Development of the triangular shell finite elements for analysis of shell structures

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- 1. Introduction
- 2. MITC3-HR shell element
- 3. MITC3+ shell element
- 4. Mode analysis
- 5. Conclusions



1-1. Shell structures

- 3D structures with one dimension(thickness), small compared to the other two dimensions
- Curved geometry, light weight, hold applied loads very effectively



Nature







1-1. Shell structures

- Engineering applications : ship, car, airplane, etc.
- Finite Element Method has been dominantly used for the analysis of shells.
 - It is still a challenging issue to develop more reliable and effective shell finite elements for the analysis of general shell structures.





1-1. Shell structures

Shell behaviors



Asymptotic categories

- Three different asymptotic categories are shown with deceasing thickness, depending on the geometry, loading and boundary conditions.
 - Bending-dominated, membrane-dominated, mixed behaviors

•
$$\rho \cong \frac{\log E(\varepsilon + \Delta \varepsilon) - \log E(\varepsilon)}{\log(\varepsilon + \Delta \varepsilon) - \log \varepsilon}$$
 * *E*: strain energy, $\varepsilon = thickness/Length$

- ρ =1 (membrane-dominated), ρ =3 (bending-dominated), 1< ρ <3 (mixed)

Asymptotic categories

◇ Lee PS and Bathe KJ., Comput Struct 2002;80:235-55.
 ◇ Lee PS and Noh HC, Journal of KSCE 2007;27(3A): 277-89.

Shell problems	Gaussian curvature	Asymptotic behavior (ρ)
Fully clamped plate Scodelis-Lo roof shell	Zero Zero	Bending-dominated (ρ =3.0) Mixed (ρ =1.75)
Free cylindrical shell	Zero	Bending-dominated (ρ =3.0) Membrane-dominated (ρ =1.0)
Clamped hemispherical cap	Positive	Membrane-dominated (ρ =1.0) Membrane-dominated (ρ =1.0)
Free hyperboloid shell Fixed hyperboloid shell	Negative Negative	Bending-dominated (ρ =3.0) Membrane-dominated (ρ =1.0)



< Negative Gaussian curvature>



< Zero Gaussian curvature >



< Positive Gaussian curvature>

1-2. Shell elements

Flat shell elements

- Simply superimpose a plate bending stiffness and a plane stress membrane stiffness
 - Membrane behavior is not coupled with the plate bending behavior.

Continuum mechanics based shell elements (degenerated shell elements)

- Degenerated from 3 dimensional solid element
 - Bending, membrane and transverse shearing behaviors are fully coupled.
 - Formulation is straightforward for both linear and nonlinear analyses.
 - Dominantly used in most commercial FE software

♦ AHMAD S et al., Analysis of thick and thin shell structures by curved finite elements. Int J Numer Methods Eng 1970;2:419-51.

< 3-node flat shell elements>



< 3-node continuum mechanics based shell elements >



1-3. Locking

- If the finite element discretization cannot accurately approximate the pure bending displacement fields of shells, the solution accuracy deteriorates in bending dominated and mixed shell behaviors
- Locking phenomenon can be severe when the shell thickness decreases



• **DISP3** : 3-node displacement-based triangular element



In the bending-dominated shell problem Strain energy using 16×16 meshes

t/L	DISP4	MITC4
1/100	3.2272E-08	4.5079E-07
1/1,000	3.4363E-07	4.4696E-04
1/10,000	3.4385E-06	4.4692E-01
Order of change	t/L	$(t/L)^3$

Locking

Locking categories

Shell element	Shear locking	Membrane locking	Remark
3-node	\checkmark	-	Flat geometry
4-node, 6-node, 9-node, 16-node	\checkmark	\checkmark	

1-3. Locking

Locking treatment

- Reduced Integration \Rightarrow Spurious energy modes
- Enhanced Assumed Strain method \Rightarrow Unknowns are added
 - * Static condensation on element level
 - ◇ Simo JC and Rifai MS, Int J Numer Methods Eng 1990;29:1595-638
 - ◇ Ramm E and Andelfiger U, Int J Numer Methods Eng 1993;36:1311-37
- Assumed Strain method (ANS, MITC method)

* No spurious energy mode, simple formulation

- ♦ Dvorkin EN and Bathe KJ, Eng Comput 1984;1:77-88
- ◇ Park KC and Stanley GM, J Appl Mech 1986;53:278-90
- < 4-node quadrilateral shell element (MITC4) >



1-4. MITC shell elements

- Quadrilateral shell elements : MITC4, MITC9, MITC16
- Triangular shell elements : MITC3, MITC6









◇ Lee PS et al., Comput Struct 2004;82:945-62.
 ◇ Kim DN et al., Comput Struct 2009;87:1451-60.

