 2.780E-02 (°) |
| 3 | $6.685E{+}00~(\circ)$ | $1.899E{+}02 (\times)$ | $1.746E{+}00~(\circ)$ | $1.189E-02 (\circ)$ |
| 4 | $1.358E{+}01~(\circ)$ | $3.386E{+}02~(\times)$ | $1.173E{+}00~(\circ)$ | $0.000E{+}00~(\times)$ |
| 5 | $1.570E{+}01~(\circ)$ | $8.477E + 02 (\times)$ | $1.307E + 00 (\circ)$ | $0.000E{+}00~(\times)$ |
| : | : | : | : | : |
| : | (0) | (\times) | (0) | (×) |
| : | : | : | : | : |
| 18 | $9.199E{+}01~(\circ)$ | $7.347E + 05 (\times)$ | $7.222E-02$ (\circ) | $0.000E{+}00~(\times)$ |
| 19 | $1.060E + 02 (\circ)$ | $7.348E{+}05~(\times)$ | $5.794\text{E-}02~(\circ)$ | $0.000E{+}00~(\times)$ |
| 20 | $1.190E{+}02 (\circ)$ | $7.348E{+}05~(\times)$ | $4.072\text{E-}03~(\times)$ | $0.000E{+}00~(\times)$ |
| 21 | $1.256E{+}02~(\circ)$ | $7.350E{+}05~(\times)$ | $6.004\text{E-}03~(\times)$ | $0.000E{+}00~(\times)$ |
| 22 | $1.261E{+}02~(\circ)$ | $8.382E{+}05~(\times)$ | $4.851\text{E-}03~(\times)$ | $0.000E{+}00~(\times)$ |
| 23 | $1.456E{+}02~(\circ)$ | $8.382E{+}05~(\times)$ | $4.487\text{E-}03~(\times)$ | $0.000E{+}00~(\times)$ |
| 24 | $1.475E{+}02~(\circ)$ | $8.382E{+}05~(\times)$ | $7.815E-03(\circ)$ | $0.000E{+}00~(\times)$ |
| 25 | $1.531E{+}02~(\circ)$ | $8.382E{+}05~(\times)$ | $1.520E-02~(\circ)$ | $0.000E{+}00~(\times)$ |
| 26 | $1.657E{+}02 (\times)$ | $8.383E{+}05~(\times)$ | 1.363 E-02 (\circ) | $0.000E{+}00~(\times)$ |
| 27 | $1.706E{+}02~(\times)$ | $8.384E + 05 (\times)$ | 1.070E-02 (°) | $0.000E{+}00~(\times)$ |
| 28 | $1.708E{+}02~(\times)$ | $8.384E+05 (\times)$ | $1.130\text{E-}02~(\circ)$ | $0.000E{+}00 (\times)$ |

Table 6.5: Selected substructural modes in the square plate problem using the CB method. Note that $C_j^{(k)}$ is the value of the substructural modal contribution in Equation 6.14.

 \circ : selected, \times : unselected

6.5.3 Hemisphere shell problem

Let us consider a hemisphere shell with free boundary at both ends. Height H is 3.084m and thickness h is 0.05m. The radii R_1 and R_2 are 2m and 0.618m, respectively. For the hemisphere shell, 20 and 40 shell finite elements are used in the axial and circumferential directions, respectively. The hemisphere shell is partitioned into four substructures, see Figure 6.12.

Considering the interesting natural frequency ranges ($\omega_t^{\rm L} = 0 \text{ rad/sec}$, $\omega_t^{\rm U} = 125 \text{ rad/sec}$), $N_t^{\rm L}$ and $N_t^{\rm U}$ become 1 and 22, respectively. Using the mode selection procedures, the dominant substructural modes are selected. The numbers of the selected substructural modes $\bar{N}_d^{(k)}$ are listed in Table 6.6. Figure 6.13 shows the excellent performance of the present mode selection method.



Figure 6.12: Hemisphere shell problem.

| d | | | | | in the noninspirere shen proble | | | |
|-------|-------|-----------------------|-------------------|-------------------|---------------------------------|-------------------|------------------------------------|--|
| | CMS | Mode selection method | $\bar{N}_d^{(1)}$ | $\bar{N}_d^{(2)}$ | $\bar{N}_d^{(3)}$ | $\bar{N}_d^{(4)}$ | $\bar{N}_d = \sum \bar{N}_d^{(k)}$ | |
| | CB | Freq.cut-off | 38 | 10 | 60 | 16 | 124 | |
| | СD | Present | 43 | 14 | 50 | 17 | 124 | |
| | | Freq.cut-off | 38 | 13 | 52 | 18 | 121 | |
| F-CMS | F-CMS | Park and Park | 33 | 14 | 52 | 22 | 121 | |
| | | Present | 34 | 17 | 49 | 21 | 121 | |

Table 6.6: Retained substructural mode numbers $\bar{N}_d^{(k)}$ in the hemisphere shell problem.


Figure 6.13: Relative natural frequency error in the hemisphere shell problem. (a) CB method, (b) F-CMS method.

6.6 Closure

A mode selection method for structural component mode synthesis was presented, which aids the structural analyst in selecting dominant substructural modes by ranking the contributions of substructural modes to target global modes. Since it is an *a posteriori* method, it can be iteratively used to arrive at the desired target accuracy of the global modes. A key idea exploited in the development of the present mode selection method is the fact that a leading term of the global mode shapes is expressed in terms of two compounded mode shapes ($\Phi_d \bar{\Phi}_{q_d}$) for both the CB and F-CMS methods. Thus, the magnitude of the j^{th} row vector of $\bar{\Phi}_{q_d}$ indicates a contribution level of the j^{th} substructural mode to the target global modes. It is this observation that has been exploited in the development of the present mode selection method.

Numerical experiments indicate that the present mode selection method is consistently more robust and accurate than the two *a priori* methods, viz., frequency cut-off method and flexibility-based mode selection method in Equation 6.23, as evidenced by the results obtained from component mode syntheses of a square plate, a cylindrical shell, and a hemisphere shell. An important by-product of the present mode selection method was a strategy to handle inaccurately approximated global modes in reduced models. Full potential of the present error control strategy for subsequent model updating remains to be explored further.

An immediate further fine tuning of the present method will come from a rational way of determining the weighting factor (κ_j) introduced in Equation 6.15. It should also be noted that the present mode selection method assumes that the interface degrees of freedom are preserved. The interface degrees of freedom constitute a considerable part of the reduced model for complex large structures, especially for models involving three-dimensional interfaces. A companion technique for reducing the interface degrees of freedom is under active development. These and other auxiliary issues such as more efficient solution procedures are actively pursued and will be presented in the future.

Chapter 7. Error estimation method for model reduction methods

7.1 Error estimation method for Guyan reduction

In the DOFs based reduction method, a small proportion of the dominant DOFs, known as "master", is only retained for the reduced model, and the other DOFs, known as "slave", are eliminated. Therefore, the order of original model can be dramatically reduced. These procedures implies that the accuracy of the reduced model strongly depends on the selection of the master DOFs, and then the key challenge is to develop the robust node selection method to properly select the master DOFs. The most node selection method is the sequential elimination method (SEM) which is based on the ratio of stiffness to mass matrices in the diagonal terms [56, 57], and then the related researches have been performed using analytical ways or energy estimation methods [58, 59, 60]. Using those methods, one can rationally select the master DOFs, and reduce the computational time. A question, then, arises: How do we evaluate the reliability of the reduced model? It means that the development of the error estimation method is required in the DOFs based reduction methods. This requirement is well revealed by Hughes's statement [61]: "A disadvantage of reduction techniques such as the Irons-Guyan procedure is that there is no guarantee that the eigenvalues and eigenvectors of the reduced problem will be good approximations of those of the original problem."

The objective of this study is to develop an error estimator that accurately predicts relative eigenvalue errors in finite element models reduced by Guyan reduction. It is not easy to estimate relative eigenvalue errors because the exact eigenvalues of the original model are unknown. The proposed error estimator makes it is possible to estimate the reliability of reduced models.

The proposed error estimator is derived from the original eigenvalue problem, in which the exact eigenvalue and eigenvectors are divided into approximated and error parts and Kidder's transformation matrix for Guyan reduction is used to approximate the exact eigenvector [4]. Kidder's transformation matrix provides a more accurate approximation of the exact eigenvector than the original transformation matrix of Guyan reduction. The derivation procedure is simple and straightforward.

In order to test the feasibility and performance of the proposed error estimator, we calculate both the exact relative eigenvalue errors and the estimated errors using the proposed error estimator, after which the error values are compared. Various numerical examples are considered for numerical tests.

7.1.1 Derivation procedures

To evaluate the reliability of the approximated eigensolutions from Equation 2.4, the following relative eigenvalue error is generally used

$$\xi_i = \frac{\bar{\lambda}_i}{\lambda_i} - 1, \tag{7.1}$$

in which ξ_i denotes the relative eigenvalue error. Since the reference eigenvalue λ_i is obtained from the original eigenvalue problem or experimentally measured, Equation 7.1 is quite expensive and impractical in engineering practice.

In this section, we introduce a method to estimate the relative eigenvalue error in Equation 7.1 without knowing the exact eigenvalue λ_i .

Since λ_i and $(\phi)_i$ is the solution of the eigenvalue problem in Equation 2.2, we have

$$\frac{1}{\lambda_i}(\boldsymbol{\phi})_i^T \mathbf{K}(\boldsymbol{\phi})_i = (\boldsymbol{\phi})_i^T \mathbf{M}(\boldsymbol{\phi})_i,$$
(7.2)

where λ_i and $(\phi)_i$ also satisfy the mass-orthonormality and stiffness-orthogonality properties, and then these can be expressed as the approximations and error terms as

$$\lambda_i = \bar{\lambda}_i + \delta \lambda_i, \tag{7.3a}$$

$$(\phi)_i = (\bar{\phi})_i + (\delta\phi)_i, \tag{7.3b}$$

in which $\delta \lambda_i$ and $(\delta \phi)_i$ are errors of the eigenvalue and eigenvector, respectively. Using Equation 7.3(b) in Equation 7.2, we obtain (specific derivation is presented in Appendix)

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i - \frac{1}{\lambda_i}(\delta \boldsymbol{\phi})_i^T \mathbf{K}(\delta \boldsymbol{\phi})_i + (\delta \boldsymbol{\phi})_i^T \mathbf{M}(\delta \boldsymbol{\phi})_i = 0.$$
(7.4)

Here, the approximated eigenvector $(\bar{\phi})_i$ can be expressed by using Kidder's transformation matrices

$$(\bar{\boldsymbol{\phi}})_{i} = \mathbf{T}_{K}(\boldsymbol{\phi}_{1})_{i} = [\mathbf{T}_{G} + \mathbf{T}_{r}](\boldsymbol{\phi}_{1})_{i},$$

$$\mathbf{T}_{r} = \begin{bmatrix} \mathbf{0} \\ \lambda_{i} \left[\mathbf{K}_{22}^{-1} \mathbf{M}_{21} - \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \mathbf{K}_{21} \right] + \lambda_{i}^{2} \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \mathbf{M}_{21} \end{bmatrix}.$$
(7.5)

