대형 유한요소 모델의 동적 해석을 위한 모델축소기법에 관한 연구

On the model reduction methods for dynamic analysis of large finite element models

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1. Introduction

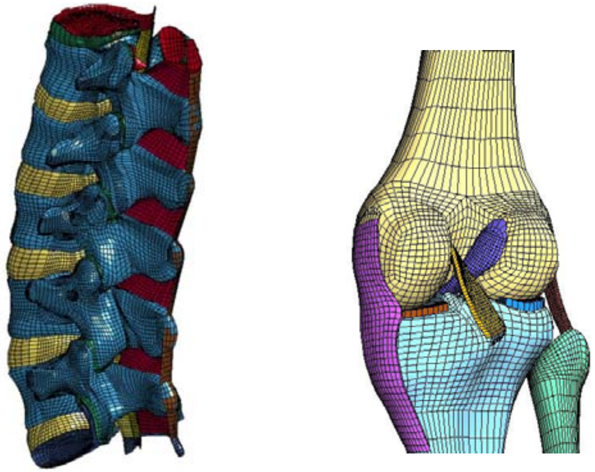
FEM in engineering fields

Offshore Engineering

Aerospace Engineering

Earthquake Engineering

Impact Engineering



BIO Engineering

Today, FEM is closely related with the <u>human safety</u> and <u>health</u> in real life !

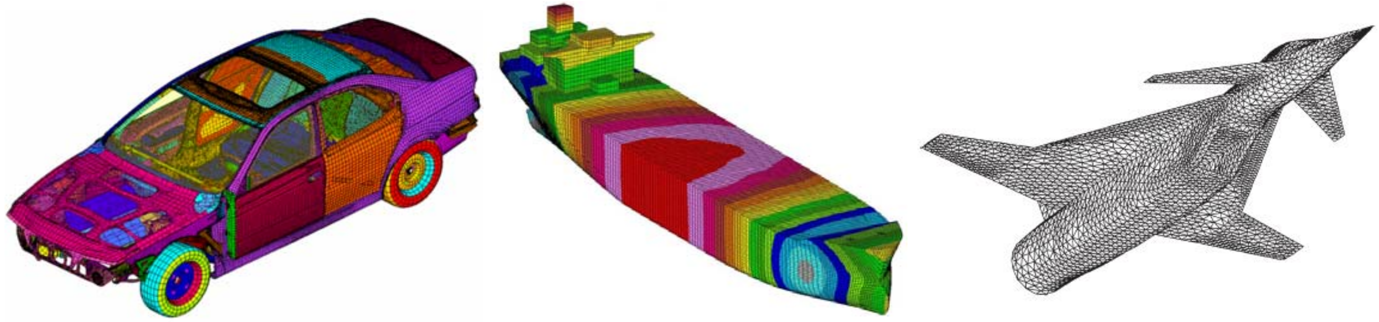
Motivations of model reduction

<u>Finite element model</u>

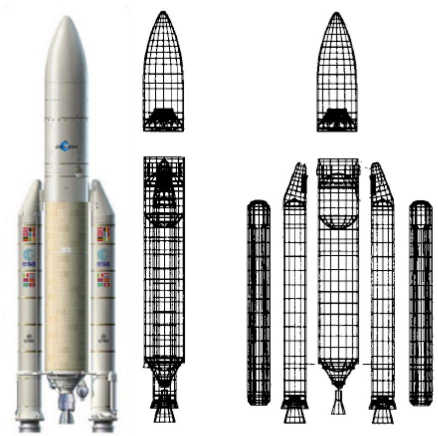
Complexity ↑

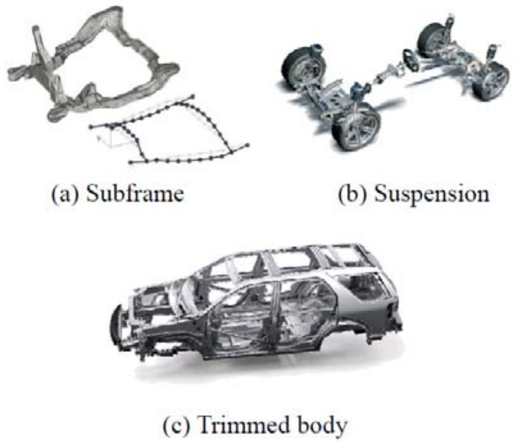
Model size (M_g, K_g) ↑

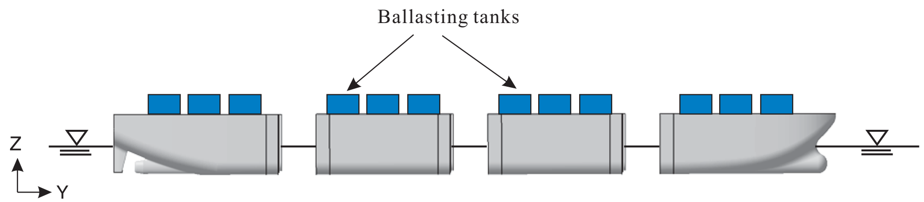
Computing time ↑



Reducing the computational cost has been an important issue.





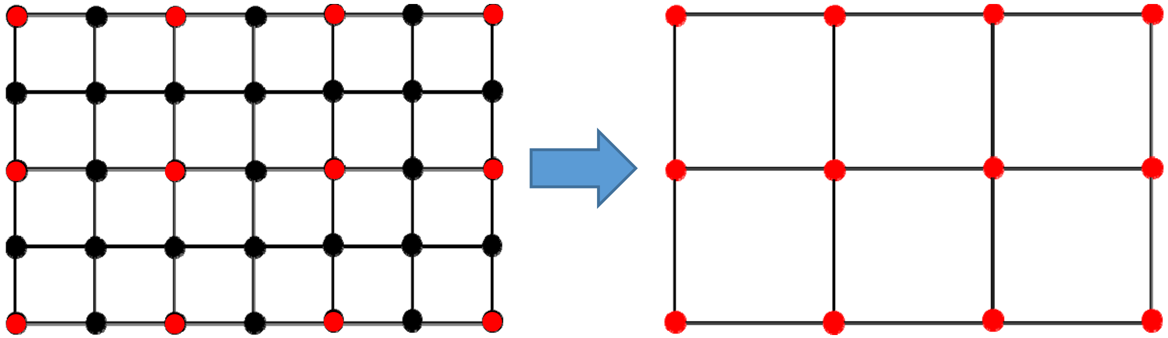


Separate into many pieces and reduce every pieces: "Divide and Conquer" paradigm

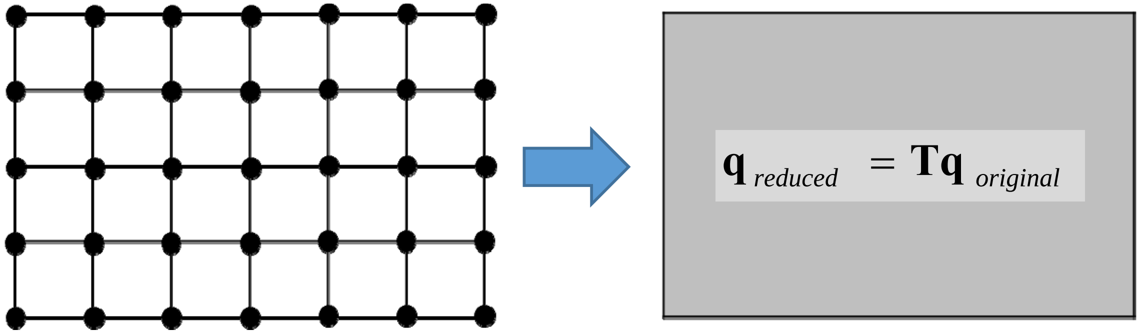
Model reduction methods have been studied for more than 50 years.
Still important research field.

Model reduction concepts

❖ **There are 2 ways to make the reduced model.**



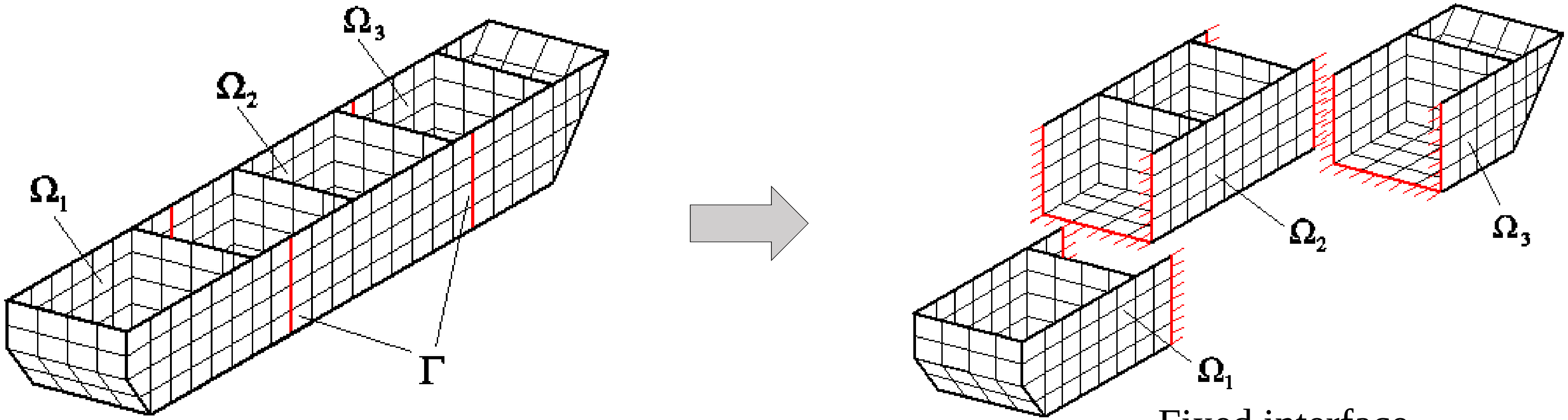
1. DOFs based reduction methods:
Ex) Guyan reduction, IRS method, Super-element etc..