Isotropic triangular shell elements

Isotropic element :

Response is independent of node numbering sequences (3-node : $1 \rightarrow 2 \rightarrow 3$, $2 \rightarrow 3 \rightarrow 1$, $3 \rightarrow 1 \rightarrow 2$)

1-5. Triangular mesh

- Relatively easy to generate meshes even for complex shell analyses
- 3-node shell element is so effective. \Rightarrow Auto mesh generation, computational efficiency









1-6. 3-node triangular shell element



In the bending-dominated shell problem using the isotropic shell elements

- **DISP3** : 3-node displacement-based triangular element
- MITC3 : 3-node MITC triangular element
- MITC4 : 4-node MITC quadrilateral element

< Convergence studies >



Performance of shell elements DISP3 < MITC3 < MITC4

Currently, no available

"Effective 3-node continuum mechanics based shell element"

Purpose

Development of the effective 3-node shell finite element



Requirements

- No spurious zero energy mode
- Spatially isotropic behavior
- Pass the patch tests
- Reliable results for membrane and bending dominated shell problems

Zero energy mode test

• The unsupported single shell element should pose six zero eigenvalues, corresponding to the six physical rigid body modes

Isotropic element test

- The response of the single shell finite element should be the same regardless of the node numbering
 - Different sequences of node numberings : $1 \rightarrow 2 \rightarrow 3$, $2 \rightarrow 3 \rightarrow 1$, $3 \rightarrow 1 \rightarrow 2$



Patch tests

- The minimum number of degrees of freedom is constrained to prevent rigid body motions
- Nodal forces that should result in constant stress conditions are applied
- The constant stress should be calculated to pass the patch test



< Triangular mesh used for the patch tests >

Convergence studies Reliable results

- Bending dominated shell problems
 - Fully clamped plate, 60° skew plate, free cylindrical shell, free hyperboloid shell
- Membrane dominated shell problems
 - Clamped cylindrical shell, clamped hyperboloid shell



Benchmark problems

- A mesh of 96 \times 96 element mesh of MITC9 shell elements

	Boundary condition	Asymptotic behavior	t/L	Strain energy ratio (%)		
Benchmarks				Bending	Membrane	Transverse shear
Cylindrical shell	Fixed-Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	1.94 0.32 0.88	98.02 99.68 99.92	0.03 0.00 0.00
	Free-Free	Bending- dominated	1/100 1/1,000 1/10,000	99.77 99.93 99.98	0.22 0.07 0.02	0.01 0.00 0.00
Fixe Hyperboloid shell Fre	Fixed-Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	4.16 1.16 0.35	95.78 98.84 99.65	0.06 0.00 0.00
	Free-Free	Bending- dominated	1/100 1/1,000 1/10,000	99.11 99.99 100.0	0.83 0.01 0.00	0.06 0.00 0.00

1. Introduction

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2-1. MITC3 shell element



Geometry interpolation

$$\vec{x}(r,s,t) = \sum_{i=1}^{3} h_i(r,s)\vec{x}_i + \frac{t}{2}\sum_{i=1}^{3} a_i h_i(r,s)\vec{V}_n^i$$

where $h_1 = 1 - r - s$, $h_2 = r$, $h_3 = s$

Displacement interpolation

$$\vec{u}(r,s,t) = \sum_{i=1}^{3} h_i(r,s)\vec{u}_i + \frac{t}{2}\sum_{i=1}^{3} a_i h_i(r,s) \left(-\vec{V}_2^i \alpha_i + \vec{V}_1^i \beta_i\right)$$

Assumed transverse shear strain field

$$\widetilde{e}_{rt}^{MITC3} = e_{rt}^{(1)} + cs, \quad \widetilde{e}_{st}^{MITC3} = e_{st}^{(2)} - cr$$

where

$$c = (e_{rt}^{(3)} - e_{rt}^{(1)}) - (e_{st}^{(3)} - e_{st}^{(2)})$$

2-2. MITC3-HR shell element

Hellinger-Reissner functional for 3-node shell elements

$$\Pi_{HR}^{S} = \int_{V} \left(\frac{1}{2}\vec{\varepsilon}_{bm}^{T}C_{bm}\vec{\varepsilon}_{bm} - \frac{1}{2}\vec{\gamma}^{A^{T}}C_{\gamma}\vec{\gamma}^{A} + \vec{\gamma}^{A^{T}}C_{\gamma}\vec{\gamma}\right)dV - \int_{V}\vec{u}^{T}\vec{f}^{B}dV + \text{boundary terms}$$

$$\vec{\varepsilon}_{bm} = \begin{pmatrix} \varepsilon_{\bar{r}\bar{r}} \\ \varepsilon_{\bar{s}\bar{s}} \\ 2\varepsilon_{\bar{r}\bar{s}} \end{pmatrix} = \boldsymbol{B}_{bm}\vec{U} \qquad \vec{\gamma} = \begin{pmatrix} 2\varepsilon_{\bar{s}\bar{t}} \\ 2\varepsilon_{\bar{r}\bar{t}} \end{pmatrix} = \boldsymbol{B}_{\gamma}\vec{U} \qquad \vec{\gamma}^{A} = \begin{pmatrix} 2\varepsilon_{\bar{s}\bar{t}} \\ 2\varepsilon_{\bar{r}\bar{t}} \end{pmatrix} = \boldsymbol{B}_{\gamma}^{A}\vec{\alpha}$$

$$\vec{\varepsilon}_{bm} \quad \text{: Bending and membrane strains from the 3-node displacement-based shell formulation}$$

$$\vec{\gamma} \quad \text{: Transverse shear strains from the 3-node displacement-based shell formulation}$$

$$\vec{\gamma}^{A} \quad \text{: Approximated transverse shear strains with unknowns}$$