Using Equation 7.5 in Equation 7.4, we obtain

$$\frac{1}{\lambda_{i}}(\boldsymbol{\phi}_{1})_{i}^{T}\left[\bar{\mathbf{T}}_{G}+\bar{\mathbf{T}}_{r}\right]^{T}\mathbf{K}\left[\bar{\mathbf{T}}_{G}+\bar{\mathbf{T}}_{r}\right](\boldsymbol{\phi}_{1})_{i}$$

$$-(\boldsymbol{\phi}_{1})_{i}^{T}\left[\bar{\mathbf{T}}_{G}+\bar{\mathbf{T}}_{r}\right]^{T}\mathbf{M}\left[\bar{\mathbf{T}}_{G}+\bar{\mathbf{T}}_{r}\right](\boldsymbol{\phi}_{1})_{i}-\frac{1}{\lambda_{i}}(\delta\boldsymbol{\phi})_{i}^{T}\left[\mathbf{K}-\lambda_{i}\mathbf{M}\right](\delta\boldsymbol{\phi})_{i}=0.$$

$$(7.6)$$

After expanding Equation 7.6 and applying the mass-orthonormality and stiffness-orthogonality conditions for the reduced eigenvalue problem, the following equation is obtained

$$\frac{\bar{\lambda}_{i}}{\lambda_{i}} - 1 = 2(\boldsymbol{\phi}_{1})_{i}^{T} \bar{\mathbf{T}}_{G}^{T} \left[\mathbf{M} - \frac{1}{\lambda_{i}} \mathbf{K} \right] \bar{\mathbf{T}}_{r}(\boldsymbol{\phi}_{1})_{i} + (\boldsymbol{\phi}_{1})_{i}^{T} \bar{\mathbf{T}}_{r}^{T} \left[\mathbf{M} - \frac{1}{\lambda_{i}} \mathbf{K} \right] \bar{\mathbf{T}}_{r}(\boldsymbol{\phi}_{1})_{i} + \frac{1}{\lambda_{i}} (\delta \boldsymbol{\phi})_{i}^{T} \left[\mathbf{K} - \lambda_{i} \mathbf{M} \right] (\delta \boldsymbol{\phi})_{i},$$
(7.7)

in which the left-hand side is the relative eigenvalue error in Equation 7.1.

When the approximated original eigenvector $(\bar{\phi})_i$ are close enough to the exact original eigenvectors $(\phi)_i$, it is possible to assume that

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i \approx 1 \text{ and } \frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i \gg \frac{1}{\lambda_i}(\delta \boldsymbol{\phi})_i^T \mathbf{K}(\delta \boldsymbol{\phi})_i,$$
(7.8a)

$$(\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i \approx 1 \text{ and } (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i \gg (\delta \boldsymbol{\phi})_i^T \mathbf{M}(\delta \boldsymbol{\phi})_i.$$
 (7.8b)

Under the assumption in Equation 7.8, we neglect the last term on the right-hand side of Equation 7.7

$$\frac{\bar{\lambda}_i}{\lambda_i} - 1 \approx 2(\boldsymbol{\phi}_1)_i^T \bar{\mathbf{T}}_G^T \left[\mathbf{M} - \frac{1}{\lambda_i} \mathbf{K} \right] \bar{\mathbf{T}}_r(\boldsymbol{\phi}_1)_i + (\boldsymbol{\phi}_1)_i^T \bar{\mathbf{T}}_r^T \left[\mathbf{M} - \frac{1}{\lambda_i} \mathbf{K} \right] \bar{\mathbf{T}}_r(\boldsymbol{\phi}_1)_i.$$
(7.9)

Finally, we propose an error estimator η_i for the relative eigenvalue error

$$\eta_i = 2(\boldsymbol{\phi}_1)_i^T \bar{\mathbf{T}}_G^T \left[\mathbf{M} - \frac{1}{\bar{\lambda}_i} \mathbf{K} \right] \bar{\mathbf{T}}_r(\boldsymbol{\phi}_1)_i + (\boldsymbol{\phi}_1)_i^T \bar{\mathbf{T}}_r^T \left[\mathbf{M} - \frac{1}{\bar{\lambda}_i} \mathbf{K} \right] \bar{\mathbf{T}}_r(\boldsymbol{\phi}_1)_i,$$
(7.10)

with

$$\mathbf{T}_{r} = \begin{bmatrix} \mathbf{0} \\ \bar{\lambda}_{i} \left[\mathbf{K}_{22}^{-1} \mathbf{M}_{21} - \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \mathbf{K}_{21} \right] + \bar{\lambda}_{i}^{2} \mathbf{K}_{22}^{-1} \mathbf{M}_{22} \mathbf{K}_{22}^{-1} \mathbf{M}_{21} \end{bmatrix},$$
(7.11)

in which, to approximate λ_i contained in Equation 7.10, we use the i^{th} approximated eigenvalue $\bar{\lambda}_i$ instead of λ_i .

In general, reduced models more accurately approximate lower modes than higher modes. Therefore, the assumption in Equation 7.8 would not be well applied to the estimation of relative eigenvalue errors corresponding to higher modes. For this reason, the proposed error estimator η_i will give a more accurate estimation for relative eigenvalue errors corresponding to lower modes.

In Equations 7.10 and 7.11, we can easily identify the fact that the computational cost of the error estimator η_i is low. In those equations, we reuse the matrix \mathbf{K}_{22}^{-1} previously calculated, and the required matrix operations are merely simple additions and multiplications.



7.1.2 Numerical examples

In this section, we test the performance of the proposed error estimation method. Three different structural problems are considered: cylindrical panel, Square plate with circular holes, and shaft-shaft interaction problems.

Cylindrical panel problem

We here apply the present error estimator to a cylindrical panel with free boundary condition, see Figure 7.1. Length L is 0.8m, radius R is 0.5m, and thickness h is 0.005m. Young's modulus E is 69GPa, Poisson's ratio ν is 0.35, and the structural density ρ_s is 2700kg/m³. The cylindrical panel is modeled by a 24 × 16 mesh of shell finite elements (1275 DOFs).

In two numerical cases, we differently select the nodes as shown in Figure 7.2, and then every DOFs in the selected nodes are defined as the master DOFs. Figure 7.3 shows that the excellent performances of the present error estimation method.



Figure 7.1: Cylindrical panel.



Figure 7.2: Selected nodes of the cylindrical panel. (a) 35 nodes, (b) 59 nodes.



Figure 7.3: Exact and estimated relative errors in eigenvalues in the cylindrical panel. (a) 35 nodes, (b) 59 nodes.

Hyperboloid panel problem

We here consider a hyperboloid panel of height H = 4.0m and thickness h = 0.05m. Young's modulus E is 69GPa, Poisson's ratio ν is 0.35, and density ρ_s is 2700kg/m³. The mid-surface of this shell structure is described by

$$x^{2} + y^{2} = 2 + z^{2}; \ z \in [-2, 2].$$
 (7.12)

We use a mesh of 24×16 MITC4 shell elements, see Figure 7.4. Two numerical cases with differently selected master DOFs are considered as shown in Figure 7.5. The excellent performance of the proposed error estimator is observed in Figure 7.6.



Figure 7.4: Hyperboloid panel.



Figure 7.5: Selected nodes of the hyperboloid panel. (a) 35 nodes, (b) 59 nodes.





Figure 7.6: Exact and estimated relative errors in eigenvalues in the hyperboloid panel. (a) 35 nodes, (b) 59 nodes.

Shaft-shaft interaction problem

We here consider two cylindrical shafts connected with fillets of radius 0.002m and, in this example, the free boundary condition is imposed, see Figure 7.7. Height H is 0.08m and thickness h is 0.0005m. The radii R_1 and R_2 are 0.01m and 0.0075m, respectively. Young's modulus E is 207GPa, Poisson's ratio ν is 0.29, and the structural density ρ_s is 2700kg/m³. For this example, 534 elements and 555 nodes are used.

We here consider two different numbers of arbitrary selected nodes, and these are presented in Figure 7.8. The exact and estimated eigenvalue errors are presented in Figure 7.9, and this result clearly show the robustness of the proposed error estimation method.



Figure 7.7: Shaft-shaft interaction problem.



Figure 7.8: Selected nodes of the shaft-shaft interaction problem. (a) 39 nodes, (b) 70 nodes.





Figure 7.9: Exact and estimated relative errors in eigenvalues in the shaft-shaft interaction problem. (a) 39 nodes, (b) 70 nodes.

7.2 Error estimation method for CB method

An important issue in CMS methods is how to evaluate the reliability of the reduced model compared to the global (original) model. Although the reliability of the reduced model can be directly assessed by errors in its approximated global eigenvalues, it is basically difficult to calculate the errors because the exact global eigenvalues are unknown. To handle this issue, various error estimation methods have been developed (see, e.g., Bourquin [62], Yang et al. [46], Elssel and Voss [47], Jakobsson and Larson [63]). However, those error estimation methods show qualitative tendencies rather than meaningful quantities in eigenvalue errors.

In order to accurately estimate individual eigenvalue errors in reduced models, we here propose an error estimator. The error estimator is derived from the global (original) eigenvalue problem, in which the global eigenvalue and eigenvector are divided into approximated and error parts and a newly defined transformation matrix $\bar{\mathbf{T}}_{ECB}$ is used to approximate the global eigenvector in the CB method.

In the original CB formulation, the transformation matrix $\mathbf{\bar{T}}_{CB}$ is constructed by using only dominant substructural modes, and the residual substructural modes are truncated without any consideration. However, considering the residual substructural modes, the transformation matrix $\mathbf{\bar{T}}_{CB}$ can be enhanced. Therefore, when the enhanced transformation matrix $\mathbf{\bar{T}}_{ECB}$ is used instead of the original one, the global eigenvectors can be approximated more accurately.

The derivation procedure shows that the relative eigenvalue error can be approximated by the proposed error estimator. To evaluate the error estimator proposed, no heavy computation is required. That is, only simple additions and multiplications of known matrices are necessary.



7.2.1 Derivation procedures

The exact global eigensolutions are expressed by the approximated global eigensolutions and their error terms as

$$\lambda_i = \bar{\lambda}_i + \delta \lambda_i, \tag{7.13a}$$

$$(\boldsymbol{\phi}_q)_i = (\bar{\boldsymbol{\phi}}_q)_i + (\delta \boldsymbol{\phi}_q)_i, \tag{7.13b}$$

where $\delta \lambda_i$ and $(\delta \phi_a)_i$ are errors in the *i*th eigenvalue and eigenvector, respectively.

Due to the linear independency of the exact global eigenvectors, the approximated global eigenvector $(\bar{\phi}_a)_i$ can be represented by a linear combination of the exact global eigenvectors

$$(\bar{\phi}_g)_i = \sum_{k=1}^{N_g} \alpha_k(\phi_g)_k,$$
(7.14)

where α_k are coefficients for the linear combination.