2. Modes based reduction methods:
Ex) CB method, AMLS method etc..

The Craig-Bampton method

Step 1. Partitioning and interface handling



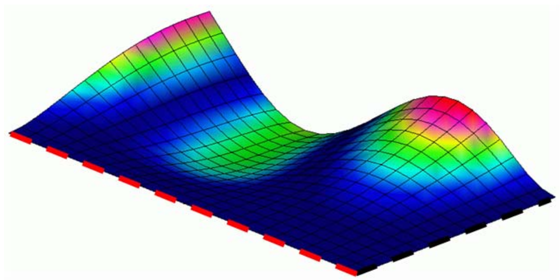
Partitioned structure

Fixed interface
(e.g. CB method, AMLS method)

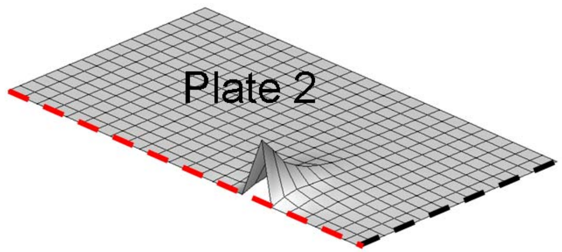
Step 2. Construct transformation matrix

$$\mathbf{T}_0 = [\boldsymbol{\Phi} \quad \boldsymbol{\Psi}],$$

$\boldsymbol{\Phi}$: Fixed interface normal mode



$\boldsymbol{\Psi}$: Interface constraint mode



The Craig-Bampton method

✓ Fixed interface normal modes

$$\mathbf{K}^{(i)}\mathbf{\Phi}^{(i)} = \mathbf{\Lambda}^{(i)}\mathbf{M}^{(i)}\mathbf{\Phi}^{(i)}, \quad \mathbf{\Phi}^{(i)} = [\mathbf{\Phi}_i^d \quad \mathbf{\Phi}_i^r]$$

s : substructural
b : interface boundary
d : dominant
r : residual

$$\Rightarrow \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_s \\ \mathbf{0}_b \end{bmatrix}, \quad \mathbf{\Phi}_s = [\mathbf{\Phi}_s^d \quad \mathbf{\Phi}_s^r]$$

Step 3. Reduced transformation matrix

$$\mathbf{T}_0 = \left[\begin{array}{cc:c} \mathbf{\Phi}_s^d & \mathbf{\Phi}_s^r & \mathbf{\Psi}_c \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_b \end{array}\right] \quad \Rightarrow \quad \overline{\mathbf{T}}_0 = \left[\begin{array}{c:c} \mathbf{\Phi}_s^d & \mathbf{\Psi}_c \\ \mathbf{0} & \mathbf{I}_b \end{array}\right]$$

Truncated :
Large amount of global modes

: Reduced transformation matrix

[Note]

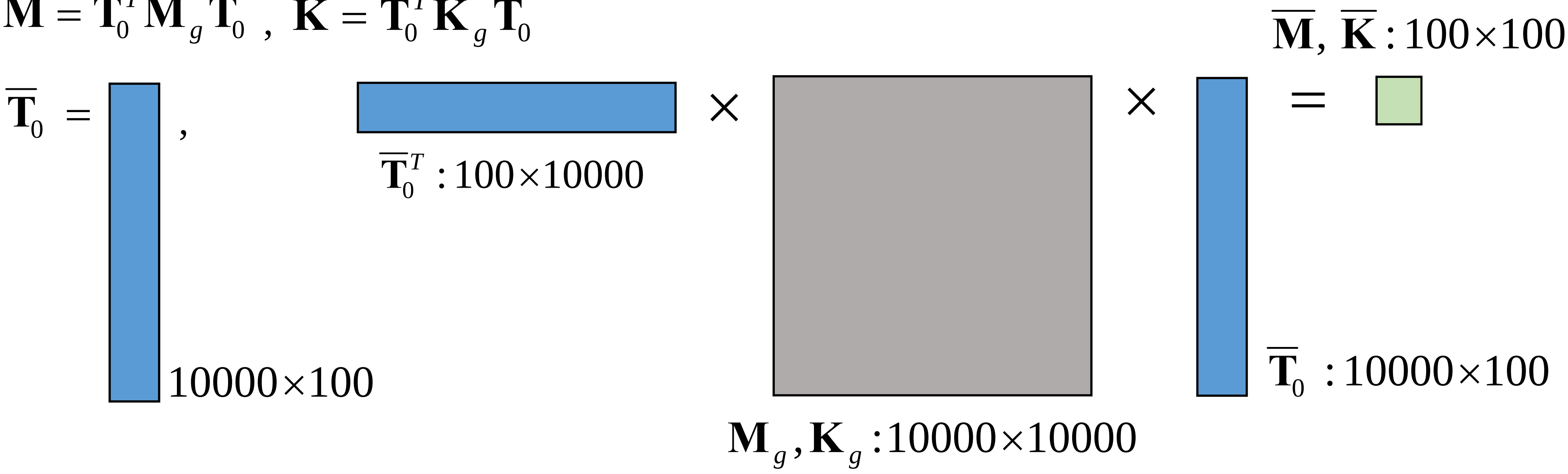
More accurate method → Depend on transformation matrix.

The Craig-Bampton method

Step 4. **Rayleigh-Ritz procedure**

$\overline{\mathbf{M}}, \overline{\mathbf{K}}$ is much smaller than $\mathbf{M}_g, \mathbf{K}_g$.

$$\overline{\mathbf{M}} = \overline{\mathbf{T}}_0^T \mathbf{M}_g \overline{\mathbf{T}}_0 \ , \ \overline{\mathbf{K}} = \overline{\mathbf{T}}_0^T \mathbf{K}_g \overline{\mathbf{T}}_0$$



Step 5. Reduced eigenvalue problem

$$\overline{\mathbf{K}}(\overline{\boldsymbol{\varphi}})_i = \bar{\lambda}_i \overline{\mathbf{M}}(\overline{\boldsymbol{\varphi}})_i \Rightarrow \textbf{Approximated eigenvalue } \bar{\lambda}_i$$

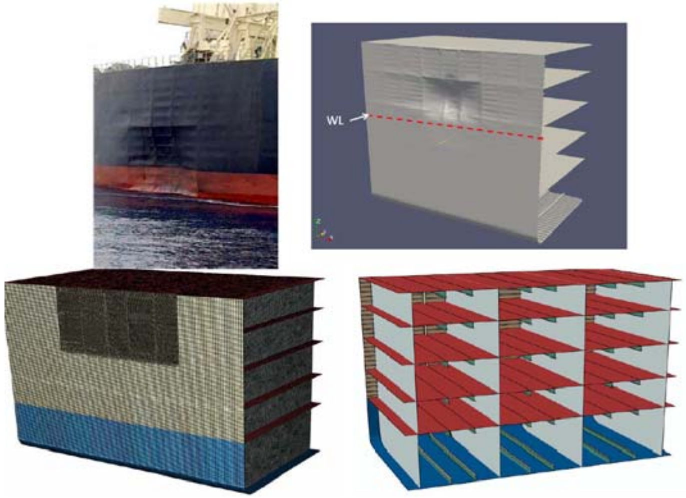
Step 6. Back transformation procedure

$$(\boldsymbol{\varphi}_g)_i \approx (\overline{\boldsymbol{\varphi}}_g)_i = \overline{\mathbf{T}}_0 (\overline{\boldsymbol{\varphi}}_p)_i$$

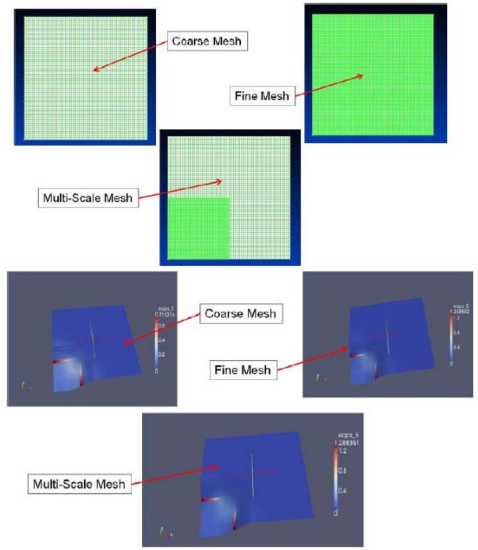
$\overline{\mathbf{T}}_0$ is a key factor in model reduction.

Our Aims → Accurate & Fast.

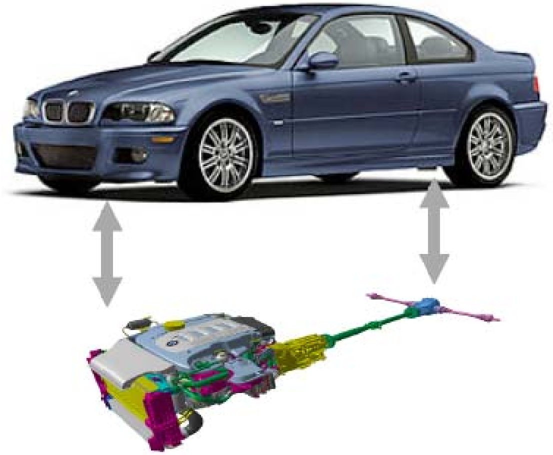
Applications

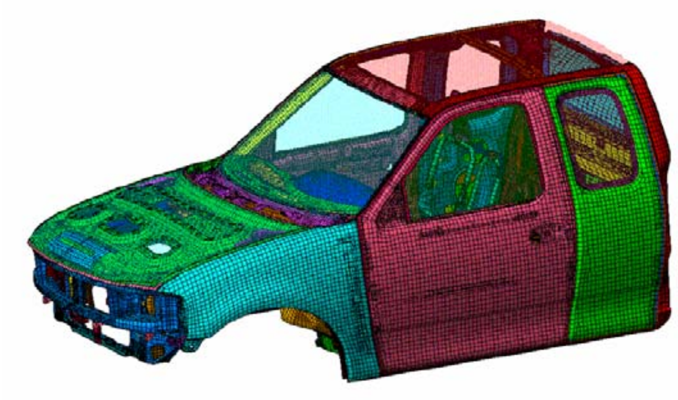


Shock response & damage prediction

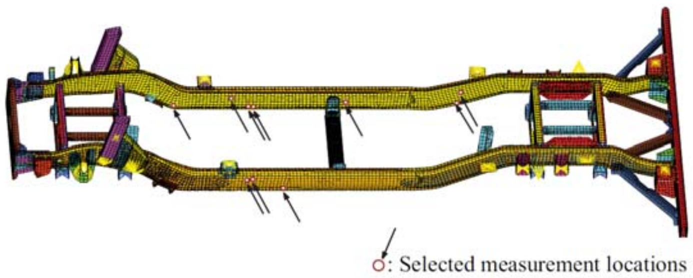


Multi-scale model analysis

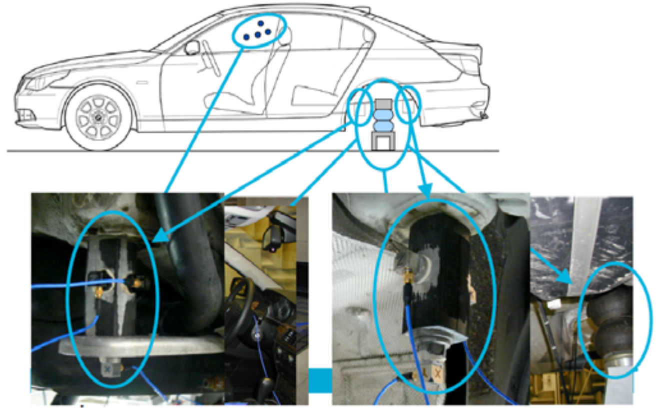




Structural dynamic analysis
and “Eigenvalue solver”
(Especially AMLS method)