Modified Hellinger-Reissner functional for the MITC3-HR shell element

$$\Pi_{HR}^{M} = \int_{V} \left(\frac{1}{2}\vec{\varepsilon}_{bm}^{MITC3^{T}}C_{bm}\vec{\varepsilon}_{bm}^{MITC3} - \frac{1}{2}\vec{\gamma}^{A^{T}}C_{\gamma}\vec{\gamma}^{A} + \vec{\gamma}^{A^{T}}C_{\gamma}\vec{\gamma}^{MITC3}\right)dV - \int_{V}\vec{u}^{T}\vec{f}^{B}dV + \text{boundary terms}$$

$$\vec{\varepsilon}_{bm}^{MITC3} = \begin{pmatrix} \varepsilon_{\bar{r}\bar{r}}^{M} \\ \varepsilon_{\bar{s}\bar{s}}^{M} \\ 2\varepsilon_{\bar{r}\bar{s}}^{M} \end{pmatrix} = \boldsymbol{B}_{bm}^{MITC3}\vec{U} \qquad \vec{\gamma}^{MITC3} = \begin{pmatrix} 2\varepsilon_{\bar{s}\bar{t}}^{M} \\ 2\varepsilon_{\bar{r}\bar{t}}^{M} \end{pmatrix} = \boldsymbol{B}_{\gamma}^{MITC3}\vec{U} \qquad \vec{\gamma}^{A} = \begin{pmatrix} 2\varepsilon_{\bar{s}\bar{t}}^{A} \\ 2\varepsilon_{\bar{r}\bar{t}}^{A} \end{pmatrix} = \boldsymbol{B}_{\gamma}^{A}\vec{\alpha}$$

$$\vec{\varepsilon}_{bm}^{MITC3}$$
: Bending and membrane strains from the MITC3 shell formulation

 $\vec{\gamma}^{MITC3}$: Transverse shear strains from the **MITC3 shell formulation**

 $\vec{\gamma}^A$

: Approximated transverse shear strains introduced on the rotated contravariant base vectors



Shell elements

- MITC3 : 3-node triangular shell element
- MITC3-HR : 3-node triangular shell element based on the Hellinger-Reissner principle
- MITC4 : 4-node quadrilateral shell element

Results of basic tests

Shell element	Zero energy mode test	Isotropic test	Patch test
MITC3	Pass	Pass	Pass
MITC3-HR	Pass	Pass	Pass
MITC4	Pass	Pass	Pass

2-4. Convergence studies

S-norm for convergence studies

Relative error
$$E_h = \frac{\left\|\vec{u}_{ref} - \vec{u}_h\right\|_s^2}{\left\|\vec{u}_{ref}\right\|_s^2}$$
 $\left\|\vec{u}_{ref} - \vec{u}_h\right\|_s^2 = \int_{\Omega_{ref}} \Delta \vec{\varepsilon}^T \Delta \vec{\tau} d\Omega_{ref}$
where
 $\Delta \vec{\varepsilon} = \vec{\varepsilon}_{ref} - \vec{\varepsilon}_h$ $\Delta \vec{\tau} = \vec{\tau}_{ref} - \vec{\tau}_h$ $\vec{x}_{ref} = \Pi_{ref}(\vec{x}_h)$
 \vec{u}_{ref} : the reference solution obtained by a very fine mesh practically
(a mesh of 96×96 MITC9 shell elements)
 \vec{u}_h : the solution of the finite element discretization with *N*×*N* meshes
 $(N = 4, 8, 16, 32 \text{ and } 64)$
 Π : One-to-one mapping
For optimal convergence behavior for a 3-node shell element

 $E_h \cong Ch^k$

h is the element size, *C* must be constant, k=2

2-4. Convergence studies

• Hyperboloid shell problems



$$x^{2} + z^{2} = 1 + y^{2}; y \in [-1, 1]$$
 (a)









•
$$E = 2.0 \times 10^{11}$$
, $v = 1/3$

- Clamped-clamped :

membrane-dominated

- Free-Free :

bending dominated

(b)

Clamped hyperboloid shell problem / uniform mesh



* The bold line represents the optimal convergence rate.

• Free hyperboloid shell problem / uniform mesh



* The bold line represents the optimal convergence rate.

• Free hyperboloid shell problem / uniform mesh



2-4. Convergence studies

Clamped hyperboloid shell problem / distorted mesh



* The bold line represents the optimal convergence rate.

• Free hyperboloid shell problem / distorted mesh



* The bold line represents the optimal convergence rate.

MITC3-HR shell element

Summary of convergence studies using the s-norm

Benchmarks	Boundary condition	Shell behavior	Mesh pattern	Performance
Square plate	Fully clamped	Bending-dominated	Uniform	MITC3 < MITC3-HR < MITC4
			Distorted	MITC3 < MITC3-HR < MITC4
60° skew plate	Simply supported	Bending-dominated	Distorted	MITC3 < MITC3-HR < MITC4
Cylindrical shell	Clamped-Clamped	Membrane- dominated	Uniform	MITC3 \approx MITC3-HR \approx MITC4
			Distorted	$\textbf{MITC3} \approx \textbf{MITC3-HR} \approx \textbf{MITC4}$
	Free-Free	Bending-dominated	Uniform	$\textbf{MITC3} \approx \textbf{MITC3-HR} \approx \textbf{MITC4}$
			Distorted	MITC3 < MITC4 < MITC3-HR
Hyperboloid shell	Clamped-Clamped	Membrane- dominated	Uniform	MITC3 \approx MITC3-HR \approx MITC4
			Distorted	MITC3 \approx MITC3-HR \approx MITC4
	Free-Free	Bending-dominated	Uniform	MITC3 < MITC3-HR < MITC4
			Distorted	MITC3 < MITC4 < MITC3-HR

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Improving the MITC3 shell finite element by using the Hellinger-Reissner principle

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ABSTRACT

The objective of this study is to improve the performance of the MITC3 shell finite element. The Hellinger–Reissner (HR) variational principle is modified in the framework of the MITC method, and a special approximated transverse shear strain field is proposed. The MITC3-HR shell finite element improved by using the Hellinger–Reissner functional passes all the basic tests (zero energy mode test, patch test, and isotropic element test). Convergence studies considering a fully clamped plate problem, a sixty-degree skew plate problem, cylindrical shell problems, and hyperboloid shell problems demonstrate the improved predictive capability of the new 3-node shell finite element.