As more substructural modes are contained in the reduced model, the approximated global eigenvector $(\bar{\phi}_g)_i$ calculated from the reduced eigenvalue problem becomes closer to the exact global eigenvector $(\phi_g)_i$. When the approximated global eigenvectors are close enough to the exact global eigenvector, we can assume

$$\alpha_i \approx 1,$$
 (7.15a)

$$|\alpha_i| \gg |\alpha_i - 1|, \ |\alpha_1|, \ |\alpha_2|, \cdots |\alpha_{i-1}|, \ |\alpha_{i+1}|, \cdots, |\alpha_{N_g}|.$$
(7.15b)

Since the exact eigensolutions $(\lambda_i, (\phi_g)_i)$ are obtained from the global eigenvalue problem in Equation 3.2, the following equation is given

$$\frac{1}{\lambda_i} (\boldsymbol{\phi}_g)_i^T \mathbf{K}_g (\boldsymbol{\phi}_g)_i = (\boldsymbol{\phi}_g)_i^T \mathbf{M}_g (\boldsymbol{\phi}_g)_i.$$
(7.16)

The global eigensolutions λ_i and $(\phi_g)_i$ satisfy the mass-orthonormality and stiffness-orthogonality conditions in Equation 3.3(a) and (b), respectively. Using Equation 7.13(b) in Equation 7.16, we obtain

$$\frac{1}{\lambda_{i}} \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right]^{T} \mathbf{K}_{g} \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right] \\
= \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right]^{T} \mathbf{M}_{g} \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right].$$
(7.17)

Using Equation 7.14 in Equation 7.17, the left-hand side of Equation 7.17 can be rewritten as follows:

$$\frac{1}{\lambda_{i}} \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right]^{T} \mathbf{K}_{g} \left[(\bar{\boldsymbol{\phi}}_{g})_{i} + (\delta \boldsymbol{\phi}_{g})_{i} \right]$$

$$= \frac{1}{\lambda_{i}} (\bar{\boldsymbol{\phi}}_{g})_{i}^{T} \mathbf{K}_{g} (\bar{\boldsymbol{\phi}}_{g})_{i} - 2(\alpha_{i} - 1) - \left[(\alpha_{i} - 1)^{2} + \sum_{\substack{k=1\\k \neq i}}^{N_{g}} \alpha_{k}^{2} \frac{\lambda_{k}}{\lambda_{i}} \right].$$
(7.18)

Similarly, the right-hand side of Equation 7.17 also becomes

$$\left[(\bar{\phi}_g)_i + (\delta \phi_g)_i \right]^T \mathbf{M}_g \left[(\bar{\phi}_g)_i + (\delta \phi_g)_i \right]$$

$$= (\bar{\phi}_g)_i^T \mathbf{M}_g (\bar{\phi}_g)_i - 2(\alpha_i - 1) - \left[(\alpha_i - 1)^2 + \sum_{\substack{k=1\\k \neq i}}^{N_g} \alpha_k^2 \right].$$

$$(7.19)$$

Using Equations 7.18 and 7.19 in Equation 7.17, the leading order terms $2(\alpha_i - 1)$ are canceled, and the following equation is obtained

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{K}_g(\bar{\boldsymbol{\phi}}_g)_i - (\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{M}_g(\bar{\boldsymbol{\phi}}_g)_i - \sum_{\substack{k=1\\k\neq i}}^{N_g} \alpha_k^2 \left(\frac{\lambda_k}{\lambda_i} - 1\right) = 0.$$
(7.20)

In the CB method, using the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ in Equation 4.10, the approximated global eigenvector $(\bar{\phi}_g)_i$ can be defined

$$(\bar{\boldsymbol{\phi}}_g)_i = \bar{\mathbf{T}}_{ECB}(\bar{\boldsymbol{\phi}}_p)_i \quad \text{with } \bar{\mathbf{T}}_{ECB} = \bar{\mathbf{T}}_{CB} + \bar{\mathbf{T}}_r.$$
 (7.21)

Using Equation 7.21, Equation 7.20 is rewritten by

$$\frac{1}{\lambda_{i}}(\bar{\boldsymbol{\phi}}_{p})_{i}^{T}\left[\bar{\mathbf{T}}_{CB}+\bar{\mathbf{T}}_{r}\right]^{T}\mathbf{K}_{g}\left[\bar{\mathbf{T}}_{CB}+\bar{\mathbf{T}}_{r}\right](\bar{\boldsymbol{\phi}}_{p})_{i} \\
-(\bar{\boldsymbol{\phi}}_{p})_{i}^{T}\left[\bar{\mathbf{T}}_{CB}+\bar{\mathbf{T}}_{r}\right]^{T}\mathbf{M}_{g}\left[\bar{\mathbf{T}}_{CB}+\bar{\mathbf{T}}_{r}\right](\bar{\boldsymbol{\phi}}_{p})_{i} -\sum_{\substack{k=1\\k\neq i}}^{N_{g}}\alpha_{k}^{2}\left(\frac{\lambda_{k}}{\lambda_{i}}-1\right) = 0.$$
(7.22)

After expanding Equation 7.22 and using the mass-orthonormality and stiffness-orthogonality in Equation 3.5, we obtain

$$\frac{\bar{\lambda}_i}{\lambda_i} - 1 = 2(\bar{\phi}_p)_i^T \bar{\mathbf{T}}_{CB}^T \left[\mathbf{M}_g - \frac{1}{\lambda_i} \mathbf{K}_g \right] \bar{\mathbf{T}}_r (\bar{\phi}_p)_i
+ (\bar{\phi}_p)_i^T \bar{\mathbf{T}}_r^T \left[\mathbf{M}_g - \frac{1}{\lambda_i} \mathbf{K}_g \right] \bar{\mathbf{T}}_r (\bar{\phi}_p)_i + \sum_{\substack{k=1\\k \neq i}}^{N_g} \alpha_k^2 \left(\frac{\lambda_k}{\lambda_i} - 1 \right),$$
(7.23)

where the left-hand side is the relative eigenvalue error. Therefore, Equation 7.23 shows that the relative eigenvalue error can be expressed by three scalar terms.

The last term on the right-hand side of Equation 7.23 is much smaller than the other terms due to α_k^2 under the assumption in Equation 7.15. Neglecting the last term, we obtain

$$\frac{\bar{\lambda}_i}{\lambda_i} - 1 \approx 2(\bar{\boldsymbol{\phi}}_p)_i^T \bar{\mathbf{T}}_{CB}^T \left[\mathbf{M}_g - \frac{1}{\lambda_i} \mathbf{K}_g \right] \bar{\mathbf{T}}_r(\bar{\boldsymbol{\phi}}_p)_i \\
+ (\bar{\boldsymbol{\phi}}_p)_i^T \bar{\mathbf{T}}_r^T \left[\mathbf{M}_g - \frac{1}{\lambda_i} \mathbf{K}_g \right] \bar{\mathbf{T}}_r(\bar{\boldsymbol{\phi}}_p)_i,$$
(7.24)

which can be used to estimate the relative eigenvalue error.

Finally, let us define the error estimator η_i as

$$\eta_{i} = 2(\bar{\boldsymbol{\phi}}_{p})_{i}^{T} \bar{\mathbf{T}}_{CB}^{T} \left[\mathbf{M}_{g} - \frac{1}{\bar{\lambda}_{i}} \mathbf{K}_{g} \right] \bar{\mathbf{T}}_{r} (\bar{\boldsymbol{\phi}}_{p})_{i} + (\bar{\boldsymbol{\phi}}_{p})_{i}^{T} \bar{\mathbf{T}}_{r}^{T} \left[\mathbf{M}_{g} - \frac{1}{\bar{\lambda}_{i}} \mathbf{K}_{g} \right] \bar{\mathbf{T}}_{r} (\bar{\boldsymbol{\phi}}_{p})_{i},$$
(7.25)

with

$$\bar{\mathbf{T}}_{r} = \bar{\lambda}_{i} \begin{bmatrix} \mathbf{0} & \mathbf{F}_{rs} \left[-\mathbf{M}_{s} \mathbf{K}_{s}^{-1} \mathbf{K}_{c} + \mathbf{M}_{c} \right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$
(7.26)

It is very important that, for λ_i in Equation 7.33 and ω^2 contained in $\bar{\mathbf{T}}_r$ in Equation 4.11, the *i*th approximated eigenvalue $\bar{\lambda}_i$ calculated from the reduced eigenvalue problem is used to calculate η_i . However, the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ cannot be used to improve the CB method in its present form because ω^2 in the transformation matrix is unknown. Therefore, the enhanced transformation matrix $\bar{\mathbf{T}}_{ECB}$ is only used for error estimation.

We note that, for higher modes, the assumption in Equation 7.15 could not be well satisfied and $\lambda_k/\lambda_i - 1$ could increase, in particular, when selected dominant modes are not many enough. Therefore, the proposed error estimator will give better accuracy for lower modes in general. It is also valuable to note that the computational cost of the error estimator proposed is not heavy because simple matrix additions and multiplications are required in Equations 7.34 and 7.26. The concept of the error estimator have been also employed for Guyan reduction, AMLS and F-CMS methods [64, 65, 66].

7.2.2 Numerical examples

In this section, we test the performance of the proposed error estimator. Four different structural problems are considered: rectangular plate, shaft-shaft interaction, hemisphere shell, and stiffened plate problems.

The conventional frequency cut-off mode selection method and recently developed mode selection method [67] (see Section 6) are employed to select the dominant substructural modes.

When the frequency cut-off mode selection method is used, the performance of the present error estimator is compared with the previous one developed by Elssel and Voss [47]

$$\eta_i' = \frac{\bar{\lambda}_i}{|\lambda_r - \bar{\lambda}_i|},\tag{7.27}$$

where λ_r is the smallest residual eigenvalue of substructures. The error estimator η'_i was proposed as an upper bound of the relative eigenvalue error

$$0 \le \xi_i \le \eta_i'. \tag{7.28}$$

Note that almost no computational cost is required for evaluating this error estimator.

The mode selection method developed by Kim et al [67] uses the eigenvector relation between substructures and global structure. Since this mode selection method can rank the substructural modal contributions to the global modes, it can improve the solution accuracy compared to the frequency cutoff mode selection method. The proposed error estimator is also tested using this new mode selection method.



Simple plate problem

Let us consider a simple plate with free boundary, see Figure 4.1. Here, 15 and 30 substructural modes are retained for two numerical cases ($N_d = 15$, $N_d = 30$), and the numbers of retained substructural modes are listed in Table 7.1. Figures 7.10 and 7.11 present the exact and estimated relative eigenvalue errors, respectively. Figure 7.10 shows that the present error estimator outperforms the error estimator by Elssel and Voss [47] when the frequency cut-off mode selection method is used. Figure 7.11 also shows the excellent performance of the present error estimator when the mode selection method by Kim et al. [67] is used. In Table 7.2, we list the exact and estimated relative eigenvalue errors corresponding to Figure 7.10(a).

Table 7.1: Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem.

| Mode selection method | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | N_d |
|-----------------------|------|-------------|-------------|-------|
| Freq. cut-off | 1 | 10 | 5 | 15 |
| | 2 | 21 | 9 | 30 |
| Kim et al. [67] | 1 | 10 | 5 | 15 |
| | 2 | 21 | 9 | 30 |



Figure 7.10: Exact and estimated relative eigenvalue errors in the simple plate problem. The frequency cut-off mode selection method is used. (a) $N_d = 15$, (b) $N_d = 30$.