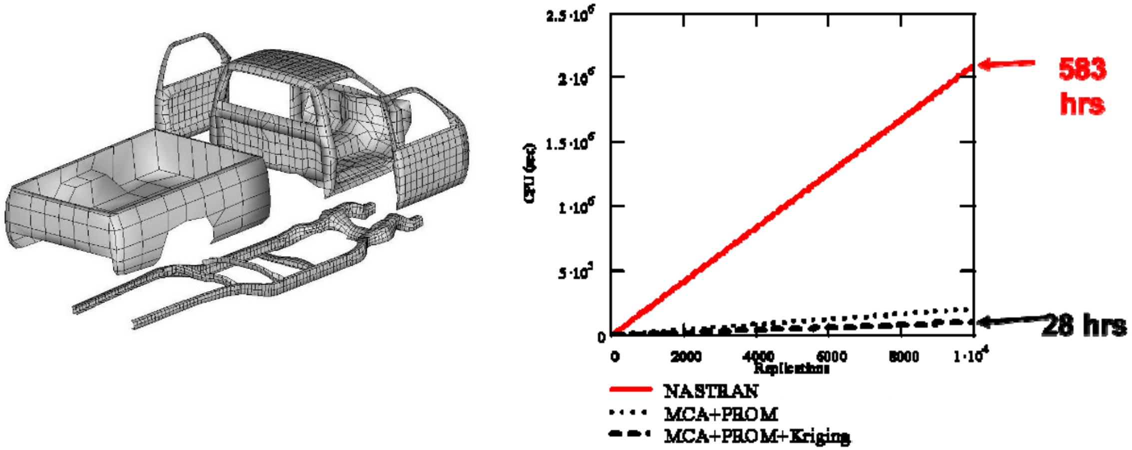


Health monitoring
and sensor positioning

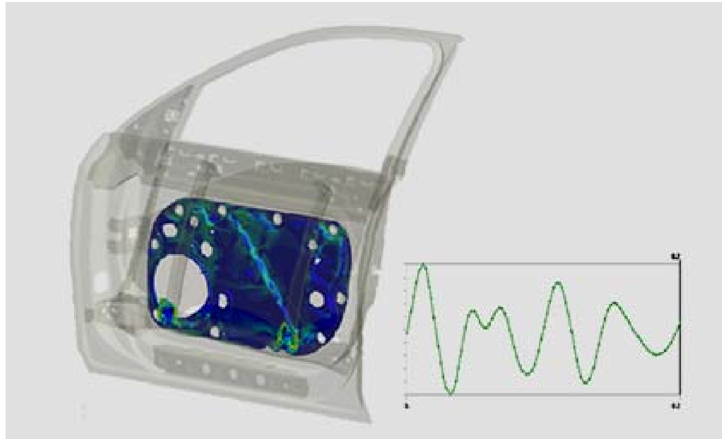


Experimental dynamic analysis

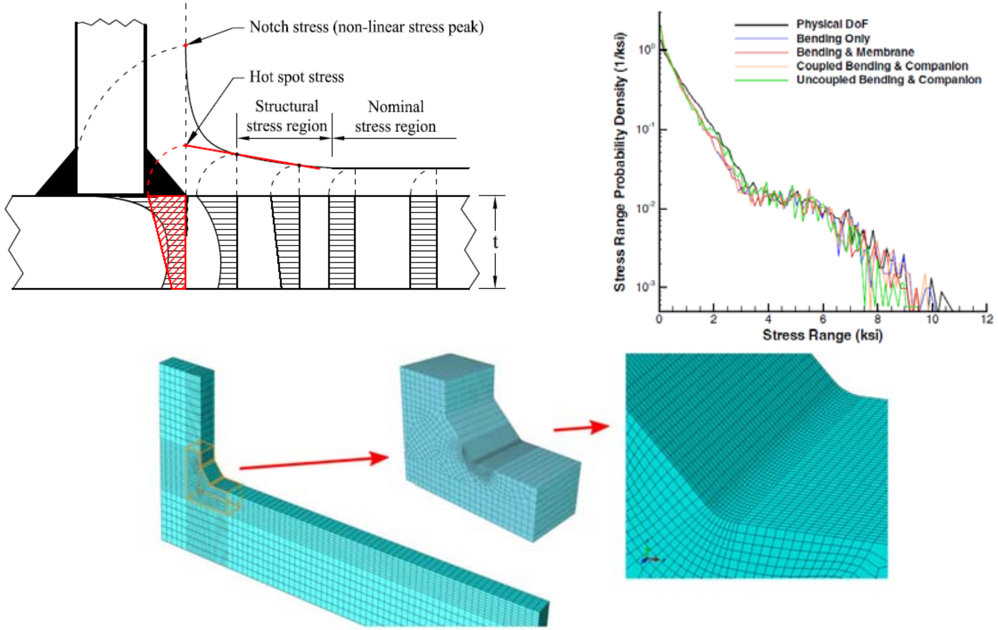
Applications



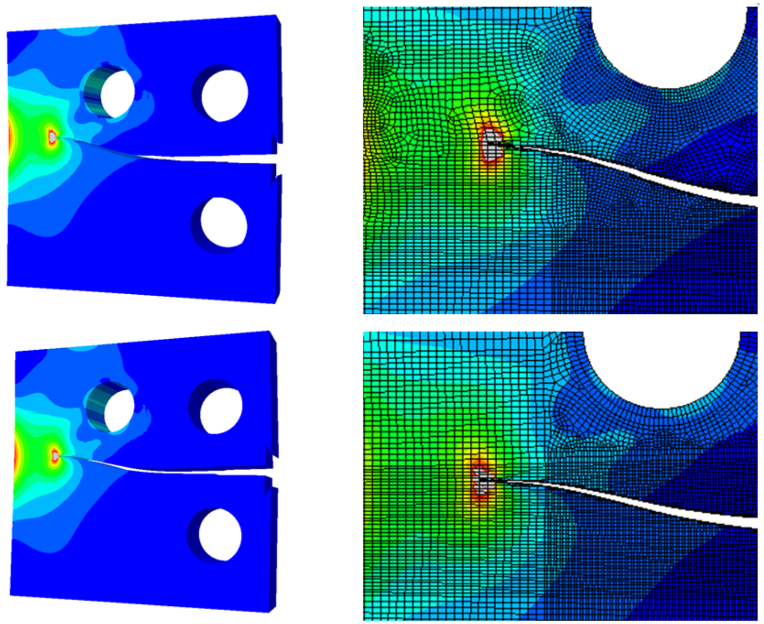
Design optimization through re-analysis



Transient analysis
(Alternative for implicit / explicit solvers)



Spectral fatigue analysis



Crack propagation analysis

2. History & Issues

History of the model reduction method

1965

- Hurty

: A pioneer of CMS method

- Use '*component modes*' and '*synthesis*'.
- Fixed interface CMS
- Rigid body +Constraint

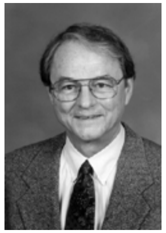
+ Fixed interface normal modes

1968

- Craig-Bampton (CB) method
- Modified Hurt's approach.

:Constraint + Fixed interface normal modes

- Simple & Accurate
- Most popular



Craig

1969

- Goldman

: Free interface normal modes

1971

- MacNeal

: Hybrid method

- *Fixed-Free* interface normal modes

MacNeal

1975

- Rubin

: Extended MacNeal's approach (Approximating *Free* interface normal modes)

- Considering the residual effects

1976

- Craig-Chang

: Modified Rubin's idea (*Free* interface normal modes)

- Considering residual flexibility (Accuracy ↑)

1984

- Ookuma & Nagamatsu

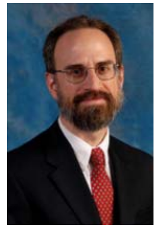
: Developed *multiple* CMS

- Multiple partitioning (Efficiency ↑)

1987

- J.K. Bennighof
- '*Component mode iteration*' method

:Using subspace iteration method at the substructure level



Bennighof

1988

- Craig & Hale

: Used Block Krylov Ritz vectors instead of component normal modes

- Less computational effort

1960s 1970s 1980s

[1960s ~ 1980s]

✓ Various interface handling techniques. (Fixed → Free → Fixed + Free)

✓ To calculate more accurate substructural normal modes. (SIM, Block Krylov)

History of the model reduction method

1990
- El-Sayed & Hsiung

: Employing parallel processing
(Efficiency ↑)

1992
- Singh & Suarez

: A new ***Free* interface** CMS
- Using higher order modal combination technique
(Accuracy ↑)

1996
- Parametric model reduction
- Model updating


E. Balmes

1992
- CMS + Guyan reduction
- Substructuring for solution efficiency


Bouhaddi

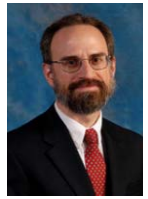
1997
- Greg Hulbert
- Using the quasi-static mode instead of constraint mode


G. Hulbert

2002
- Dual CB method
- Assembly using interface force
- Experimental substructuring


D. Rixen

2004
- J.K. Bennighof
- AMLS method
- Automated multi-level substructuring
- Computer-aid formulation: Extremely fast, eigensolver


Bennighof

2005
- C. Yang & NG Esmond
- ‘*Priori*’ error bound for AMLS method (Only for single level)

2006
- H. Voss : Modified Yang’s approach
- ‘*Priori*’ error bound for CB/AMLS methods
- Check the reliability of the reduced model


H. Voss

1990s

2000s

[1990s ~ 2000s]

✓ First try for ‘Parallel’ computing.

✓ New interface handling techniques.

✓ More accurate reduced model.

[2010s ~]

✓ Expansion to Experiment.