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1. Introduction

Shell structures have been widely used in many engineering applications, and the finite element method has been dominantly adopted for the analysis of shells. Since the effectiveness of a shell finite element analysis depends highly on the predictive capability The MITC3 triangular shell finite element shows much better predictive capability than the displacement-based 3-node triangular shell finite elements and other 3-node isotropic triangular shell finite elements [15,18]. However, the locking alleviation by MITC3 is not as large as that by MITC4; that is, the accuracy of the solutions is not as good as that of the MITC4 quadrilateral shell finite

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3-1. MITC3+ shell element



- The cubic bubble node for the rotations is positioned on the flat geometry
 - Enrich the bending displacements
 - Membrane locking is not present, effective in geometrically nonlinear analysis
 - Providing compatibility with a 4-node shell element
 - Static condensation can be carried out on the element level for the bubble node

New assumed transverse shear strain field

 $\hat{e}_{rt} = \hat{e}_{rt}^{const.} + \hat{e}_{rt}^{linear} \qquad \hat{e}_{st} = \hat{e}_{st}^{const.} + \hat{e}_{st}^{linear}$

3-1. MITC3+ shell element



Geometry interpolation

$$\vec{x}(r,s,t) = \sum_{i=1}^{3} h_i(r,s)\vec{x}_i + \frac{t}{2}\sum_{i=1}^{4} a_i f_i(r,s)\vec{V}_n^i$$

with $a_4\vec{V}_n^4 = \frac{1}{3}(a_1\vec{V}_n^1 + a_2\vec{V}_n^2 + a_3\vec{V}_n^3),$
 $f_4 = 27rs(1-r-s),$
 $f_1 = h_1 - \frac{1}{3}f_4, \quad f_2 = h_2 - \frac{1}{3}f_4, \quad f_3 = h_3 - \frac{1}{3}f_4$

Displacement interpolation

$$\vec{u}(r,s,t) = \sum_{i=1}^{3} h_i(r,s)\vec{u}_i + \frac{t}{2}\sum_{i=1}^{4} a_i f_i(r,s) \left(-\vec{V}_2^i \alpha_i + \vec{V}_1^i \beta_i\right)$$

New assumed transverse shear strain field

- No spurious zero energy mode, isotropic, pass patch tests and presents excellent convergence behavior
 - Tying points for the covariant transverse shear strains should be inside of the element
 - The stiffness of in-plane twisting mode must be reduced


3-1. MITC3+ shell element





New tying scheme Along edge directions



Constant transverse shear strain field

$$\hat{e}_{rt}^{const.} = \hat{e}_{rt} = \frac{2}{3} (e_{rt}^{(B)} - \frac{1}{2} e_{st}^{(B)}) + \frac{1}{3} (e_{rt}^{(C)} + e_{st}^{(C)})$$
$$\hat{e}_{st}^{const.} = \hat{e}_{st} = \frac{2}{3} (e_{st}^{(A)} - \frac{1}{2} e_{rt}^{(A)}) + \frac{1}{3} (e_{rt}^{(C)} + e_{st}^{(C)})$$

3-1. MITC3+ shell element



Shell elements

3-node elements	Remark	Transverse shear strains
MITC3i	Standard interpolation function	New assumed transverse shear strain field
MITC3+	Cubic bubble function on rotations	New assumed transverse shear strain field

Results of basic tests

Shell element	Zero energy mode test	Isotropic test	Patch test
MITC3i	Pass	Pass	Pass
MITC3+	Pass	Pass	Pass

• Fully clamped plate problem under uniform pressure



• Fully clamped plate problem under uniform pressure



* The bold line represents the optimal convergence rate. The solid and dotted lines correspond to the results obtained by the mesh patterns in (a) and (b), respectively.

• Hyperboloid shell problems



$$x^{2} + z^{2} = 1 + y^{2}; y \in [-1, 1]$$









$$E = 2.0 \times 10^{11}, v = 1/3$$

- Clamped-clamped :

membrane-dominated

- Free-Free :

bending dominated

(b)

(a)

Clamped hyperboloid shell problem / uniform mesh



* The bold line represents the optimal convergence rate.

• Free hyperboloid shell problem / uniform mesh



* The bold line represents the optimal convergence rate.

Relative error using 16×16 meshes

t/L	MITC3	MITC3+	MITC4
1/1,000	74.24%	0.47%	0.22%
1/10,000	99.65%	0.48%	0.22%

Clamped hyperboloid shell problem / distorted mesh



* The bold line represents the optimal convergence rate.

• Free hyperboloid shell problem / distorted mesh



* The bold line represents the optimal convergence rate.