Figure 7.11: Exact and estimated relative eigenvalue errors in the simple plate problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 15$, $\tilde{N}_d = 30$, (b) $N_d = 30$, $\tilde{N}_d = 60$.

| Mada numban | Erroot | Estimated | Estimated |
|-------------|--------------|-------------------------|--------------|
| Mode number | Exact | (Elssel and Voss) | (Present) |
| 1 | 1.29749E-04 | 5.40922E-03 | 1.39749E-04 |
| 2 | 7.11199 E-05 | 8.03123E-03 | 7.31199E-05 |
| 3 | 7.87912 E-04 | 4.13343E-02 | 7.87912E-04 |
| 4 | 1.55629 E-03 | 4.60452 E-02 | 1.65629E-03 |
| 5 | 2.78188 E-03 | 1.07879E-01 | 2.78188 E-03 |
| 6 | 5.38288E-03 | 1.43772 E-01 | 5.58288E-03 |
| 7 | 2.75964 E-03 | 1.55603E-01 | 2.76964 E-03 |
| 8 | 2.46485 E-03 | 2.36205 E-01 | 2.46485 E-03 |
| 9 | 3.35683E-03 | 3.53565E-01 | 3.45683E-03 |
| 10 | 5.82414 E-03 | 5.09857 E-01 | 6.74143 E-03 |
| 11 | 4.48544 E-02 | $1.04450 \text{E}{+}00$ | 5.48544E-02 |
| 12 | 1.36776E-01 | 2.40284E + 00 | 1.36776E-01 |
| 13 | 9.63322E-02 | 7.17569E + 00 | 9.63322E-02 |
| | | | |

Table 7.2: Exact and estimated eigenvalue errors in the rectangular plate problem (The frequency cut-off mode selection method is used and $N_d = 15$).



Shaft-shaft interaction problem

We here consider two cylindrical shafts connected with fillets of radius 0.002m and no boundary condition is imposed, see Figure 7.12. Height H is 0.08m, and thickness h is 0.5×10^{-3} m. The radii R_1 and R_2 are 0.01m and 0.0075m, respectively. Young's modulus E is 207GPa, Poisson's ratio ν is 0.29, and density ρ_s is 2700kg/m³. For this example, 534 elements and 555 nodes are used, and the finite element model is partitioned into two substructures ($N_s = 2$).

Two different numbers of retained substructural modes ($N_d = 20$, $N_d = 40$) are considered as listed in Table 7.3. Figures 7.13 and 7.14 show the excellent performance of the present error estimator, which also shows much better accuracy than the previous error estimator by Elssel and Voss [47].



Figure 7.12: Shaft-shaft interaction problem.

Table 7.3: Retained substructural mode numbers $N_d^{(k)}$ in the shaft-shaft interaction problem.

| Mode selection method | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | N_d |
|-----------------------|------|-------------|-------------|-------|
| Freq. cut-off | 1 | 13 | 7 | 20 |
| | 2 | 24 | 16 | 40 |
| Kim et al. [67] | 1 | 13 | 7 | 20 |
| | 2 | 26 | 14 | 40 |



Figure 7.13: Exact and estimated relative eigenvalue errors in the shaft-shaft interaction problem. The frequency cut-off mode selection method is used. (a) $N_d = 20$, (b) $N_d = 40$.



Figure 7.14: Exact and estimated relative eigenvalue errors in the shaft-shaft interaction problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 20$, $\tilde{N}_d = 40$, (b) $N_d = 40$, $\tilde{N}_d = 80$.

Hemisphere shell problem

Let us consider a hemisphere shell structure with free boundary at both ends, see Figure 7.15. Height H is 3.084m, and thickness h is 0.05m. The radii R_1 and R_2 are 2m and 0.618m, respectively. Young's modulus E is 69GPa, Poisson's ratio ν is 0.35, and density ρ_s is 2700kg/m³. In this problem, 20 and 40 shell finite elements are used in the axial and circumferential directions, respectively. The hemisphere shell is partitioned into four substructures ($N_s = 4$).

We consider 25 and 80 substructural modes ($N_d = 25$, $N_d = 80$), see Table 7.4. As shown in Figures 7.16 and 7.17, the present error estimator very accurately estimates the relative eigenvalue errors in the two numerical cases.



Figure 7.15: Hemisphere shell problem.

| Mode selection method | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | $N_d^{(3)}$ | $N_d^{(4)}$ | N_d |
|-----------------------|------|-------------|-------------|-------------|-------------|-------|
| Freq. cut-off | 1 | 3 | 2 | 12 | 8 | 25 |
| | 2 | 17 | 10 | 32 | 21 | 80 |
| Kim et al. [67] | 1 | 2 | 2 | 13 | 8 | 25 |
| | 2 | 16 | 10 | 33 | 21 | 80 |

Table 7.4: Retained substructural mode numbers $N_d^{(k)}$ in the hemisphere shell problem.



Figure 7.16: Exact and estimated relative eigenvalue errors in the hemisphere shell problem. The frequency cut-off mode selection method is used. (a) $N_d = 25$, (b) $N_d = 80$.



Figure 7.17: Exact and estimated relative eigenvalue errors in the hemisphere shell problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 25$, $\tilde{N}_d = 50$, (b) $N_d = 80$, $\tilde{N}_d = 160$.

Stiffened plate problem

We here apply the present error estimator to a stiffened plate with free boundary, see Figure 4.5. We use 25 and 75 substructural modes ($N_d = 25$, $N_d = 75$) in two numerical cases, and the numbers of dominant substructural modes $N_d^{(k)}$ are listed in Table 7.5. The exact and estimated eigenvalue errors are plotted in Figures 7.18 and 7.19, and the graphs clearly show the robustness of the present error estimator.

| Table 1.5. Retained Substi | ucturar | mode i | umbers | 5 I d | m the st | mened | plate p | nomen |
|----------------------------|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------|
| Mode selection method | Case | $N_d^{(1)}$ | $N_d^{(2)}$ | $N_d^{(3)}$ | $N_d^{(4)}$ | $N_d^{(5)}$ | $N_d^{(6)}$ | N_d |
| Free out off | 1 | 11 | 4 | 1 | 1 | 4 | 4 | 25 |
| Freq. cut-on | 2 | 27 | 16 | 6 | 6 | 10 | 10 | 75 |
| K_{im} at al $[67]$ | 1 | 9 | 3 | 2 | 1 | 5 | 5 | 25 |
| $\mathbf{K} = \{0\}$ | 2 | 22 | 15 | 8 | 8 | 11 | 11 | 75 |

Table 7.5: Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem.





Figure 7.18: Exact and estimated relative eigenvalue errors in the stiffened plate problem. The frequency cut-off mode selection method is used. (a) $N_d = 25$, (b) $N_d = 75$.



Figure 7.19: Exact and estimated relative eigenvalue errors in the stiffened plate problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 25$, $\tilde{N}_d = 50$, (b) $N_d = 75$, $\tilde{N}_d = 150$.

Computational cost

In order to investigate the computational cost required for the error estimator proposed, computation times are measured for the exact and estimated relative eigenvalue errors. MATLAB is used for the computation with a personal computer (Intel duo-core E6750, 2.66GHz CPU, 8.0GB RAM).

Table 7.6 presents the computation times for the relative eigenvalue errors in the four numerical examples, in which the computations are performed only for the lowest eigenvalues (mode number = 1). Note that, for the exact relative eigenvalue error, we consider the solution time of the exact lowest eigenvalue and the calculation time of the exact relative eigenvalue error. The results show that the proposed error estimator is computationally efficient.

| | DO | \mathbf{Fs} | Computation time (sec) | | | | |
|------------------------------|------------|--------------------|------------------------|-------------------|--------------------|--|--|
| | 77 | $\bar{\mathbf{N}}$ | Deer et | Estimated | Estimated | | |
| | N_g | N_p | Exact | (Elssel and Voss) | (Present) | | |
| Rectangular plate | 072 | 26 | 9.160F-01 | 1 5495 05 | 1.026E.02 | | |
| (Freq. cut-off, $N_d = 15$) | 215 | 30 | 2.100E-01 | 1.048£-00 | 1.950E-05 | | |
| Shaft-shaft interaction | 0.775 | 00 | 2.72 E ± 0.0 | 1.007E.05 | 9 196E 09 | | |
| (Freq. cut-off, $N_d = 20$) | 2,775 | 90 | 3.735E+00 | 1.897E-05 | 2.180E-02 | | |
| Hemisphere shell | 4 900 | 495 | 1 799E + 01 | 2 5075 05 | 4 7995 09 | | |
| (Freq. cut-off, $N_d = 25$) | 4,200 | 420 | 1.722E+01 | 3.397E-05 | 4. <i>(22</i> E-02 | | |
| Stiffened plate | 0.051 | 402 | F 141E + 00 | 0.191E-06 | 9 419E 09 | | |
| (Freq. cut-off, $N_d = 25$) | 3,351 | 423 | 0.141E+00 | 9.131E-00 | 2.413E-02 | | |

Table 7.6: Computation times for the exact and estimated relative eigenvalue errors.

7.3 Error estimation method for the F-CMS method

Recently, to construct more reliable and accurate reduced models, Park and Park has proposed the flexibility based component mode synthesis (F-CMS) method [28]. However, it is not possible to estimate the reliability of models reduced by the F-CMS method. The objective of this study is to develop a method that can accurately estimate individual eigenvalue errors for the F-CMS method. The error estimation method is derived from the global eigenvalue problem, in which the global eigenvalue and eigenvector are decomposed into approximated and error parts. The derivation procedure shows that the proposed error estimator can approximate the relative eigenvalue errors. Of course, the exact global eigenvalues are unknown in this error estimation method.

7.3.1 Derivation procedures

Since the F-CMS method shares the general description with the CB method, derivation procedures of the error estimation method in the F-CMS method is also similar with the one in the CB method. Especially, the fore procedures are totally same with the CB method until Equation 7.20 as

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{K}_g(\bar{\boldsymbol{\phi}}_g)_i - (\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{M}_g(\bar{\boldsymbol{\phi}}_g)_i - \sum_{\substack{k=1\\k\neq i}}^{N_g} \alpha_k^2 \left(\frac{\lambda_k}{\lambda_i} - 1\right) = 0.$$
(7.29)

Here, we assume that the global stiffness matrix \mathbf{K}_g can be divided into the approximated global stiffness matrix and its error

$$\mathbf{K}_g = \bar{\mathbf{K}}_g + \delta \mathbf{K}_g,\tag{7.30}$$

in which $\delta \mathbf{K}_g$ is error in the global stiffness matrix, and $\bar{\mathbf{K}}_g$ is the approximated global stiffness matrix which satisfies

$$(\bar{\boldsymbol{\phi}}_g)_i^T \bar{\mathbf{K}}_g (\bar{\boldsymbol{\phi}}_g)_i = \bar{\lambda}_i. \tag{7.31}$$

Using Equations 7.30 and 7.31, Equation 7.29 can be rewritten

$$\frac{\bar{\lambda}_i}{\lambda_i} - (\bar{\phi}_g)_i^T \mathbf{M}_g(\bar{\phi}_g)_i = \left| \sum_{\substack{k=1\\k\neq i}}^{N_g} \alpha_k^2 \left(\frac{\lambda_k}{\lambda_i} - 1 \right) \right| - \frac{1}{\lambda_i} (\bar{\phi}_g)_i^T \delta \mathbf{K}_g(\bar{\phi}_g)_i.$$
(7.32)

It is not easy to identify the magnitude and sign of the terms in the right-hand side of Equation 7.32. However, the terms are obviously much smaller than the terms of left-hand side under the assumption in Equations 7.15 and 7.30. Neglecting the right-hand side in Equation 7.32, we obtain

$$\frac{\lambda_i}{\lambda_i} - 1 \approx (\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{M}_g(\bar{\boldsymbol{\phi}}_g)_i - 1, \tag{7.33}$$

where the left-hand side is the relative eigenvalue error and the right-hand side is the mass-orthonormality error of the approximated global eigenvector $(\bar{\phi}_g)_i$. Consequently, the preceding derivation procedure shows that the mass-orthonormality error is an approximation of the relative eigenvalue error ξ_i . Therefore, using the relation, we can estimate the relative eigenvalue error ξ_i without knowing the exact global eigenvalue λ_i . Now, let us define the error estimation parameter η_i as

$$\eta_i = (\bar{\boldsymbol{\phi}}_g)_i^T \mathbf{M}_g(\bar{\boldsymbol{\phi}}_g)_i - 1.$$
(7.34)

7.3.2 Numerical examples

In this section, we test the performance of the proposed error estimation method in the F-CMS method. Three different structural problems are considered: rectangular plate, hemisphere shell, and stiffened plate problems. The present error estimation method is tested for the F-CMS method in comparison with previous two error estimation methods proposed by Elssel and Voss [47] and Kim et al. [67].