✓ New paradigm: Eigensolver, Interest for ‘Efficiency’

✓ Studies on “Error estimator”: Interest for ‘Reliability’

Primal issues for half a century

❖ **We can identify that, there have been "<u>Three Primal Issues</u>" in the research field.**

Issue 1. Solution "<u>Reliability</u>"

❖ Error bound (= Error estimation): 1. For **<u>CB</u>** method 2. For **<u>AMLS</u>** method

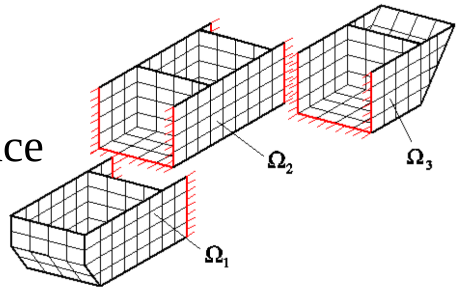
➡ **Topic 1 : Error estimators for the CB/AMLS methods**

Issue 2. Solution "<u>Accuracy</u>"

❖ Interface handling techniques

1. **<u>Fixed</u>** interface 2. **<u>Free</u>** interface

3. **<u>Hybrid</u>** interface



❖ Substructural modes calculation

1. **<u>SIM</u>** 2. **<u>Block Krylov Ritz vectors</u>**

3. **<u>Quasi-static mode</u>**

➡ **Topic 2 : Enhanced AMLS method**

Issue 3. Solution "<u>Efficiency</u>"

❖ Improving solution efficiency

1. **<u>Parallel</u>** techniques 2. **<u>Automated multi-level</u>** substructuring

3. Using **<u>sparsity pattern</u>**

➡ **Topic 3 : Algebraic dynamic condensation method**

3. Research topics

Topic 1-1. Simplified error estimator for the CB method

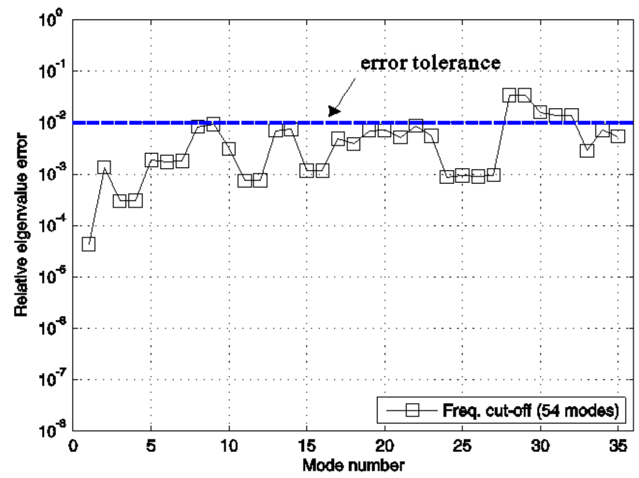
Motivation

❖ **To verify the <u>solution reliability</u> of the reduced model,**

Generally, we use "**Relative eigenvalue errors**"

$$\xi_i = \frac{\bar{\lambda}_i - \lambda_i}{\lambda_i} = \frac{\bar{\lambda}_i}{\lambda_i} - 1$$

However, to obtain the exact eigenvalue λ_i, we should solve the **global eigenvalue problem** $\mathbf{K}_g(\boldsymbol{\varphi}_g)_i = \lambda_i \mathbf{M}_g(\boldsymbol{\varphi}_g)_i$.



< Relative eigenvalue errors >

Since 2000s…

<u>Finite element model</u>

Complexity ↑

Model size ($\mathbf{M}_g$, $\mathbf{K}_g$) ↑

$\mathbf{K}_g(\boldsymbol{\varphi}_g)_i = \lambda_i \mathbf{M}_g(\boldsymbol{\varphi}_g)_i$ **: Requires large computational cost.**

✓ **What about 1,000,000 DOFs or more DOFs problem ?**

→ It is not easy to get λ_i .

❖ **For the reduced model of the large FE model: Interest for "<u>Solution Reliability</u> ↑"**

Previous error estimation method

❖ **Several meaningful error estimators were developed.**

1. Yang *et al*. (2005), "*priori*" error bound for AMLS method

$$\rho(\omega) = \frac{\lambda_1}{\omega - \lambda_1} < \tau$$: Only for single level substructuring

: ω is the approximated eigenvalue, λ_1 is the smallest eigenvalue under consideration

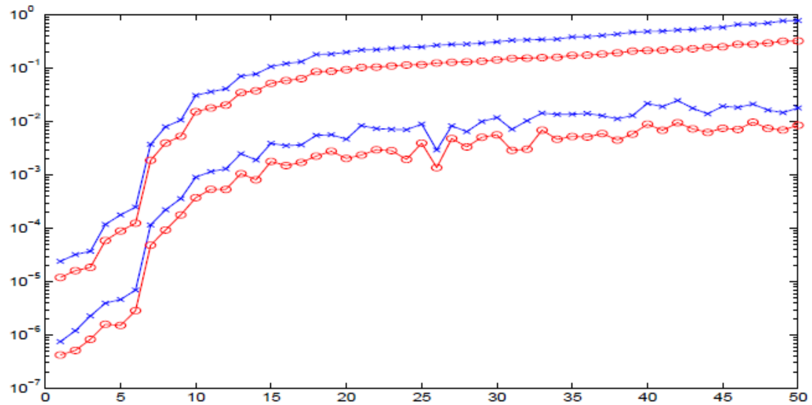
τ is the given tolerance

2. Elssel and Voss (2006), "*priori*" error bound for CB and AMLS methods

CB: $\hat{\mu}_i = \dfrac{\bar{\lambda}_i}{\left|\lambda_r - \bar{\lambda}_i\right|}$, **AMLS:** $\hat{\mu}_i = \prod_{k=0}^{p}\left(1 + \dfrac{\bar{\lambda}_i}{\lambda_k - \bar{\lambda}_i}\right) - 1$

: λ_r is the smallest residual eigenvalue of substructures.

✓ **Scalar operation: Very fast. But, poor accuracy.**



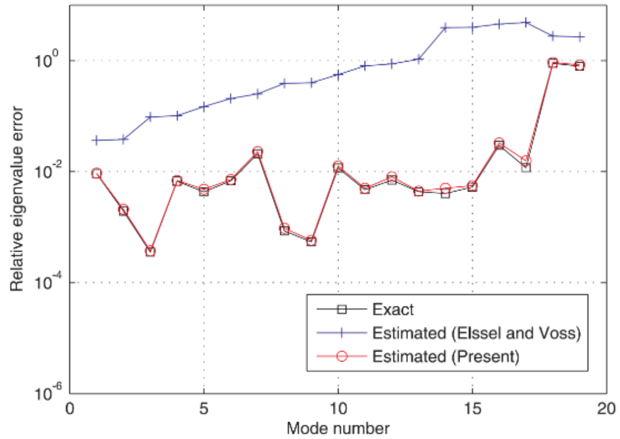
Previous error estimation method

3. Kim and Lee (2015), "*posteriori*" error estimator for the CB method

✓ Derive the "**enhanced transformation matrix**" using residual flexibility matrix $\mathbf{F}_{rs}$.

$$\overline{\mathbf{T}}_1 \quad \text{with} \quad \overline{\mathbf{T}}_1 = \overline{\mathbf{T}}_0 + \overline{\mathbf{T}}_a$$

$$\mu_i = 2(\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_0^T \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_a (\overline{\boldsymbol{\varphi}}_p)_i + (\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_a^T \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_a (\overline{\boldsymbol{\varphi}}_p)_i$$



✓ **Very accurate.**
✓ **Global matrix operations, complicated.**

❖ What is the drawback / limitation?

✓ Yang, Elssel and Voss's : Accuracy ↓ .

✓ Kim and Lee's : Accurate, but efficiency ↓ .

Global matrix computing, Requiring large memory

→ Cannot apply to large FE models (over 100,000 DOFs).

Design focus

1. **Requirements** for a new error estimator ?

- ✓ **Accuracy & Efficiency**. Adoptable for real engineering problems.

2. **Key ideas** ?

- ✓ **Way to see in a part**: **Submatrix level** formulation and computing.
- ✓ Using matrix property: **Sparsity** (**Compact computing**),
  Symmetry (**Half calculation**), **Orthogonality** (**Give zero matrix**).

3. What is the "**Strengths**", which is expected ?

- ✓ Formulations would be expressed in submatrix form: **Computationally efficient**.
- ✓ Then, the **substructural error contributions** for global mode can be calculated.

Derivation procedure

1. Original formulation of error estimator

$$\eta_i = 2(\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_0^{\mathrm{T}} \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_r (\overline{\boldsymbol{\varphi}}_p)_i + (\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_r^T \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_r (\overline{\boldsymbol{\varphi}}_p)_i$$

1 2 3 4

$$\Rightarrow \quad \eta_i = \overline{\boldsymbol{\varphi}}_i^T [\, 2\,\overline{\lambda}_i \overline{\mathbf{T}}_0^T \mathbf{M}_g \mathbf{T}_a - 2\,\overline{\mathbf{T}}_0^T \mathbf{K}_g \mathbf{T}_a + \overline{\lambda}_i^2 \mathbf{T}_a^T \mathbf{M}_g \mathbf{T}_a - \overline{\lambda}_i \mathbf{T}_a^T \mathbf{K}_g \mathbf{T}_a \,] \overline{\boldsymbol{\varphi}}_i$$

✓ **Submatrix form :** $\mathbf{M}_g = \begin{bmatrix} \mathbf{M}_s & \mathbf{M}_c \\ \mathbf{M}_c^{\mathrm{T}} & \mathbf{M}_b \end{bmatrix}$, $\mathbf{K}_g = \begin{bmatrix} \mathbf{K}_s & \mathbf{K}_c \\ \mathbf{K}_c^{\mathrm{T}} & \mathbf{K}_b \end{bmatrix}$, $\overline{\mathbf{T}}_0 = \begin{bmatrix} \mathbf{\Phi}_s^d & \mathbf{\Psi}_c \\ \mathbf{0} & \mathbf{I}_b \end{bmatrix}$, $\mathbf{T}_a = \begin{bmatrix} \mathbf{0} & \mathbf{F}_{rs}\hat{\mathbf{M}}_c \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

1 $\overline{\mathbf{T}}_0^T \mathbf{M}_g \mathbf{T}_a = \begin{bmatrix} \mathbf{0} & \mathbf{A}_c \\ \mathbf{0} & \mathbf{A}_b \end{bmatrix}$ with $\mathbf{A}_c = \underline{(\mathbf{\Phi}_s^d)^T \mathbf{M}_s \mathbf{F}_{rs}} \hat{\mathbf{M}}_c$, $\mathbf{A}_b = \hat{\mathbf{M}}_c^T \mathbf{F}_{rs} \hat{\mathbf{M}}_c$

"**Orthogonal**" property
$(\mathbf{\Phi}_s^d)^T \mathbf{M}_s \mathbf{F}_{rs} = \mathbf{0}$

$$\Rightarrow \quad \therefore \overline{\mathbf{T}}_0^T \mathbf{M}_g \mathbf{T}_a = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_b \end{bmatrix}$$