Relative error using 16×16 meshes

t/L	MITC3	MITC3+	MITC4
1/1,000	91.94%	5.02%	91.11%
1/10,000	99.9%	5.59%	99.91%

Strain energy ratio / uniform mesh

- A mesh of 16×16 element mesh of **MITC3+** shell elements

					:	Strain ener	gy ratio (9	%)	
Benchmarks	Boundary condition	Asymptotic behavior	t / L	t / L Bendir		Bending Membra		Transve shea	erse r
				MITC3+	Ref.	MITC3+	Ref.	MITC3+	Ref.
Cylindrical shell	Fixed- Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	1.89 0.21 0.19	1.94 0.32 0.88	98.05 99.76 99.77	98.02 99.68 99.92	0.06 0.03 0.03	0.03 0.00 0.00
	Free- Free	Bending- dominated	1/100 1/1,000 1/10,000	99.76 99.98 100.0	99.77 99.93 99.98	0.23 0.02 0.00	0.22 0.07 0.02	0.01 0.00 0.00	0.01 0.00 0.00
Hyperboloid shell	Fixed- Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	3.72 0.90 0.27	4.16 1.16 0.35	96.21 99.07 99.69	95.78 98.84 99.65	0.08 0.03 0.04	0.06 0.00 0.00
	Free- Free	Bending- dominated	1/100 1/1,000 1/10,000	99.17 99.99 99.99	99.11 99.99 100.0	0.81 0.01 0.00	0.83 0.01 0.00	0.03 0.00 0.01	0.06 0.00 0.00

* Reference solutions are obtained by the 96×96 element mesh of MITC9 shell elements.

Strain energy ratio / distorted mesh

- A mesh of 16×16 element mesh of **MITC3+** shell elements

					:	Strain ener	gy ratio (9	%)	
Benchmarks	Boundary condition	Asymptotic behavior	t / L	Bend	ing	Memb	orane	Transve shea	erse r
				MITC3+	Ref.	MITC3+	Ref.	MITC3+	Ref.
Cylindrical shell	Fixed- Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	2.94 0.66 0.74	1.94 0.32 0.88	96.67 98.89 98.82	98.02 99.68 99.92	0.40 0.45 0.44	0.03 0.00 0.00
	Free- Free	Bending- dominated	1/100 1/1,000 1/10,000	98.80 99.61 99.53	99.77 99.93 99.98	1.13 0.39 0.39	0.22 0.07 0.02	0.06 0.00 0.08	0.01 0.00 0.00
Hyperboloid shell	Fixed- Fixed	Membrane- dominated	1/100 1/1,000 1/10,000	3.38 0.59 0.65	4.16 1.16 0.35	96.41 99.24 99.18	95.78 98.84 99.65	0.21 0.17 0.17	0.06 0.00 0.00
	Free- Free	Bending- dominated	1/100 1/1,000 1/10,000	98.94 99.85 99.75	99.11 99.99 100.0	0.91 0.15 0.04	0.83 0.01 0.00	0.15 0.01 0.22	0.06 0.00 0.00

* Reference solutions are obtained by the 96×96 element mesh of MITC9 shell elements.

MITC3+ shell element

Summary of convergence studies using the s-norm

Benchmarks	Boundary condition	Shell behavior	Mesh pattern	Performance
Cauara plata	Fully clampad	Reading dominated	Uniform	MITC3 < MITC3+ \approx MITC4
Square plate	Square plate in uny clamped	Bending-dominated	Distorted	MITC3 < MITC3+ \approx MITC4
60° skew plate	Simply supported	Bending-dominated	Distorted	MITC3 < MITC3+ \approx MITC4
	Clamped-	Membrane-	Uniform	MITC3 \approx MITC3+ \approx MITC4
Cylindrical	Clamped	dominated	Distorted	MITC3 \approx MITC3+ \approx MITC4
shell	Free-Free	Donding dominated	Uniform	MITC3 \approx MITC3+ \approx MITC4
		Bending-dominated	Distorted	MITC3 < MITC4 < MITC3+
	Clamped-	Membrane-	Uniform	MITC3 \approx MITC3+ \approx MITC4
Hyperboloid	Clamped	dominated	Distorted	MITC3 \approx MITC3+ \approx MITC4
shell	Free Free		Uniform	MITC3 < MITC3+ \approx MITC4
	Free-Free	Bending-dominated	Distorted	MITC3 < MITC4 < MITC3+

3-node shell elements

Shell elements	DOFs	Membrane Strains	Bending strains	Bending strains Transverse shear strains	
MITC3+	15	Displacement-based	Displacement-based	Assumed strain	Static condensation
A3 (ANSYS)	18	Reduced Integration (stabilization scheme)	Reduced Integration (stabilization scheme)	Assumed strain	Collapsed (4node→3node)
S3 (ABAQUS)	18	• Enhanced Assumed Strain	Displacement-based	Reduced Integration (stabilization scheme)	Collapsed (4node→3node)
TRIC	18	Axial strain modesDrilling rotational modes	 Symmetric bending modes Antisymmetric bending + shear modes 		Natural mode method

3-4. Classical convergence studies

• Fully clamped plate problem under uniform pressure



- $\mathsf{Displacement}(\mathsf{w}_\mathsf{C})$ is normalized by the analytical solution





3-4. Classical convergence studies

• Pinched cylindrical shell with rigid diagrams

- R = L = 1.0, $E = 3.0 \times 10^7$, t = 0.01, v = 0.3
- $Displacement(w_C)$ is normalized by the analytical solution





32×32 meshes of MITC3+

Shell behavior	Strain energy ratio (%)							
	Bending		Membrane		Transverse shear			
	MITC3+	Ref.	MITC3+	Ref.	MITC3+	Ref.		
Mixed	60.60	61.40	38.76	37.35	0.63	1.25		

* Reference solutions are obtained by the 96×96 element mesh of MITC9 shell elements.