The error estimation method proposed by Elssel and Voss [47] was developed as an upper bound of the relative eigenvalue error τ_i .

$$0 \le \xi_j \le \tau_j = \frac{\bar{\lambda}_j}{\lambda_r - \bar{\lambda}_j},\tag{7.35}$$

where λ_r is the smallest residual eigenvalue of substructures. This method was originally developed for the CB and AMLS methods based on the frequency cut-off mode selection method. However, since this method can evaluate the individual eigenvalue error, we here apply it to the F-CMS method.

The error estimation method by Kim et al. [67] is defined in Equation 6.22. This error estimation method and its performance are presented in detail in Section 6.3.



Simple plate problem

Let us consider a simple plate with free boundary, see Figure 4.1. Since the error estimation method by Elssel and Voss in Equation 7.35 originally developed for the CB and AMLS methods, we first test its performance in the CB method. Using the frequency cut-off rule, 10 and 20 substructural modes are retained for two numerical cases ($N_d = 10$, $N_d = 20$), and retained substructural mode numbers are listed in Table 7.7. Figure 7.20 shows the relative eigenvalue error ξ_j and the error estimation parameter τ_j in the CB method. The numerical results show that τ_j provide the upper bound for the relative eigenvalue error ξ_j and the difference between τ_j and ξ_j is about one or two digits in log scale as presented in reference [47].

Figures 7.21 and 7.22 present the exact and estimated relative errors in eigenvalues when the F-CMS method is employed. To select the dominant substructural modes, we adopt the frequency cut-off mode selection method and the mode selection method proposed by Kim et al. [67]. The numbers of retained substructural modes are listed in Table 7.7.

The performance of the error estimation by Elssel and Voss is similar in the both CB and F-CMS methods. The numerical results show the excellent performance of the present error estimation method for both numerical cases compared with the previous error estimation methods. It is interesting to note that the proposed error estimation provides an upper bound of the relative eigenvalue error in the problem considered here.

| Mode selection method | Numerical case | CB | | CB | | | |
|-----------------------|----------------|---------------|---------------|-------|-------------|---------------|-------|
| Mode selection method | Numericai case | $N_{d}^{(1)}$ | $N_{d}^{(2)}$ | N_d | $N_d^{(1)}$ | $N_{d}^{(2)}$ | N_d |
| Free cut off | Case 1 | 7 | 3 | 10 | 7 | 3 | 10 |
| rieq.cut-on | Case 2 | 13 | 7 | 20 | 13 | 7 | 20 |
| | Case 1 | - | - | - | 8 | 2 | 10 |
| Kim et al. [67] | Case 2 | - | - | - | 13 | 7 | 20 |

Table 7.7: Retained substructural mode numbers $N_d^{(k)}$ in the simple plate problem.

Hemisphere shell problem

Let us consider a hemisphere shell with free boundary condition at both ends, see Figure 7.15. Using the F-CMS method, we consider two numerical cases for the numbers of retained substructural modes, 54 and 76 substructural modes $(N_d = 54, N_d = 76)$ as listed in Table 7.8. Figures 7.23 and 7.24 show that the excellent performances of the proposed error estimation method compared with the previous error estimation methods. As in the previous numerical example, the proposed error estimation acts as an upper bound for the exact relative eigenvalue error in general.

 $N_{d}^{(2)}$ $N_{d}^{(3)}$ $N_{d}^{(4)}$ Numerical case $N_d^{(1)}$ Mode selection method N_d Case 1 128 201454Freq.cut-off Case 2 1813271876

12

15

8

11

21

29

13

21

54

76

Case 1

Case 2

Table 7.8: Retained substructural mode numbers $N_{A}^{(k)}$ in the hemisphere shell problem.

Stiffened plate problem

Kim et al. [67]

Here, the proposed error estimation method is used for a stiffened plate with free boundary condition, see Figure 4.5.

In this problem, we consider two cases for the numbers of retained substructural modes: 32 and 52 substructural modes ($N_d = 32, N_d = 52$), see Table 7.9. As shown in Figures 7.25 and 7.26, the proposed error estimation method can very accurately estimate the relative eigenvalue error in the reduced model constructed by the F-CMS method. Also, the method provides a tight upper bound rather than lower bound in this problem.

| Mode selection method | Numerical case | $N_d^{(1)}$ | $N_d^{(2)}$ | $N_d^{(3)}$ | $N_d^{(4)}$ | $N_d^{(5)}$ | $N_d^{(6)}$ | N_d |
|---|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------|
| Mode selection method Numerical cas Freq.cut-off Case 1 Case 2 Case 1 | Case 1 | 8 | 8 | 4 | 4 | 4 | 4 | 32 |
| | Case 2 | 12 | 12 | 7 | 7 | 7 | 7 | 52 |
| | Case 1 | 9 | 7 | 4 | 4 | 4 | 4 | 32 |
| Kim et al. $[67]$ | Case 2 | 12 | 14 | 7 | 7 | 6 | 6 | 52 |
| | | | | | | | | |

Table 7.9: Retained substructural mode numbers $N_d^{(k)}$ in the stiffened plate problem.

7.4 Closure

In this thesis, we have proposed an error estimator to accurately estimate the relative eigenvalue errors in the model reduction methods. To derive the error estimator, enhanced transformation matrix might be defined first, and then, using the enhanced transformation matrix, we can derive an error estimator for model reduction methods from the global eigenvalue problem. The performance of the proposed error estimator was tested in the CB and F-CMS methods, and Guyan reduction. Although the present error estimator requires an enhanced transformation matrix, the required matrix operations are simply additions and multiplications of known matrices. For this reason, the present error estimator possesses not only improved accuracy but also computational efficiency.

The error estimator proposed in this study would be used to develop the mode selection algorithms for the reduced-order modeling and the solution techniques for eigenvalue problems in structural dynamics. It should be also noted that the proposed enhanced transformation matrices also can be used to develop the enhanced formulations of the original model reduction methods [42, 68].



Figure 7.20: Exact and estimated relative errors in eigenvalues in the simple plate problem using the CB method. The frequency cut-off mode selection method is used. (a) $N_d = 10$, (b) $N_d = 20$.



Figure 7.21: Exact and estimated relative errors in eigenvalues in the simple plate problem. The frequency cut-off mode selection method is used. (a) $N_d = 10$, (b) $N_d = 20$.


Figure 7.22: Exact and estimated relative errors in eigenvalues in the simple plate problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 10$, $\tilde{N}_d = 20$, (b) $N_d = 20$, $\tilde{N}_d = 40$.



Figure 7.23: Exact and estimated relative errors in eigenvalues in the hemisphere shell problem. The frequency cut-off mode selection method is used. (a) $N_d = 54$, (b) $N_d = 76$.



Figure 7.24: Exact and estimated relative errors in eigenvalues in the hemisphere shell problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 54$, $\tilde{N}_d = 108$, (b) $N_d = 76$, $\tilde{N}_d = 152$.



Figure 7.25: Exact and estimated relative errors in eigenvalues in the stiffened plate problem. The frequency cut-off mode selection method is used. (a) $N_d = 32$, (b) $N_d = 52$.



Figure 7.26: Exact and estimated relative errors in eigenvalues in the stiffened plate problem. The mode selection method proposed by Kim et al. [67] is used. (a) $N_d = 32$, $\tilde{N}_d = 128$, (b) $N_d = 52$, $\tilde{N}_d = 208$.

Chapter 8. Conclusions

The objectives in this work were to develop enhanced methodologies, general mode selection criterion, and error estimators for model reduction methods.

In Chapters 2 and 3, we focused on reviewing the DOFs and mode based model reduction methods before we explained the newly developed methodologies in following chapters. After explaining the general description of each model reduction method, formulations of the classical methods were presented. For DOFs based reduction methods, we presented Guyan reduction, Kidder's and Meirovitch's approaches, and the improved reduced system (IRS) method. For mode based reduction methods, we focused on component mode synthesis (CMS), in particular, the Craig-Bampton (CB) and flexibility based CMS methods.

In Chapter 4, we presented a new component mode synthesis (CMS) method by improving the wellknown CB method. Unlike in the original CB method, residual substructural modes were considered in constructing the transformation matrix. As a result, the original CB transformation matrix was enhanced by an additional dynamic term, in which the unknown eigenvalue was approximated using O'Callanhan's approach. Using the enhanced transformation matrix, global (original) structural models can be more precisely reduced and the accuracy of the reduced models dramatically improved with little additional computational cost. The excellent performance of the enhanced CB method was demonstrated using numerical examples.

In Chapter 5, an extended interface reduction technique for the F-CMS method was presented. To obtain the precise formulation, we proposed three-level reduction procedures for the internal, localized Lagrange multipliers and interface DOFs. Unlike in previous research, static correction was used in reduction level 3 for the interface DOFs, which allowed additional residual dynamic terms survived in the final eigenvalue problem. Consequently, one result of the present research is an enhanced eigenvalue problem compared with the previous eigenvalue problem. Numerical examples demonstrated the improvement of the solution accuracy with reducing the matrix size in the present formulation. In addition, the eigenvector relationships between the global and reduced models were completely defined in the whole reduction levels.

In Chapter 6, a mode selection method for structural component mode synthesis was presented, which aids the structural analyst in selecting dominant substructural modes by ranking the contributions of substructural modes to target global modes. Since it is an a posteriori method, it can be iteratively used to arrive at the desired target accuracy of the global modes. A key idea exploited in the development of the present mode selection method is the fact that a leading term of the global mode shapes is expressed in terms of two compounded mode shapes for both the CB and F-CMS methods. Numerical experiments indicate that the present mode selection method is consistently more robust and accurate than the two a priori methods, viz., frequency cut-off method and flexibility based mode selection method. An important by product of the present mode selection method is a strategy to handle inaccurately approximated global modes in reduced models.

In Chapter 7, we developed an accurate error estimator for Guyan reduction which can estimate relative eigenvalue errors in reduced models. The proposed error estimator is derived from the original eigenvalue problem using Kidder's transformation matrix. The resulting error estimator is simple and computationally inexpensive. The excellent performance of the proposed error estimator was demonstrated through various numerical examples.