✓ **Zero matrices: Give cost saving !**

2 $\overline{\mathbf{T}}_0^T \mathbf{K}_g \mathbf{T}_a = \begin{bmatrix} \mathbf{0} & \mathbf{B}_c \\ \mathbf{0} & \mathbf{B}_b \end{bmatrix}$ with $\mathbf{B}_c = \underline{(\mathbf{\Phi}_s^d)^T \mathbf{K}_s \mathbf{F}_{rs}} \hat{\mathbf{M}}_c$, $\mathbf{B}_b = \underline{(\mathbf{\Psi}_c^T \mathbf{K}_s + \mathbf{K}_c^T)} \mathbf{F}_{rs} \hat{\mathbf{M}}_c$

"**Orthogonal**" property
$(\mathbf{\Phi}_s^d)^T \mathbf{K}_s \mathbf{F}_{rs} = \mathbf{0}$

$$\Rightarrow \quad \therefore \overline{\mathbf{T}}_0^T \mathbf{K}_g \mathbf{T}_a = \mathbf{0}$$

$\mathbf{\Psi}_c = -\mathbf{K}_s^{-1} \mathbf{K}_c$, $\mathbf{\Psi}_c^T \mathbf{K}_s + \mathbf{K}_c^T = \mathbf{0}$

Derivation procedure

1 = 4 $\quad \therefore \overline{\mathbf{T}}_0^T \mathbf{M}_g \mathbf{T}_a = \mathbf{T}_a^T \mathbf{K}_g \mathbf{T}_a$

2. The following equations are obtained

$$\eta_i = \overline{\lambda}_i \overline{\boldsymbol{\varphi}}_i^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_b \end{bmatrix} \overline{\boldsymbol{\varphi}}_i + \underline{\overline{\lambda}_i^2 \overline{\boldsymbol{\varphi}}_i^T \mathbf{T}_a^T \mathbf{M}_g \mathbf{T}_a \overline{\boldsymbol{\varphi}}_i}$$

1 3

$$\boxed{\overline{\boldsymbol{\varphi}}_i = \begin{bmatrix} \overline{\boldsymbol{\varphi}}_s \\ \overline{\boldsymbol{\varphi}}_b \end{bmatrix}_i}$$

Higher order term neglected

3. The error estimator is approximated

$$\eta_i \approx \mu_i = \overline{\lambda}_i (\overline{\boldsymbol{\varphi}}_b)_i^T \mathbf{A}_b (\overline{\boldsymbol{\varphi}}_b)_i \quad \text{with} \quad \mathbf{A}_b = \hat{\mathbf{M}}_c^T \mathbf{F}_{rs} \hat{\mathbf{M}}_c.$$

Submatrix form

$$\hat{\mathbf{M}}_c = \begin{bmatrix} \hat{\mathbf{M}}_c^{(1)} \\ \hat{\mathbf{M}}_c^{(2)} \\ \vdots \\ \hat{\mathbf{M}}_c^{(k)} \end{bmatrix}, \quad \mathbf{F}_{rs} = \begin{bmatrix} \mathbf{F}_{rs}^{(1)} & & & \mathbf{0} \\ & \mathbf{F}_{rs}^{(2)} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{F}_{rs}^{(k)} \end{bmatrix}$$

4. A new error estimator

$$\mu_i = \sum_{k=1}^{n} \overline{\lambda}_i (\overline{\boldsymbol{\varphi}}_b)_i^T \hat{\mathbf{M}}_c^{(k)T} \mathbf{F}_{rs}^{(k)} \hat{\mathbf{M}}_c^{(k)} (\overline{\boldsymbol{\varphi}}_b)_i$$

$\mathbf{F}_{rs}^{(k)}$: Symmetric, fully populated matrix.

$$\mathbf{F}_{rs}^{(k)} = \mathbf{F}_d^{(k)} + \mathbf{F}_u^{(k)} + \mathbf{F}_u^{(k)T}$$

Derivation procedure

5. Finally, the simplified error estimator is defined as

$$\boxed{\mu_i = \sum_{k=1}^{n} \mu_i^{(k)}} \quad \text{with} \quad \mu_i^{(k)} = e_1^{(k)} + 2e_2^{(k)},\ e_1^{(k)} = \bar{\lambda}_i (\bar{\boldsymbol{\varphi}}_b)_i^T \hat{\mathbf{M}}_c^{(k)T} \mathbf{F}_d^{(k)} \hat{\mathbf{M}}_c^{(k)} (\bar{\boldsymbol{\varphi}}_b)_i,\ e_2^{(k)} = \bar{\lambda}_i (\bar{\boldsymbol{\varphi}}_b)_i^T \hat{\mathbf{M}}_c^{(k)T} \mathbf{F}_u^{(k)} \hat{\mathbf{M}}_c^{(k)} (\bar{\boldsymbol{\varphi}}_b)_i$$

✓ **Represented by <u>a simple summation of the substructural errors estimated</u>.**

➡ **<u>Attractive feature</u> of the proposed method.**

Previous vs New error estimator: Operation counts

❖ Previous error estimator

$$\eta_i = 2(\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_0^{\mathrm{T}} \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_r (\overline{\boldsymbol{\varphi}}_p)_i + (\overline{\boldsymbol{\varphi}}_p)_i^T \overline{\mathbf{T}}_r^T \left[\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g \right] \overline{\mathbf{T}}_r (\overline{\boldsymbol{\varphi}}_p)_i \quad \text{with} \quad \overline{\mathbf{T}}_r = \begin{bmatrix} \mathbf{0} & \mathbf{F}_{rs} \hat{\mathbf{M}}_c \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Step	Items	Counts
1	$\mathbf{F}_{rs} = (\mathbf{K}_s)^{-1} - (\boldsymbol{\Phi}_d)(\boldsymbol{\Lambda}_d)^{-1}(\boldsymbol{\Phi}_d)^T$	ns^2
2	$\overline{\mathbf{T}}_r$	$s(2s-1)bn$
3	$\mathbf{M}_g - \frac{1}{\overline{\lambda}_i} \mathbf{K}_g$	$2N^2$
4	η_i	$(2N-1)(N+d)+(2d-1)(N+1)+1$
Total counts	$ns^2 + s(2s-1)bn + \underline{2N^2 + (2N-1)(N+d) + (2d-1)(N+1) + 1}$	

Polynomials of total DOFs, N

Number of total DOFs : N

Number of substructures : n

Number of each substructural DOFs : s

Size of the interface : b

Size of the reduced model : d

Previous vs New error estimator: Operation counts

❖ New error estimator

$$\mu_i = \sum_{k=1}^{n} \mu_i^{(k)} \quad \text{with} \quad \mu_i^{(k)} = e_1^{(k)} + 2e_2^{(k)},\ e_1^{(k)} = \bar{\lambda}_i (\bar{\boldsymbol{\varphi}}_b)_i^T \hat{\mathbf{M}}_c^{(k)T} \mathbf{F}_d^{(k)} \hat{\mathbf{M}}_c^{(k)} (\bar{\boldsymbol{\varphi}}_b)_i,\ e_2^{(k)} = \bar{\lambda}_i (\bar{\boldsymbol{\varphi}}_b)_i^T \hat{\mathbf{M}}_c^{(k)T} \mathbf{F}_u^{(k)} \hat{\mathbf{M}}_c^{(k)} (\bar{\boldsymbol{\varphi}}_b)_i$$

Step	Items	Counts
1	$\mathbf{F}_{rs}^{(k)} = (\mathbf{K}^{(k)})^{-1} - (\boldsymbol{\Phi}_d^{(k)})(\boldsymbol{\Lambda}_d^{(k)})^{-1}(\boldsymbol{\Phi}_d^{(k)})^T$	ns^2
2	$e_1^{(k)}$ and $e_2^{(k)}$	$2n[(s+1)(2b-1)+(b+s)(2s-1)+1]$
3	$\mu_i^{(k)}$	$2n-1$
Total counts	$ns^2 + 2n[(s+1)(2b-1)+(b+s)(2s-1)+1] + 2n-1$ **: Polynomials of “s”**	

Number of total DOFs : N

Number of substructures : n

Number of each substructural DOFs : s

Size of the interface : b

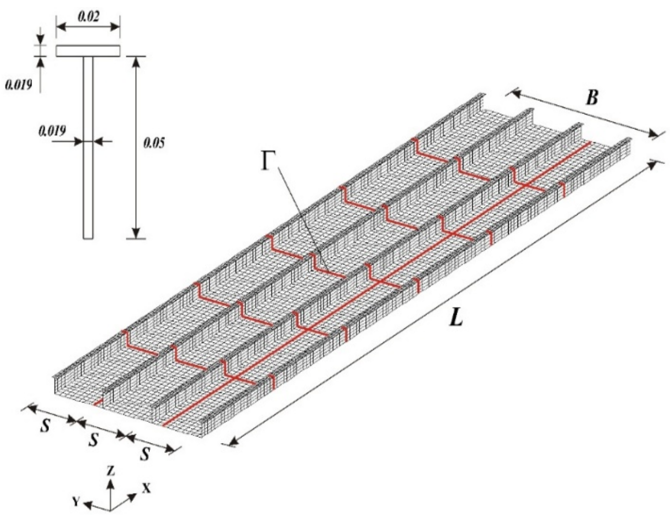
Size of the reduced model : d

Previous vs New error estimator: Operation counts

❖ Operation counts

EEM	Operation counts
Previous	$ns^2 + s(2s-1)bn + 2N^2 + (2N-1)(N+d) + (2d-1)(N+1) + 1$
New	$ns^2 + 2n[(s+1)(2b-1) + (b+s)(2s-1) + 1] + 2n - 1$

✓ For stiffened plate problem



Number of total DOFs, N=52662

Number of substructures : n=18

Number of each substructural DOFs : s=2730

Size of the interface : b=3522

Size of the reduced model : d=3582

Operation counts :

Previous vs New

$9.63\times10^{11} > 1.09\times10^{9}$

➢ **New error estimator only requires 0.22 % operation counts of that of previous error estimator.**

Numerical examples

Computer spec. : Intel core (TM) i7-3770, 3.40 GHz CPU, 32GB RAM.

FE Model: 4-node MITC shell element.