3-4. Classical convergence studies

• Hemispherical shell subjected to alternating radial forces

- $R = 10.0, E = 6.825 \times 10^7, t = 0.04, v = 0.3$
- $Displacement(u_A)$ is normalized by the analytical solution



16×16 meshes of MITC3+

Shell behavior	Strain energy ratio (%)							
	Bending		Membrane		Transverse shear			
	MITC3+	Ref.	MITC3+	Ref.	MITC3+	Ref.		
Bending- dominated	99.44	99.18	0.54	0.71	0.01	0.11		

* Reference solutions are obtained by the 96×96 element mesh of MITC9 shell elements.

3-5. Geometric nonlinear analysis

※ Total Lagrangian formulation (large displacements and large rotations)

- Same discretization assumptions employed in the linear formulation
- Collaborated with Jeon HM



< Cantilever plate under end moment >



< Hemispherical shell subjected to alternating radial forces >



< Clamped semi-cylindrical shell under point load >



< Slit annular plate under end shear force >

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The MITC3+ shell element and its performance

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encompassing convergence tests.

ARTICLE INFO

ABSTRACT

Article history: Received 2 January 2014 Accepted 22 February 2014 Available online 19 March 2014 In this paper, we present an effective new 3-node triangular shell finite element, called the MITC3+ element. The new shell element is based on the concepts earlier published for the MITC3 shell element (Lee and Bathe, 2004) [1] but is enriched by a cubic bubble function for the rotations. A new assumed transverse shear strain field is developed for the element. The shell element passes the three basic tests (the isotropy, patch and zero energy mode tests) and shows excellent convergence behavior in basic and

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1. Introduction

Keywords:

Shell structures

3-Node element

MITC method

Convergence

Shell finite elements

During the last decades, the finite element method has been widely used for the analysis of shell structures. However, although a great effort has been expended to develop an effective 3-node 3-node triangular MITC3 element is useful, the element is not optimal in its convergence behaviors [1,11].

An important point is that to generate meshes for a triangular shell element is relatively easy, even for complex shell analyses. In addition, an effective 3-node shell element would be attractive The MITC3+ shell element in geometric nonlinear analysis Hyeong-Min Jeon^a Youngyu Lee^a Phill-Seung Lee^{a,*}, Klaus-Jürgen Bathe^b

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ABSTRACT

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Keywords: Shell element 3-node triangular element Mixed finite element method MITC method Geometric nonlinear analysis Large displacements and rotations In this paper, we present the MITC3+ shell finite element for geometric nonlinear analysis and demonstrate its performance. The MITC3+ shell element, recently proposed for linear analysis [1], represents a further development of the MITC3 shell element. The total Lagrangian formulation is employed allowing for large displacements and large rotations. Considering several analysis problems, the nonlinear solutions using the MITC3+ shell element are compared with those obtained using the MITC3 and MITC4 shell elements. We conclude that the MITC3+ shell element shows, in the problems considered, the same excellent performance in geometric nonlinear analysis as already observed in linear analysis. © 2014 Elsevier Ltd. All rights reserved.

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1. Introduction

Due to significant efforts over the past decades, the finite element method has become a powerful tool for the linear and nondue to the highly sensitive and complex behavior of shell structures (categorized as bending dominated, membrane dominated and mixed behaviors), in particular, when the shell thickness is small [3,4]. Then a shell finite element discretization frequently

- 1. Introduction
- 2. MITC3-HR shell element
- 3. MITC3+ shell element

4. Mode analysis

5. Conclusions

4-1. Single right-angled triangular element



t = 1/10,000

v = 0.3

Shell elements

- **DISP3** : 3-node displacement-based triangular element
- MITC3 : 3-node MITC triangular element
- MITC3+ : 3-node MITC triangular element enriched by bubble function on rotations

Eigenvalues Static mode analysis									
Mode	DISP3		MITC3		MITC3+				
7	2.8000E+01	Т	6.6764E-07	B1	6.6685E-07	B1			
8	2.8000E+01	B1	8.1455E-07	B2	7.9621E-07	B2			
9	2.8000E+01	B2	2.4924E-06	B3	2.4921E-06	B3			
10	2.8000E+01	B3	3.6928E+01	Т	8.3107E-06	B1 +			
11	4.4800E+02	S1	4.6707E+02	S1	1.3599E-05	Т			
12	8.3813E+02	М	8.3813E+02	М	1.4128E-05	B2+			
13	1.1200E+03	S2	1.1760E+03	S2	4.6667E+02	S1			
14	1.3440E+03	М	1.3440E+03	М	8.3813E+02	М			
15	3.0019E+03	М	3.0019E+03	М	1.1760E+03	S2			
16	-		-		1.3440E+03	М			
17	-		-		3.0019E+03	М			

B: Bending modes, T: In-plane twisting mode,

S: Transverse shearing modes, M: Membrane modes,

B+: Bending modes due to the bubble function enrichment.