Also, we presented error estimators for the CB and F-CMS methods, respectively. Since CMS methods have similar general descriptions with DOFs based reduction methods, we were able to employ the derivation procedures for the error estimator used in Guyan reduction for the CB and F-CMS methods as well. Sections 7.2 and 7.3 include specific derivation procedures and adequate error estimators for the CB and F-CMS methods, respectively. Those feasibilities and performances were validated using various numerical examples.

As an extension of this work, we recommend the following future works:

- In Chapter 4, we presented a fundamental idea to enhance conventional CMS methods, and employed it for the CB method. This idea could also be employed for other CMS methods such as the automated multi-level substructuring (AMLS) and the F-CMS methods.
- In Chapter 6, we presented a new mode selection method and simple strategy to handle inaccurately approximated global modes. This mode selection method could be used to develop optimal iterative mode selection algorithms for the reduced-order modeling and for the solution techniques for eigenvalue problems in structural dynamics. In addition, full potential of the present error control strategy for subsequent model updating remains to be explored further.
- In Chapter 7, we developed an error estimator for Guyan reduction. It is important to note that, using a similar derivation procedure, error estimators for other DOFs based reduction methods such as the IRS method could be developed. In addition, when the proposed error estimator is utilized together with DOFs selection methods, efficient iterative algorithms to obtain more accurate reduced models could be developed. Also, error estimators for the CB and F-CMS method were presented. Using the same conceptual idea, error estimators for other CMS methods such as the dual CB and the proposed enhanced CMS methods could be developed. Furthermore, with the proposed mode selection method in Chapter 6, iterative mode selection algorithms could be optimized.

Appendices



Chapter A. Derivation of F_{rs} and F_{rm}

In the CB method, The mass, stiffness matrices and the eigensolutions of the structural system satisfy the following conditions:

$$\Phi_s^T \mathbf{M}_s \Phi_s = \mathbf{I}_s, \quad \Phi_s^T \mathbf{K}_s \Phi_s = \mathbf{\Lambda}_s,$$
$$\mathbf{\Lambda}_s = \operatorname{diag} \left[\mathbf{\Lambda}_d, \quad \mathbf{\Lambda}_r \right], \quad \Phi_s = \left[\Phi_d \quad \Phi_r \right]. \tag{A.1}$$

Let $\hat{\Phi}_s$ be the transformed eigenvectors

$$\hat{\mathbf{\Phi}}_s = \mathbf{M}_s^{1/2} \mathbf{\Phi}_s. \tag{A.2}$$

Using Equation A.2, Equation A.1 can be rewritten as

$$\hat{\boldsymbol{\Phi}}_{s}^{T}\hat{\boldsymbol{\Phi}}_{s} = \mathbf{I}_{s},\tag{A.3}$$

$$\hat{\boldsymbol{\Phi}}_{s}^{T} \mathbf{M}_{s}^{-1/2} \mathbf{K}_{s} \mathbf{M}_{s}^{-1/2} \hat{\boldsymbol{\Phi}}_{s} = \boldsymbol{\Lambda}_{s}.$$
(A.4)

Note that $\hat{\Phi}_s$ is a unitary matrix for the orthogonality in Equation A.3. Thus, we can get the following relation from Equation A.4:

$$\mathbf{M}_{s}^{-1/2}\mathbf{K}_{s}\mathbf{M}_{s}^{-1/2} = \mathbf{\hat{\Phi}}_{s}\mathbf{\Lambda}_{s}\mathbf{\hat{\Phi}}_{s}^{T}.$$
(A.5)

The inverse of Equation A.5 results in

$$\begin{bmatrix} \mathbf{M}_{s}^{-1/2} \mathbf{K}_{s} \mathbf{M}_{s}^{-1/2} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\mathbf{\Phi}}_{d} & \hat{\mathbf{\Phi}}_{r} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{r}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{d}^{T} \\ \hat{\mathbf{\Phi}}_{r}^{T} \end{bmatrix}$$
$$= \hat{\mathbf{\Phi}}_{d} \mathbf{\Lambda}_{d}^{-1} \hat{\mathbf{\Phi}}_{d}^{T} + \hat{\mathbf{\Phi}}_{r} \mathbf{\Lambda}_{r}^{-1} \hat{\mathbf{\Phi}}_{r}^{T}.$$
(A.6)

From Equations A.2 and A.6, we can get \mathbf{F}_{rs} as

$$\mathbf{F}_{rs} = \hat{\mathbf{\Phi}}_r \mathbf{\Lambda}_r^{-1} \hat{\mathbf{\Phi}}_r^T = \mathbf{M}_s^{-1/2} \left[\mathbf{M}_s^{-1/2} \mathbf{K}_s \mathbf{M}_s^{-1/2} \right]^{-1} \mathbf{M}_s^{-1/2} - \mathbf{\Phi}_d \mathbf{\Lambda}_d^{-1} \mathbf{\Phi}_d^T.$$
(A.7)

Then, the square of Equation A.6 provided with Equation A.3 yields The inverse of Equation A.5 results in

$$\begin{bmatrix} \mathbf{M}_{s}^{-1/2} \mathbf{K}_{s} \mathbf{M}_{s}^{-1/2} \end{bmatrix}^{-2} = \begin{bmatrix} \mathbf{\hat{\Phi}}_{d} & \mathbf{\hat{\Phi}}_{r} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{d}^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{r}^{-2} \end{bmatrix} \begin{bmatrix} \mathbf{\hat{\Phi}}_{d}^{T} \\ \mathbf{\hat{\Phi}}_{r}^{T} \end{bmatrix}$$
$$= \mathbf{\hat{\Phi}}_{d} \mathbf{\Lambda}_{d}^{-2} \mathbf{\hat{\Phi}}_{d}^{T} + \mathbf{\hat{\Phi}}_{r} \mathbf{\Lambda}_{r}^{-2} \mathbf{\hat{\Phi}}_{r}^{T}.$$
(A.8)

Similarly, From Equations A.2 and A.8, we can get \mathbf{F}_{rm} as

$$\mathbf{F}_{rm} = \hat{\mathbf{\Phi}}_r \mathbf{\Lambda}_r^{-2} \hat{\mathbf{\Phi}}_r^T = \mathbf{M}_s^{-1/2} \left[\mathbf{M}_s^{-1/2} \mathbf{K}_s \mathbf{M}_s^{-1/2} \right]^{-2} \mathbf{M}_s^{-1/2} - \mathbf{\Phi}_d \mathbf{\Lambda}_d^{-2} \mathbf{\Phi}_d^T.$$
(A.9)

In the F-CMS method, above procedures are almost same to calculate \mathbf{F}_{rs} and \mathbf{F}_{rm} . However, since the F-CMS method use the free interface handling technique, it possesses the rigid body motion. Therefore, Equation A.1 is changed as

$$\boldsymbol{\Lambda}_{s} = \operatorname{diag} \left[\boldsymbol{0}, \quad \boldsymbol{\Lambda}_{d}, \quad \boldsymbol{\Lambda}_{r} \right], \quad \boldsymbol{\Phi}_{s} = \left[\mathbf{R} \quad \boldsymbol{\Phi}_{d} \quad \boldsymbol{\Phi}_{r} \right], \tag{A.10}$$

and then, to obtain the inverse stiffness matrix, we use the generalized inverse which is the pseudo-inverse technique denoted by the superscript +. Also, the interface residual flexibilities, \mathbf{F}_{rbs} and \mathbf{F}_{rbm} , can be also calculated form the above procedures with the interface Boolean matrix **B**.

Chapter B. Woodbury matrix identity

In linear algebra, the inverse of a rank-k correction of some matrix can be computed by doing a rank-k correction to the inverse of the original matrix. It was named as the Woodbury matrix identity [43].

The Woodbury matrix identity is

$$\left[\mathbf{A} + \mathbf{U}\mathbf{C}\mathbf{V}\right]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left[\mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right]^{-1}\mathbf{V}\mathbf{A}^{-1},$$
(B.1)

where **A**, **U**, **C** and **V** denote matrices of the correct size. Specifically, **A** is $n \times n$, **U** is $n \times k$, **C** is $k \times k$ and **V** is $k \times n$.

Deriving the Woodbury matrix identity is easily done by solving the following block-wise matrix inversion problem.

$$\begin{bmatrix} \mathbf{A} & \mathbf{U} \\ \mathbf{V} & -\mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}.$$
 (B.2)

Then, Equation B.2 can be expressed

$$\mathbf{AX} + \mathbf{UY} = \mathbf{I},\tag{B.3}$$

$$\mathbf{V}\mathbf{X} - \mathbf{C}^{-1}\mathbf{Y} = \mathbf{0},\tag{B.4}$$

which is equivalent to

$$[\mathbf{A} + \mathbf{U}\mathbf{C}\mathbf{V}]\,\mathbf{X} = \mathbf{I}.\tag{B.5}$$

Using Equation B.3, we find

$$\mathbf{X} = \mathbf{A}^{-1} \left[\mathbf{I} - \mathbf{U} \mathbf{Y} \right], \tag{B.6}$$

and then, substituting Equation B.6 into Equation B.4, the following equation is obtained

$$\mathbf{V}\mathbf{A}^{-1}\left[\mathbf{I} - \mathbf{U}\mathbf{Y}\right] = \mathbf{C}^{-1}\mathbf{Y}.$$
(B.7)

Expanding and rearranging Equation B.7, we have

$$\mathbf{V}\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U} \end{bmatrix} \mathbf{Y}.$$
 (B.8)

Finally, we substitute Equation B.8 into Equation B.3, and we have

$$\mathbf{AX} + \mathbf{U} \left[\mathbf{C}^{-1} + \mathbf{V} \mathbf{A}^{-1} \mathbf{U} \right]^{-1} \mathbf{V} \mathbf{A}^{-1} = \mathbf{I},$$
(B.9)

and then, Equation B.9 can be rewritten as

$$\mathbf{X} = \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{U} \left[\mathbf{C}^{-1} + \mathbf{V} \mathbf{A}^{-1} \mathbf{U} \right]^{-1} \mathbf{V} \mathbf{A}^{-1}.$$
 (B.10)

Note that \mathbf{X} is the inverse matrix of $[\mathbf{A} + \mathbf{UCV}]$ defined in Equation B.5. Therefore, the Woodbury matrix identity can be derived

$$[\mathbf{A} + \mathbf{U}\mathbf{C}\mathbf{V}]^{-1} = \mathbf{X} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}[\mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}]^{-1}\mathbf{V}\mathbf{A}^{-1}.$$
 (B.11)

Chapter C. Specific derivation of Equation 7.4

The original eigenvalue problem can be represented

$$(\mathbf{K} - \lambda_i \mathbf{M})(\boldsymbol{\phi})_i = \mathbf{0}.$$
 (C.1)

Here, $(\phi)_i$ can be decomposed by the approximated and error terms as

$$(\phi)_i = (\bar{\phi})_i + (\delta\phi)_i, \tag{C.2}$$

and then using Equation C.2, Equation C.1 is expressed

$$(\mathbf{K} - \lambda_i \mathbf{M})((\bar{\boldsymbol{\phi}})_i + (\delta \boldsymbol{\phi})_i) = \mathbf{0}.$$
 (C.3)