Material: Mild steel. (E=206 GPa, v = 0.3, density = 7,850 kg/m^3)

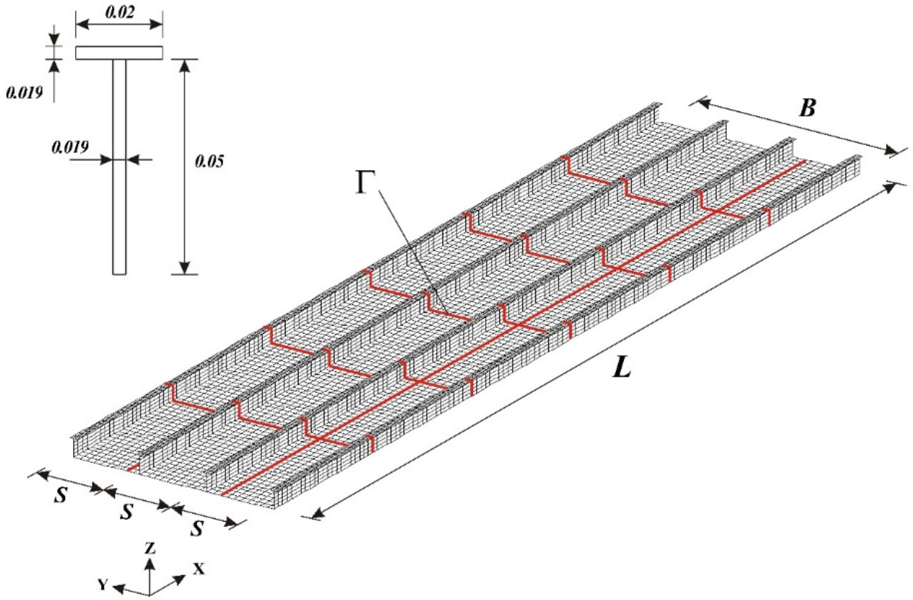
Implementation: Code implementations are done using ***MATLAB***.

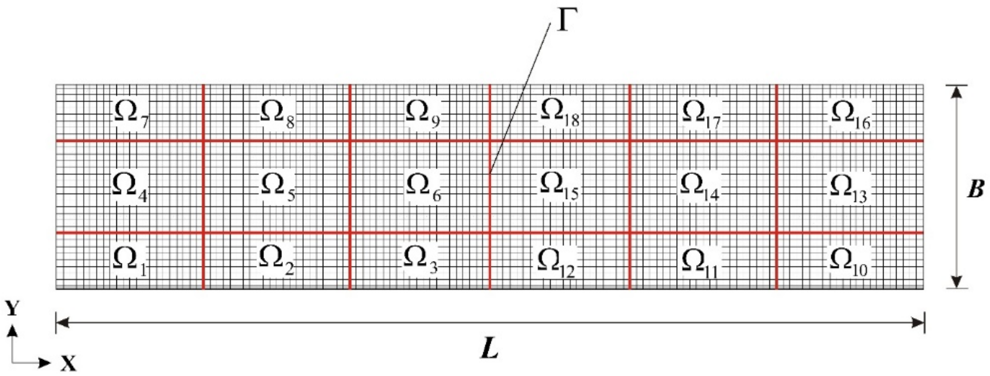
- ✓ Compared with the exact relative eigenvalue errors.
- ✓ Also, compared with the previous error estimation methods

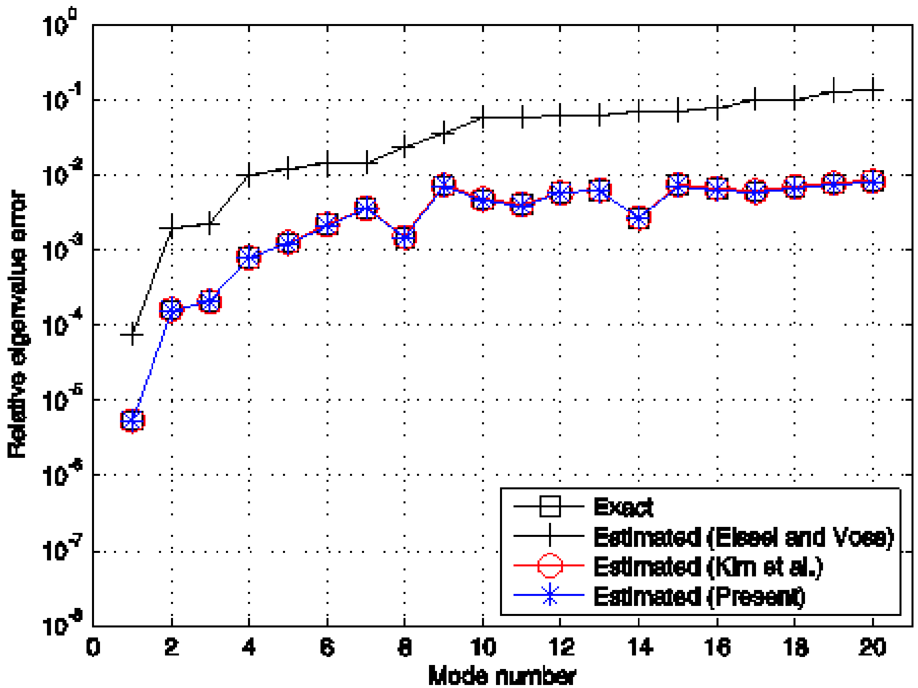
❖ Stiffened Plate structure

Dimension (L=26 m, B=6 m, S=2 m), **18 substructures**. No B.C.

Total DOFs = **52,662** → **Retain modes = 200**. **Reduced system size = 3,582 (6.8%)**.







< Relative eigenvalue errors >

❖ Stiffened Plate structure

➢ Breakdowns of the computational cost

Items		Computation times [sec]	Ratio [%]
CB method		165.24	100.00
Previous error estimator in Eq. (29)	Calculation of the residual flexibility matrix $\mathbf{F}_{rs}$	1.89	1.14
	Calculation of $\mathbf{T}_a$ matrix	41.12	24.89
	Calculation error estimator η_i	16.23	9.82
	Total	59.24	35.85
Present error estimator in Eq. (40)	Calculation of the residual flexibility matrix $\mathbf{F}_{rs}$	1.89	1.14
	Calculation error estimator μ_i	1.87	1.13
	Total	3.76	2.27
Present error estimator considering symmetric partitioning	Calculation of the residual flexibility matrix $\mathbf{F}_{rs}$	0.79	0.48
	Calculation error estimator μ_i	0.40	0.24
	Total	1.19	0.72
Elssel and Voss's error estimator in Eq. (41)		0.000011	0.0000066

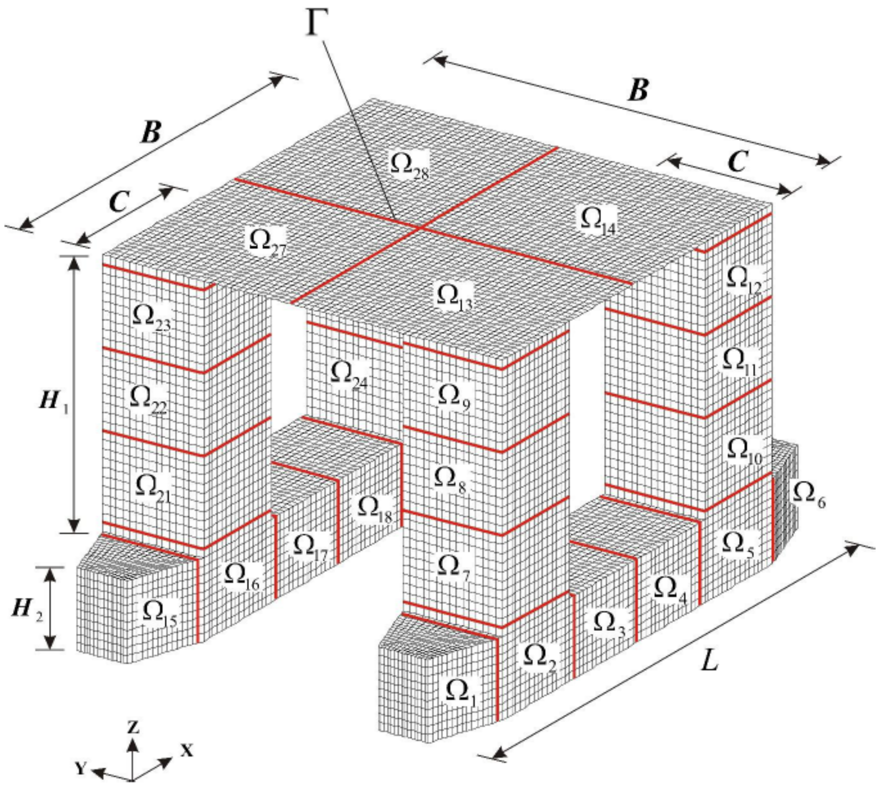
Only requires **2.27 %** of additional computation time.

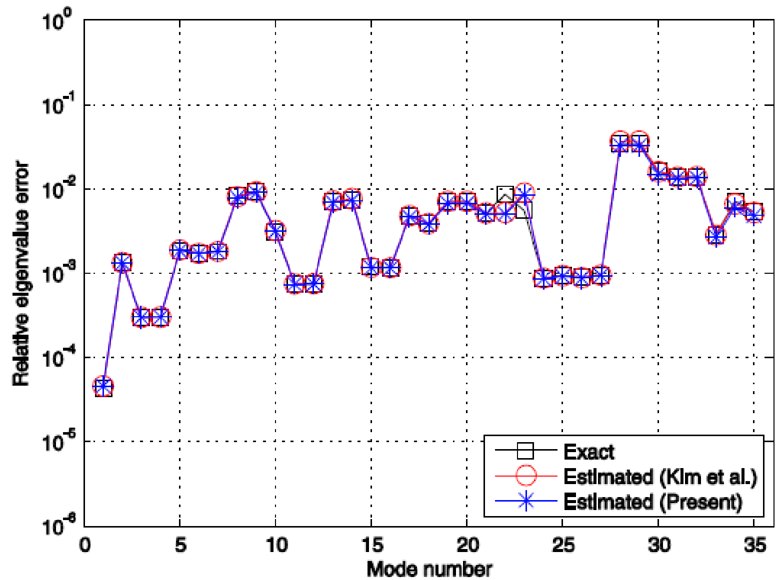
15 times faster

❖ Semi-submerged rig structure

Dimension (L=110 m, B=80 m, C=20 m, H_1=50, H_2=15), **28 substructures**. No B.C.

Total DOFs = **102,504 → Retain modes = 160**. **Reduced system size = 8,806 (8.6 %)**.





Items	Computation times [sec]	Computation times Ratio [%]
CB method	575.77	100.00
Previous error estimator in Eq. (29)	176.69	30.68
Present error estimator in Eq. (40)	11.95	2.07

Original error estimator : Cost is not small.