4-1. Single right-angled triangular element

• MITC3 shell element



Strain fields

	Bending mode B1	Bending mode B2	Bending mode B3
Eigenvalue	6.6764E-07	8.1455E-07	2.4924E-06
Bending strains	$\varepsilon_{xx} = 0.269z$	$\varepsilon_{xx} = -0.853z$	$\varepsilon_{xx} = 0.963z$
	$\varepsilon_{yy} = 0.269z$	$\varepsilon_{yy} = 0.853z$	$\varepsilon_{yy} = 0.963z$
	$\gamma_{xy} = -1.36z$	$\gamma_{xy} = 0.0$	$\gamma_{xy} = -1.42z$
Transverse shear	$\gamma_{xz} = 0.0$	$\gamma_{xz} = 0.0$	$\gamma_{xz} = 0.0$
strains	$\gamma_{yz} = 0.0$	$\gamma_{yz} = 0.0$	$\gamma_{yz} = 0.0$

4-1. Single right-angled triangular element

• MITC3+ shell element



Section

Strain fields

	Bending mode B1+	Bending mode B2+	In-plane twisting mode (d=1/10,000)
Eigenvalue	8.3107E-06	1.4128E-05	1.3599 E-05
Bending strains	$\varepsilon_{xx} = [0.019 - 21.3(s - 2rs - s^2)]z$ $\varepsilon_{yy} = [-0.019 + 21.3(r - 2rs - r^2)]z$ $\gamma_{xy} = -21.3(r - s - r^2 + s^2)z$	$\varepsilon_{xx} = [-0.001 - 21.8(s - 2rs - s^2)]z$ $\varepsilon_{yy} = [-0.001 - 21.8(r - 2rs - r^2)]z$ $\gamma_{xy} = [-0.064 - 21.8(r + s - 4rs - r^2 - s^2)]z$	$\varepsilon_{xx} = [-0.514 + 1.49(s - 2rs - s^2)]z$ $\varepsilon_{yy} = [0.514 - 1.49(r - 2rs - r^2)]z$ $\gamma_{xy} = 1.49(r - s - r^2 + s^2)z$
Transverse shear strains	$\gamma_{xz} = 0.0$ $\gamma_{yz} = 0.0$	$\begin{aligned} \gamma_{xz} &= 0.0 \\ \gamma_{yz} &= 0.0 \end{aligned}$	$\gamma_{xz} = 1.99 d (1 - 3s)$ $\gamma_{yz} = 1.99 d (-1 + 3t)$

4-2. Assemblage of two right-angled triangular elements



$E = 1.7472 \times 10^{7}$
t = 1/10,000
<i>υ</i> = 0.3

Eigenvalues

9						
Mode	MITC4		MITC3		MITC3+ (d=1/10,000)	
7	7.2000E-07	BL1	9.9556E-07	BC1	9.3805E-07	BC1
8	7.2000E-07	BL2	1.1200E-06	BC2	1.0608E-06	BC2
9	9.9556E-07	BC1	2.0800E-06	BC3	1.9629E-06	BC3
10	1.1200E-06	BC2	3.2000E-06	BL2	3.0544E-06	BL2
11	2.0800E-06	BC3	3.4167E+01	BL1	8.9316E-06	BQ1+
12	5.6000E+01	ΤQ	5.6000E+01	ΤQ	1.1912E-05	BQ2+
13	5.0400E+02	BC4	8.4000E+02	SQ1	1.4173E-05	TQ
14	8.4000E+02	SQ1	9.1783E+02	SQ2	1.5159E-05	BL1
15	8.4000E+02	SQ2	1.3440E+03	MQ	1.6660E-05	BQ3+
16	8.6400E+02	MQ	1.3440E+03	MQ	9.3333E+01	BQ4+
17	8.6400E+02	MQ	1.3440E+03	MQ	8.0267E+02	SQ1
18	1.3440E+03	MQ	1.5120E+03	BC4	8.4000E+02	SQ2
19	1.3440E+03	MQ	2.4960E+03	MQ	1.3440E+03	MQ
20	2.4960E+03	MQ	3.8400E+03	MQ	1.3440E+03	MQ
21	-		-		1.3440E+03	MQ
22	-		-		1.5493E+03	BC4
23	-		-		2.4960E+03	MQ
24	-		-		3.8400E+03	MQ

BC : Bending modes with constant bending strain fields

BL : Bending modes with linear bending strain fields

TQ : In-plane twisting mode, SQ: Transverse shearing modes

MQ : Membrane modes

BQ+ : Bending modes due to the bubble function enrichment

4-2. Assemblage of two right-angled triangular elements



Eigenvalues (MITC3+)

Mode	d=1/10	0	d=1/1,000		d=1/100,000		d=0	
7	9.3805E-07	BC1	9.3805E-07	BC1	9.3805E-07	BC1	9.3805E-07	BC1
8	1.0611E-06	BC2	1.0611E-06	BC2	1.0545E-06	BC2	1.0099E-06	BL1
9	1.9629E-06	BC3	1.9629E-06	BC3	1.1303E-06	BL1	1.0522E-06	BC2
10	3.0544E-06	BL2	3.0544E-06	BL2	1.5394E-06	τQ	1.4140E-06	τQ
11	9.7882E-06	BQ1+	9.7846E-06	BQ1+	1.9629E-06	BC3	1.9629E-06	BC3
12	1.1912E-05	BQ2+	1.1912E-05	BQ2+	3.0544E-06	BL2	3.0544E-06	BL2
13	1.6660E-05	BQ3+	1.6660E-05	BQ3+	1.0320E-05	BQ1+	1.0312E-05	BQ1+
14	1.2767E-01	τQ	1.2782E-03	τQ	1.1912E-05	BQ2+	1.1912E-05	BQ2+
15	1.2768E-01	BL1	1.2783E-03	BL1	1.6660E-05	BQ3+	1.6660E-05	BQ3+
16	9.3340E+01	BQ4+	9.3333E+01	BQ4+	9.3333E+01	BQ4+	9.3333E+01	BQ4+
17	8.0267E+02	SQ1	8.0267E+02	SQ1	8.0267E+02	SQ1	8.0267E+02	SQ1
18	8.4001E+02	SQ2	8.4000E+02	SQ2	8.4000E+02	SQ2	8.4000E+02	SQ2
22	1.5493E+03	BC4	1.5493E+03	BC4	1.5493E+03	BC4	1.5493E+03	BC4