Premultiplying $\frac{1}{\lambda_i}(\boldsymbol{\phi})_i^T$ in Equation C.3, we can obtain the following equation

$$\frac{1}{\lambda_{i}}((\bar{\boldsymbol{\phi}})_{i}^{T} + (\delta\boldsymbol{\phi})_{i}^{T})(\mathbf{K} - \lambda_{i}\mathbf{M})((\bar{\boldsymbol{\phi}})_{i} + (\delta\boldsymbol{\phi})_{i}) = \frac{1}{\lambda_{i}}(\bar{\boldsymbol{\phi}})_{i}^{T}\mathbf{K}(\bar{\boldsymbol{\phi}})_{i} - (\bar{\boldsymbol{\phi}})_{i}^{T}\mathbf{M}(\bar{\boldsymbol{\phi}})_{i} \\
+ \frac{1}{\lambda_{i}}(\bar{\boldsymbol{\phi}})_{i}^{T}(\mathbf{K} - \lambda_{i}\mathbf{M})(\delta\boldsymbol{\phi})_{i} + \frac{1}{\lambda_{i}}(\delta\boldsymbol{\phi})_{i}^{T}(\mathbf{K} - \lambda_{i}\mathbf{M})((\bar{\boldsymbol{\phi}})_{i} + (\delta\boldsymbol{\phi})_{i}) = 0.$$
(C.4)

Using Equation C.3, the fifth term in Equation C.4 is zero, and we have

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i + \frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T (\mathbf{K} - \lambda_i \mathbf{M})(\delta \boldsymbol{\phi})_i = 0.$$
(C.5)

Since the every term in Equation C.5 is scalar, it can be represented as

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i + \frac{1}{\lambda_i}(\delta \boldsymbol{\phi})_i^T (\mathbf{K} - \lambda_i \mathbf{M})(\bar{\boldsymbol{\phi}})_i = 0,$$
(C.6)

and then, using Equation C.2, we obtain

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i + \frac{1}{\lambda_i} (\delta \boldsymbol{\phi})_i^T (\mathbf{K} - \lambda_i \mathbf{M}) ((\boldsymbol{\phi})_i - (\delta \boldsymbol{\phi})_i) = 0.$$
(C.7)

After expanding Equation C.7, the following equation is obtained

$$\frac{1}{\lambda_i} (\bar{\boldsymbol{\phi}})_i^T \mathbf{K} (\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M} (\bar{\boldsymbol{\phi}})_i - \frac{1}{\lambda_i} (\delta \boldsymbol{\phi})_i^T \mathbf{K} (\delta \boldsymbol{\phi})_i + (\delta \boldsymbol{\phi})_i^T \mathbf{M} (\delta \boldsymbol{\phi})_i + \frac{1}{\lambda_i} (\delta \boldsymbol{\phi})_i^T (\mathbf{K} - \lambda_i \mathbf{M}) (\boldsymbol{\phi})_i = 0.$$
(C.8)

Using Equation C.1, the last term in Equation C.8 is zero, and we finally have

$$\frac{1}{\lambda_i}(\bar{\boldsymbol{\phi}})_i^T \mathbf{K}(\bar{\boldsymbol{\phi}})_i - (\bar{\boldsymbol{\phi}})_i^T \mathbf{M}(\bar{\boldsymbol{\phi}})_i - \frac{1}{\lambda_i}(\delta \boldsymbol{\phi})_i^T \mathbf{K}(\delta \boldsymbol{\phi})_i + (\delta \boldsymbol{\phi})_i^T \mathbf{M}(\delta \boldsymbol{\phi})_i = 0.$$
(C.9)

References

- [1] Guyan R. Reduction of stiffness and mass matrices. AIAA Journal 1965; 3(2):380.
- [2] Irons BM. Eigenvalue economisers in vibration problems. Journal of the Royal Aeronautical Society 1963;67:526.
- [3] Irons BM. Structural eigenvalue problems: elimination of unwanted variables. AIAA Journal 1965;3:961.
- [4] Kidder RL. Reduction of structural frequency equations AIAA Journal 1973; 11(6):892.
- [5] Kuhar EJ, Stahle CV. Dynamic transformation method for modal synthesis. AIAA Journal 1974; 12(5):672-678.
- [6] Leung YTL. An accurate method of dynamic condensation in structural analysis. International Journal for Numerical Methods in Engineering 1978; 21:1705-1715.
- [7] O'callahan J. A procedure for an improved reduced system (IRS) model. Proceeding the 7th International Modal Analysis Conference, Bethel, CT, 1989.
- [8] Johnson CP, Craig RR, Yargicoglu A, Rajatabhoth R. Quadratic reduction for the eigenproblem. International Journal for Numerical Methods in Engineering 1980; 15(6):911-923.
- [9] Anderson LR, Hallauer WL. A method of order reduction for structural dynamics based on Riccati iteration. AIAA Journal 1981; 19(6):796-800.
- [10] Paz M. Dynamic condensation. AIAA Journal 1984; 22(5):724-727.
- [11] Suarez LE. Dynamic condensation method for structural eigenvalue analysis. AIAA Journal 1992; 30(4):1046-1054.
- [12] Kim KO, Kang MK. Convergence acceleration of iterative modal reduction methods. AIAA Journal 2001; 39(1):134-140.
- [13] Bouhaddi N, Fillod R. A method for selecting master DOF in dynamic substructuring using the Guyan condensation method. *Computers and Structures* 1992; 45(5):941-946.
- [14] Bouhaddi N, Fillod R. Substructuring by a two level dynamic condensation method. Computers and Structures 1996; 60(3):403-409.
- [15] Kim H, Cho M. Improvement of reduction method combined with sub-domain scheme in large-scale problem. International Journal for Numerical Methods in Engineering 2007; 70:206-251.
- [16] Meirovitch L. Computational methods in structural dynamics. Sijthoff and Noordohoff: Netherlands, 1980.
- [17] Turner MJ, Clough RW, Martin HC and Topp LJ. Stiffness and deflection analysis of complex structures. J. Aernonaut. Sci. 1956; 23:805-823.

- [18] M.I. Friswell, S.D. Garvey, J.E.T. Penny, Model reduction using dynamic and iterated IRS techniques, *Journal of Sound and Vibration* 186(2) (1995) 311-323.
- [19] M.I. Friswell, S.D. Garvey, J.E.T. Penny, The convergence of the iterated IRS method, Journal of Sound and Vibration 211(1) (1998) 123-132.
- [20] Y. Xia, R. Lin, Improvement on the iterated IRS method for structural eigensolutions, Journal of Sound and Vibration 270 (4-5) (2004) 713-727.
- [21] Hurty W. Dynamic analysis of structural systems using component modes. AIAA Journal 1965; 3(4):678-685.
- [22] Craig RR, Bampton MCC. Coupling of substructures for dynamic analysis. AIAA Journal 1968; 6(7):1313-1319.
- [23] MacNeal RH. Hybrid method of component mode synthesis. Computers & Structures 1971; 1(4):581-601.
- [24] Benfield WA, Hruda RF. Vibration analysis of structures by component mode substitution. AIAA Journal 1971; 9:1255-1261.
- [25] Rubin S. Improved component-mode representation for structural dynamic analysis. AIAA Journal 1975; 13(8):995-1006.
- [26] Hintz RM. Analytical methods in component modal synthesis. AIAA Journal 1975; 13(8):1007-1016.
- [27] Bennighof JK, Lehoucq RB. An automated multilevel substructuring method for eigenspace computation in linear elastodynamics. *SIAM Journal on Scientific Computing* 2004; **25**(6):2084-2106.
- [28] Park KC, Park YH. Partitioned component mode synthesis via a flexibility approach. AIAA Journal 2004; 42(6):1236-1245.
- [29] Rixen DJ. A dual Craig-Bampton method for dynamic substructuring. J Comput Appl Math 2004;168(1-2):383-91.
- [30] D. Markovic, K.C. Park and A. Ibrahimbegovic, Reduction of substructural interface degrees of freedom in flexibility-based component mode synthesis, *International Journal of Numerical Methods* in Engineering, Vol. 70, pp. 163-180, 2007.
- [31] Craig RR. Coupling of substructures for dynamic analyses: an overview. in: Proceeding 41th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference USA: Atlanta; 2000.
- [32] Craig RR, Kurdila AJ. Fundamentals of structural dynamics. John Wiley & Sons: 2006.
- [33] Klerk DD, Rixen DJ, Voormeeren SN. General framework for dynamic substructuring: history, review, and classification of techniques. AIAA Journal 2008; 46(5):1169-1181.
- [34] Farhat C, Roux FX. A method of finite element tearing and interconnecting and its parallel solution algorithm. International Journal for Numerical Methods in Engineering 1991; 32:1205-1227.

- [35] Park KC, Kim JG, Lee PS. A mode selection criterion based on flexibility approach in component mode synthesis. Proceeding 53th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Hawaii, USA, 2012.
- [36] Kim JG, Lee KH, Lee PS. Estimating relative eigenvalue errors in the Craig-Bampton method. Computers & Structures, 2014; 139:54-64.
- [37] Dvorkin EN, Bathe KJ. A continuum mechanics based four-node shell element for general nonlinear analysis. *Engineering Computations* 1984; 1(1):77-88.
- [38] Bathe KJ, Dvorkin EN. A formulation of general shell elements-the use of mixed interpolation of tensorial components. *International Journal for Numerical Methods in Engineering* 1986; 22(3):697-722.
- [39] Lee PS, Bathe KJ. The quadratic MITC plate and MITC shell elements in plate bending. Advances in Engineering Software 2010; 41(5):712-728.
- [40] Bathe KJ, Lee PS. Measuring the convergence behavior of shell analysis schemes. Computers & Structures 2011; 89(3-4):285-301.
- [41] Hurty WC. A criterion for selecting realistic natural modes of a structure. *Technical Memorandum*, Jet Propulsion Laboratory, Pasadena, CA, 1967; 33-364.
- [42] JG Kim, SH Boo, PS Lee. The enhanced AMLS method and its performance, *Computer Methods* in Applied Mechanics and Engineering, submitted.
- [43] Max A. Woodbury, Inverting modified matrices, *Memorandum Rept. 42*, Statistical Research Group, Princeton University, Princeton, NJ, 1950; pp. 4.
- [44] Collins JD, Hart GC, Hurty WC, Kennedy B. Review and development of modal synthesis techniques. *Technical Report*, J. H. Wiggins, May 1972.
- [45] Craig RR, Chang CJ. Substructure coupling for dynamic analysis and testing. NASA Contract Report CR-2781, 1977.
- [46] Yang C, Gao W, Bai Z, Li XS, Lee LQ, Husbands P, Ng E. An algebraic substructuring method for large-scale eigenvalue calculation. SIAM Journal on Scientific Computing 2005; 27:873-892.
- [47] Elssel K, Voss H. An a priori bound for automated multilevel substructuring. SIAM Journal on Matrix Analysis and Applications 2007; 28(2):386-397.
- [48] Blelloch PA, Carney KS. Selection of component modes. Proceeding 31th AIAA Structures, Structural Dynamics, and Materials Conference, Long Beach, CA, 1990.
- [49] Kammer DC, Triller MJ. Ranking the dynamic importance of fixed interface modes using a generalization of effective mass. The International Journal of Analytical and Experimental Modal Analysis 1994; 9(2):77-98.
- [50] Kammer DC, Triller MJ. Selection of component modes for Craig-Bampton substructure representations. Journal of Vibration and Acoustics 1996; 118(2):264-270.
- [51] Barbone PE, Givoli D, Patlashenko I. Optimal modal reduction of vibrating substructures. International Journal for Numerical Methods in Engineering 2003; 57:341-369.