New error estimator : <u>15 times faster</u>

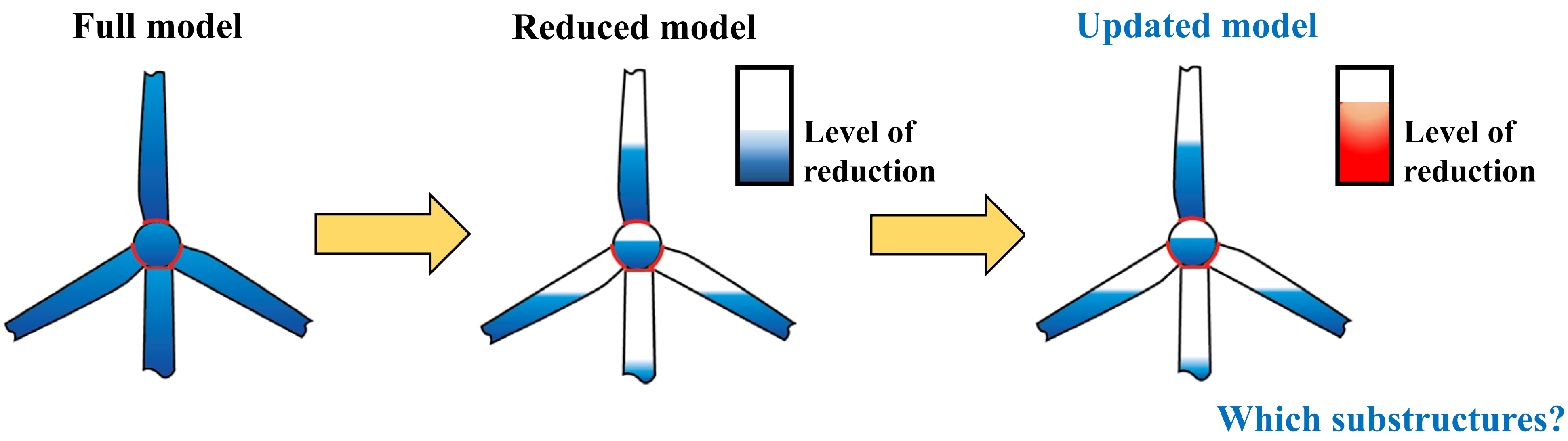
Error control strategy

❖ "**Attractive form**" **of the new error estimator : Simple summations of** $\mu_i^{(k)}$, $\mu_i = \sum_{k=1}^{n} \mu_i^{(k)}$

$\mu_i^{(k)}$: Provides the contribution of the k^{th} substructure to the i^{th} relative eigenvalue error.

➡ **The detailed errors** estimated for a certain substructure.

We can suggest an error control strategy to improve the accuracy of the global modes having relatively large errors.

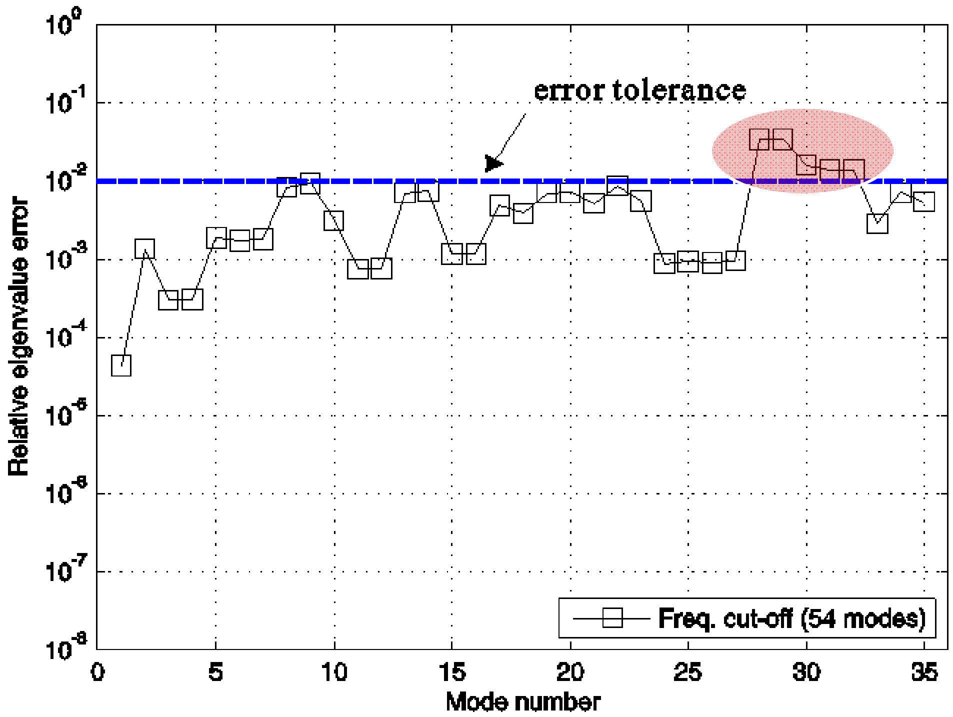


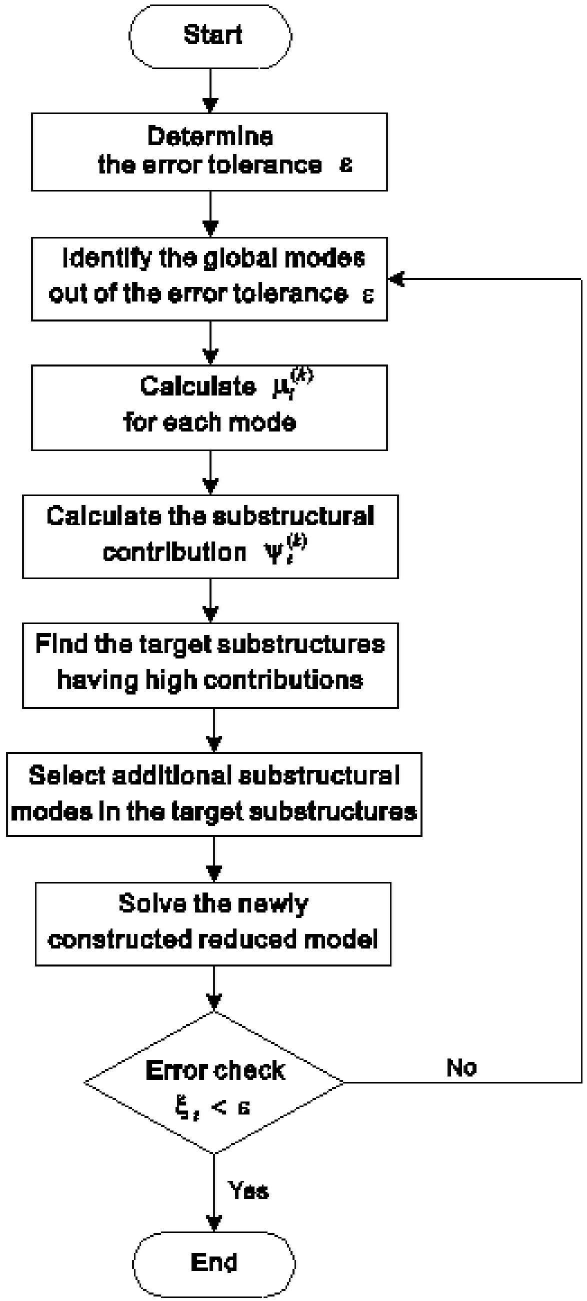
Error control strategy

❖ From $\mu_i^{(k)}$, we can define the "**substructural contribution**"

$$\psi_i^{(k)} = \frac{\mu_i^{(k)}}{\mu_i} \times 100\ [\%] \quad : \textbf{Substructural contribution}$$

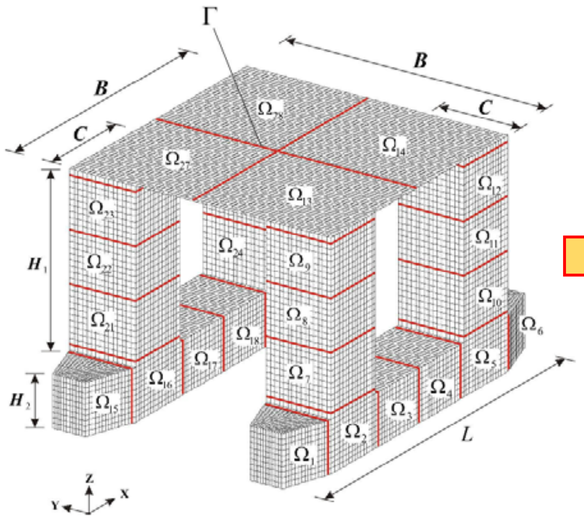
❖ Using $\psi_i^{(k)}$, we can **control the error** out of the tolerance !



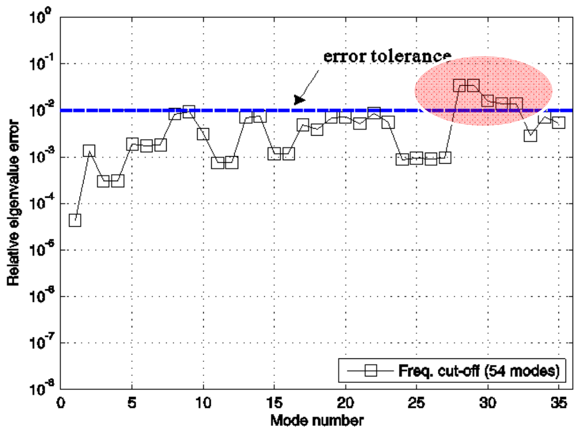


< Flow chart for error control >

Error control strategy



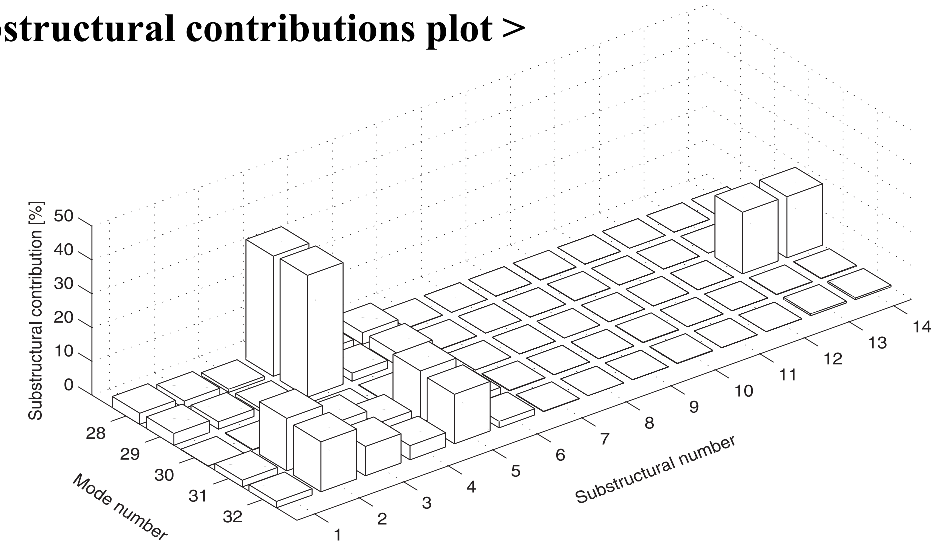
Define the error tolerance



- ✓ Out of the tolerance 10^{-2} : 28^{th} ~32^{th} modes
- ✓ Calculate the estimated substructural error $\mu_i^{(k)}$