BC: Bending modes with constant bending strain fields

BL: Bending modes with linear bending strain fields

TQ: In-plane twisting mode, SQ: Transverse shearing modes

BQ+: Bending modes due to the bubble function enrichment

for one element \downarrow

4-2. Assemblage of two right-angled triangular elements

• Bending mode BL1







Strain fields

	MITC4	MITC3 (<i>ele.1</i>)	MITC3+ (<i>ele.1</i> , d=0)
Eigenvalue	7.2000E-07	3.4167E+01	1.0099E-06
Bending strains	$\varepsilon_{xx} = -0.403 sz$ $\varepsilon_{yy} = 0.915 rz$ $\gamma_{xy} = (-0.403r + 0.915 s)z$	$\varepsilon_{xx} = -0.675z$ $\varepsilon_{yy} = 0.675z$ $\gamma_{xy} = 0.0$	$\varepsilon_{xx} = [-0.670 - 0.401(s - 2rs - s^2)]z$ $\varepsilon_{yy} = [0.670 + 0.401(r - 2rs - r^2)]z$ $\gamma_{xy} = 0.401(-r + s + r^2 - s^2)z$
Transverse shear strains	$\gamma_{xz} = 0.0$ $\gamma_{yz} = 0.0$	$\gamma_{xz} = -0.235 + 0.675s$ $\gamma_{yz} = 0.235 - 0.675r$	$\gamma_{xz} = -1.34 d(1 - 3s)$ $\gamma_{yz} = -1.34 d(-1 + 3r)$

4-3. Two-sided clamped plate



Mesh A





Mesh C

Strain energy

t/L	MITC3		MITC3-	MITC4	
	Mesh A	Mesh B	Mesh A	Mesh B	Mesh C
1/100	4.1190E-04	6.8681E-01	4.8858E-01	6.8681E-01	1.0989E+00
1/1,000	4.1209E-03	6.8681E+02	4.8840E+02	6.8681E+02	1.0989E+03
1/10,000	4.1209E-02	6.8681E+05	4.8840E+05	6.8681E+05	1.0989E+06
Order of change	t / L	$(t/L)^3$	$(t/L)^3$	$(t/L)^3$	$(t/L)^3$

4-3. Two-sided clamped plate (Mesh B)



Pure bending problem

Exact transverse shear strains

$$e_{rt} = e_{st} = 0$$
 in elements- I and II

Theoretical relationship

$$w_2 = \frac{1}{2}\alpha_2 = -\frac{1}{2}\beta_2$$

Analytical proof MITC3+ shell element

$$e_{rt} = e_{st} = 0$$

 $w_2 - \frac{1}{2}\alpha_2 = 0, \quad w_2 + \frac{1}{2}\beta_2 = 0$

• Free square plate problem

- L= 1.0, E = 2.07×10¹¹, v = 0.3, ρ = 7.8×10³





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• Free hyperboloid shell problem

- $L = 1.0, E = 2.0 \times 10^{11}, v = 1/3, \rho = 7.8 \times 10^3$















< Mode 8 >



Relative error using 60×30 meshes

t/L	MITC3	MITC3+	MITC4
1/1,000	44.71%	1.28%	49.24%
1/10,000	92.36%	1.16%	91.67%

The paper will be submitted to "Computers and Structures " soon.

On the modal behavior of the MITC3+ triangular shell elements

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Abstract₽

In this paper, we investigate the static and dynamic modal behavior of the MITC3+ triangular shell elements. First, the static mode analysis is performed for a single right-angled shell element and an assemblage of two right-angled shell elements. The detailed strain fields are investigated in bending modes, which provides insight into how shear locking happens on the mode level. We then analytically show that the MITC3+ shell element does not lock in the two-sided clamped plate problem. Considering free plate and free hyperboloid shell problems, we finally present the excellent performance of the MITC3+ shell elements in dynamic mode analysis.4¹

5. Conclusions

• An improved MITC3 shell element is developed (MITC3-HR).

- Modified Hellinger-Reissner functional with the rotated approximated transverse shear strain field
- To improve the performance of the 3-node shell element, the stiffness of the in-plane twisting mode must be reduced.
- The 3-node MITC3+ shell element is developed for general use.
 - The element is enriched by a cubic bubble function for the rotations.
 - A new assumed transverse shear strain field is proposed with the new tying scheme.
- Investigating the modal behavior of the MITC3+ shell element
 - We understand the reason why the MITC3+ shell element is significantly improved through static mode analyses and the two-sided plate problem.
 - The MITC3+ shell element is suitable for free vibration analyses.

Future works

- Improving the membrane behavior of the MITC3+ shell element
 - * No treatment for the membrane part of the MITC3+ shell element \Rightarrow In-plane shear locking
- Developing the MITC3+ solid-shell element

Education & publications

• Education

- Ph.D. Candidate : KAIST, Division of Ocean Systems Engineering (Feb 2011-)
- MS: KAIST, Division of Ocean Systems Engineering (Feb 2009 Feb 2011)
- BS : Inha University, Department of Mechanical Engineering (Mar 2002 Feb 2009)

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Thank you.