- [52] Givoli D, Barbone PE, Patlashenko I. Which are the important modes of a subsystem?. International Journal for Numerical Methods in Engineering 2004; 59:1657-1678.
- [53] Liao B, Bai Z, Gao W. The important modes of subsystems: A moment-matching approach. International Journal for Numerical Methods in Engineering 2007; 70:1581-1597.
- [54] Moore BC. Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Transactions on Automatic Control* 1981; 26:17-32.
- [55] Hibbit, Karlsson & Sorensen. ABAQUS/Standard User's Manual, Version 6.3, 2003.
- [56] Henshell RD, Ong JH. Automatic masters from eigenvalues economisation. Earthquake engineering and structural dynamics 1975; 3:375-383.
- [57] Ong JH. Improved automatic masters for eigenvalues economisation. *Finite elements in analysis and design* 1987; 3(2):149-160.
- [58] Shah VN, Raymund M. Analytical selection of masters for the reduced eigenvalue problem. International Journal for Numerical Methods in Engineering 1982; 18(1):89-98.
- [59] Kim KO, Choi YJ. Energy method for selection of degrees of freedom in condensation. AIAA Journal 2000; 38(7):1253-1259.
- [60] Cho M, Kim H. Element-based node selection method for reduction of eigenvalue problems AIAA Journal 2004; 42(8):1677-1684.
- [61] Hughes TJR. The Finite Element Method: Linear Static and Dynamic Finite Element Analysis. Dover: Mineola, NY, USA, 2000.
- [62] Bourquin F. Analysis and comparison of several component mode synthesis methods on one dimensional domains. *Numerische Mathematik* 1990; 58(1):11-33.
- [63] Jakobsson H, Larson MG. A posteriori error analysis of component mode synthesis for the elliptic eigenvalue problem. Computer Methods in Applied Mechanics and Engineering 2011; 200:2840–2847.
- [64] Kim JG, Lee PS. An accurate error estimator for Guyan reduction. Computer Methods in Applied Mechanics and Engineering 2014; 278:1-19.
- [65] Kim JG, Lee PS. An error estimation method for the flexibility-based component mode synthesis method, AIAA journal, submitted.
- [66] Boo SH, Kim JG, Lee PS. Error estimation for the automated multi-level substructuring method, International Journal for Numerical Methods in Engineering, submitted.
- [67] Kim JG, Lee PS, Park KC. A mode selection method for structural component mode synthesis. International Journal of Numerical Methods in Engineering, submitted.
- [68] Kim JG, Lee PS. An enhanced Craig-Bampton method, International Journal for Numerical Methods in Engineering, submitted.

Summary

On the finite element model reduction methods in structural dynamics

축소기법(model reduction method)은 원래의 유한요소모델(global finite element model)이 갖는 자유도(DOFs)를 효과적으로 감소시켜 구조해석 및 설계에 소요되는 시간을 줄이고자 개발되었다. 축 소기법은 방법론에 따라 크게 자유도 기반 (DOFs based)과 모드 기반(mode based) 감소법으로 구분할 수 있다.

자유도 기반 축소기법은 원래 유한요소모델의 강성행렬과 질량행렬로 부터 주요한 자유도(master DOFs)를 제외한 나머지 자유도(slave DOFs)를 응축(condensation)하여 원래 유한요소모델에 근사한 축소모델(reduced model)을 구성하는 방법을 말한다. 1960년대에 Guyan 기법이 제시된 이후 최근에는 IRS(improved reduced system, IRS) 방법과 이에 반복적 알고리즘을 더한 I-IRS (iterative IRS) 기법이 가장 널리 쓰이고 있다. 자유도 기반 축소기법은 최적의 축소모델 구성에 적용될 뿐만 아니라, 동적거 동실험의 수치모델 구축, 센서의 최적 계측 위치, 구조물의 결함 측정 등에 다양하게 이용되고 있다.

모드 기반 축소기법은 개발 초기에 자유도 기반 축소기법에서 아이디어를 얻었으나, 부구조법 (substructuring)의 적용으로 자유도 기반 축소기법에 비해 향상된 성능을 보여주었다. 모드 기반 축소 기법은 응용수학 및 모달해석 분야 등에서 다양한 명칭으로 정의되지만, 본 연구에서는 전산구조동역학 분야에서 사용하는 부분구조합성법(component mode synthesis, CMS)을 사용하였다. 부분구조합성법 은 먼저 하나의 거대 유한요소모델을 여러 개의 다루기 쉬운 부구조(substructure)로 분할하여 이에 대한 고유치 해석을 수행한다. 이후 얻어진 주요 부구조 고유모드 만을 이용해 원래의 유한요소모델에 근사 한 축소모델을 구성하게 된다. 지난 50년간 다양한 형태의 부분구조합성법 관련 연구가 진행되었으며, 이 중 Craig-Bampton (CB) 기법, 유연도기반 부분구조합성법(flexibility based CMS, F-CMS) 등은 여러 연구를 통해 그 우수성이 입증되었다. 부분구조합성법을 이용하면 수치해석에 소요되는 시간을 줄일 수 있을 뿐만 아니라, 다양한 부구조물의 결합을 통해 제작되는 자동차, 비행기, 선박 등의 구조해석에 적합하기 때문에 연구 및 산업 현장에서 널리 사용되고 있다. 또한 최근에는 란초스(Lanczos) 및 부공 간 축차(subspace iteration) 알고리즘과 더불어 거대 고유치 문제를 효과적으로 풀 수 있는 방법론의 하나로 각광을 받고 있다.

유한요소모델 축소기법의 주요한 연구이슈는 다음과 같다.

- 최적의 축소기법은 축소모델을 구축 함에 있어서 우수한 정확성(accuracy)과 계산 효율성(computational efficiency)을 동시에 담보할 수 있어야 한다.
- 축소기법을 이용해 최적의 축소모델을 구축하기 위해서는 적절한 주요 자유도 혹은 주요 부구조
 모드를 선택할 수 있는 명확한 판단기준이 필요하다.
- 축소기법을 이용해 얻어진 축소모델의 신뢰성을 확보하기 위해서는 축소모델이 원래의 유한요소 모델에 얼마나 근사한지를 판단할 수 있는 명확한 오차추정기법이 요구된다.

본 연구에서는 이와 같은 주요 이슈들을 해결하기 위해 새로운 형태의 부분구조합성법, 모드선택 및 오차추정기법 등을 개발하였다.

먼저 최적의 축소기법을 개발하기 위해 CB 기법을 향상시킨 새로운 부분구조합성법을 제안하였다. 원래의 CB 기법은 주요 부구조 모드 만을 사용한 변환행렬(transformation matrix)을 이용해 축소모델을 구성한다. 하지만 이때 고려되지 않은 여분의 부구조 모드(residual mode)를 고려하면 원래의 변환행 렬의 보정이 가능하다. 여분의 부구조 모드에 의한 영향은 잔류 유연도(residual flexibility)로 표현이 되며, 잔류 유연도는 전체 유연도에서 주요 부구조 모드에 의해 계산되는 주요 유연도의 차로 손쉽게 계산이 가능하다. 따라서 새롭게 개발된 변환행렬을 이용하면 적은 추가 계산량 만으로도 축소모델의 정확성을 획기적으로 향상시킬 수 있다. 개발된 기법은 다양한 수치 예를 통해 그 성능을 검토하였으며, 기존의 CB 기법 및 F-CMS 기법과 비교해 우수성을 입증하였다.

두번째로 F-CMS 기법의 경계 자유도(interface boundary DOFs) 감소를 위한 정식을 개발하였 다. F-CMS 기법은 부구조간의 경계조건을 자유단(free interface boundary condition)으로 정의하기 때문에 완벽하게 독립된 병렬처리(parallel computing)가 가능하며, 모달해석의 실험 연동 등에 장점이 있다. 하지만 라그랑지승수(localized Lagrange multiplier)를 사용하여 구속조건(constraint condition) 을 정의하기 때문에 고정 경계조건(fixed interface boundary condition)을 사용하는 CB 기법 등에 비해 축소모델의 크기가 크게 증가하는 단점이 있다. 이를 해결하기 위해 본 연구에서는 F-CMS 기법의 경계 자유도를 효율적으로 감소시킬 수 있는 정식을 제시하였다. 제시된 방법론을 사용하면 기존의 F-CMS 기법에 의해 정확성을 유지하면서, 모델의 크기는 크게 감소된 축소모델의 구축이 가능하다.

세번째로 본 연구에서 부분구조합성법을 위한 모드선택기법을 개발하였다. 전통적인 모드선택기법 인 주파수 cut-off 방식은 부구조의 고유주파수(natural frequency) 크기를 이용해 주요 모드를 선택하게 된다. 하지만 이러한 방식의 모드선택기법은 부구조의 기하학적/물적 특성 및 부구조간의 연결방식 등을 적절하게 반영하지 못해 축소모델의 신뢰성을 떨어뜨리는 요인이 된다. 이를 해결하기 위해 본 연구에서는 원래의 유한요소모델과 부구조가 갖는 고유벡터의 관계성을 이용해 부구조 모드를 선택할 수 있는 기법을 제안하였다. 이를 이용하면 원래의 고유벡터에 크게 영향을 미치는 부구조 모드를 선택할 보할 수 있기 때문에 기존의 모드선택기법에 비해 원 고유벡터를 더 정확하게 근사할 수 있다. 개발된 모드선택기법은 다양한 수치 예를 통해 그 성능을 검토하였으며, 이전의 연구와 비교해 우수성을 입증 하였다. 이에 더해 개발된 모드선택기법을 이용하면 부분적으로 수치오차가 큰 모드를 예측할 수 있을 뿐만 아니라, 해당 부분의 고유치 해석 결과를 선별적으로 향상 시킬 수 있다. 본 연구에서는 이에 대한 방법론과 해석절차를 추가적으로 제시하였으며, 수치 예를 통해 적용 가능성과 성능을 검토하였다.

마지막으로 본 연구에서는 축소모델의 신뢰성을 명확하게 판단할 수 있는 오차추정기법을 개발 하였다. 원래 유한요소모델의 고유값과 고유벡터는 축소모델에서 얻어진 근사값과 오차로 표현할 수 있다. 이를 원래 유한요소모델의 고유치 문제에 대입하여 정리하면, 근사 고유값의 상대오차에 대한 근사식을 얻을 수 있다. 제시된 오차추정기법은 기존의 기법과 달리 직접적으로 각 모드 별 고유치 상대오차를 판별할 수 있을 뿐만 아니라, 별도의 계산 없이 효율적으로 오차를 추정할 수 있다는 장점이 있다. 개발된 오차추정기법을 실제 축소기법에 적용하기 위해서는, 원래의 것보다 더 정밀하게 근사된 변환행렬이 요구되기 때문에 축소기법의 방법론에 따라 각각 다른 형태의 오차추정식이 유도된다. 본 연구에서는 축소기법 중 Guyan 기법, CB 기법, 그리고 F-CMS 기법의 오차추정기법을 각각 개발하여, 그 성능을 검증하였다.