Contributions analyze by $\psi_i^{(k)}$

< Substructural contributions plot >



Substructural numbers	Substructural contributions $\psi_i^{(k)}$ (%)				
	$\psi_{28}^{(k)}$	$\psi_{29}^{(k)}$	$\psi_{30}^{(k)}$	$\psi_{31}^{(k)}$	$\psi_{32}^{(k)}$
1, 15	3.41	3.57	0.04	2.05	1.93
2, 16	2.22	2.24	0.16	16.12	15.34
3, 17	0.92	0.93	0.10	9.45	9.00
4, 18	36.84	37.35	0.09	4.61	4.32
5, 19	2.40	2.42	0.16	15.96	15.18
6, 20	3.57	3.73	0.04	2.03	1.91
7, 21	0.10	0.08	0.02	0.16	0.15
8, 22	0.00	0.00	0.00	0.01	0.01
9, 23	0.00	0.00	0.01	0.01	0.01
10, 24	0.12	0.10	0.02	0.15	0.15
11, 25	0.00	0.00	0.00	0.01	0.01
12, 26	0.00	0.00	0.01	0.01	0.01
13, 27	0.00	0.00	19.42	0.14	0.50
14, 28	0.00	0.00	19.27	0.14	0.49
Total	100.00	100.00	100.00	100.00	100.00

- ✓ Define "Target substructures" having high contributions
- ✓ Target substructures : 2 (16), 4 (18), 5 (19), 13 (27), 14 (28)

Error control strategy

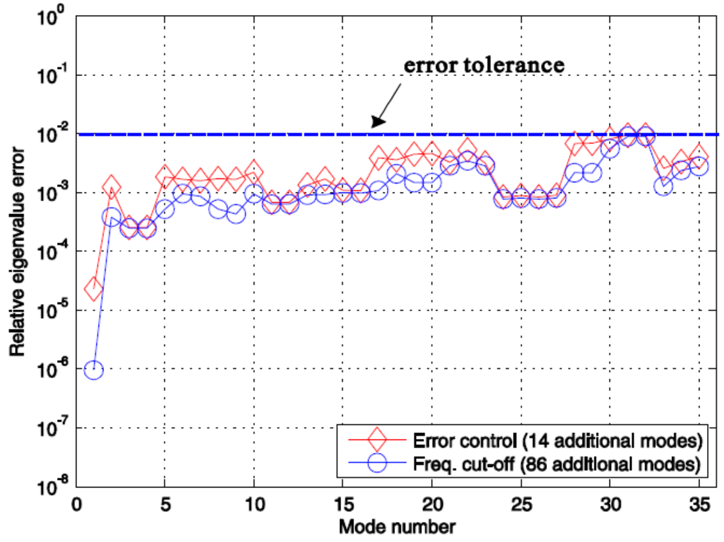
Add substructural modes

- ✓ **Target substructures 2, 5, 16, 19**

  : 2 additional modes

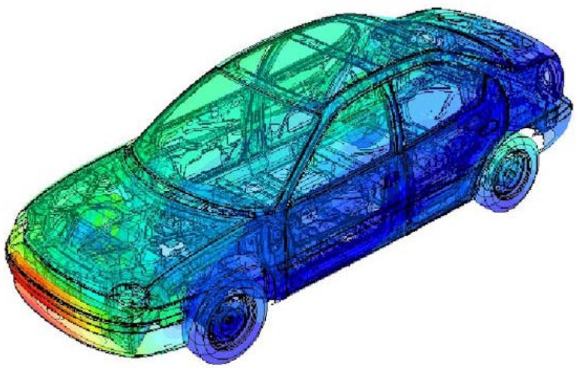
- ✓ **The other target substructures**

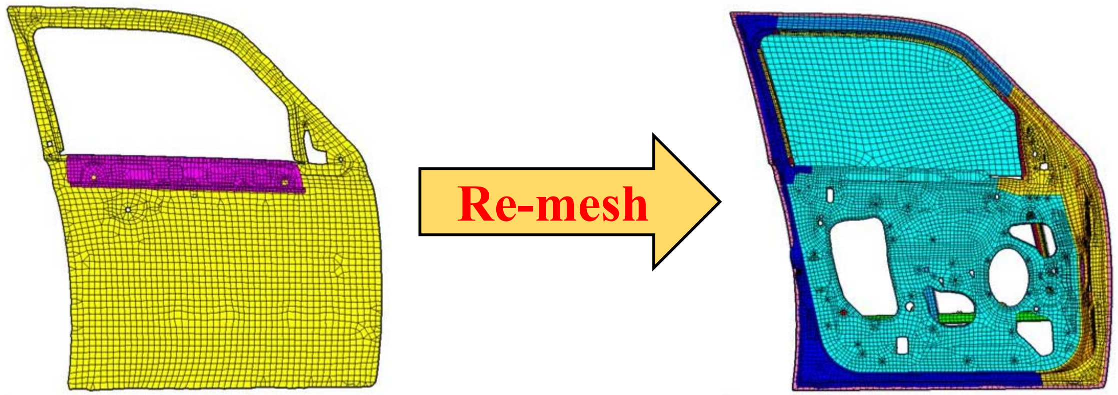
  : 1 additional mode



❖ Also, we can use this error control strategy for "**Model refinement**".

✓ Not adding substructural modes, but '**Re-meshing**' the target substructures crucial for the interested global mode.





Closure

1. We proposed a **simplified error estimator** for the CB method.

2. The estimated relative eigenvalue error is simple to calculate using **summation of the substructural errors estimated**.

3. We proposed **an error control strategy** to improve the accuracy of reduced models efficiently.

4. It would be valuable to develop the **iterative mode selection algorithms** to construct accurate reduced-order models.

Topic 1-2. Error estimator for the AMLS method

Brief introduction of the AMLS method

❖ **Component Mode Synthesis (CMS) methods**

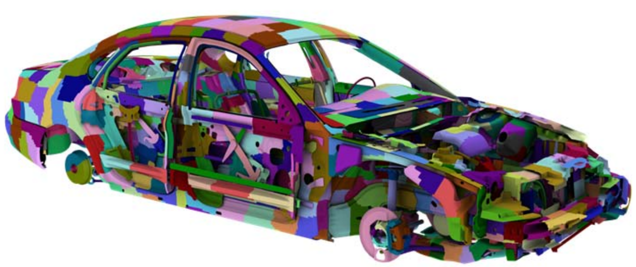
Well known methods. Used for many years in structural engineering.

But, it too suffers from **limitations due to FE model size**.

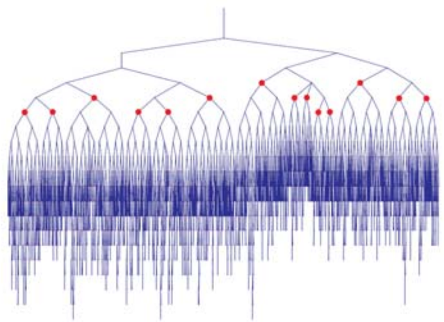
❖ **Automated Multi-Level Substructuring (AMLS) method** (J.K. Bennighof *et al* , *2004*)

✓ **Automatically** partitioning.

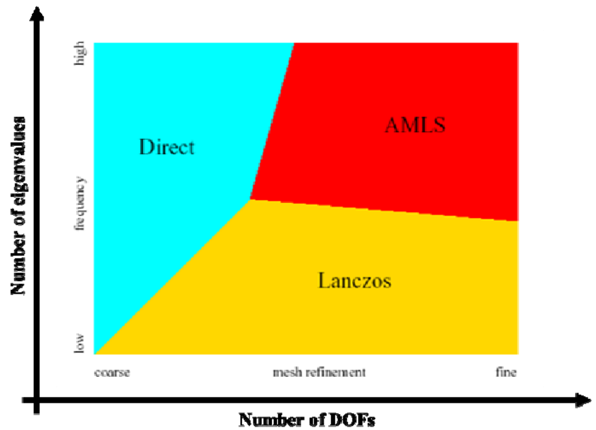
✓ **Recursively transformed** strategies, but much **more complex formulation**.



< Automated substructuring>



< Transformation tree >



✓ Successful in computing **thousands of eigenvalues in a few hours on PC**.

AMLS is a promising **alternative to "Lanczos"** for very large DOFs problem.

Enhanced transformation matrix

❖ To develop the error estimator of the AMLS method

→ Define the "enhanced transformation matrix" of the AMLS method.

1. Original AMLS transformation matrix

$$\mathbf{T}_0 = \mathbf{T}^{(1)}\mathbf{T}^{(2)}\cdots\mathbf{T}^{(N_s)} = \prod_{i=1}^{N_s}\mathbf{T}^{(i)} \Rightarrow \mathbf{T}_0 = \hat{\mathbf{\Psi}}\mathbf{\Phi}, \quad \mathbf{\Phi} = [\mathbf{\Phi}_d \quad \mathbf{\Phi}_r]$$

2. Therefore, the transformation matrix is also divided into two parts, dominant and residual parts.

$$\mathbf{T}_0 = [\mathbf{T}_0^d \quad \mathbf{T}_0^r] \quad \text{with} \quad \underline{\mathbf{T}_0^d = \hat{\mathbf{\Psi}}\mathbf{\Phi}_d}, \; \mathbf{T}_0^r = \hat{\mathbf{\Psi}}\mathbf{\Phi}_r$$

T matrix of AMLS method

3. The global displacement vector is divided into the dominant and residual parts.

$$\mathbf{u}_g = \mathbf{T}_0\boldsymbol{\eta}_p = [\mathbf{T}_0^d \quad \mathbf{T}_0^r]\begin{bmatrix}\boldsymbol{\eta}_p^d \\ \boldsymbol{\eta}_p^r\end{bmatrix} \Rightarrow \mathbf{M}_g\ddot{\mathbf{u}}_g + \mathbf{K}_g\mathbf{u}_g = \mathbf{0} \Rightarrow \left[\begin{array}{c|c}\mathbf{\Lambda}_d - \lambda\mathbf{M}_{dd} & -\lambda\mathbf{M}_{dr} \\ \hline -\lambda\mathbf{M}_{dr}^T & \mathbf{\Lambda}_r - \lambda\mathbf{M}_{rr}\end{array}\right]\begin{bmatrix}\boldsymbol{\eta}_p^d \\ \boldsymbol{\eta}_p^r\end{bmatrix} = \mathbf{0}$$

Exact transformed form.

Enhanced transformation matrix

4. Expanding the second row linear equation.

$$\begin{bmatrix} \mathbf{\Lambda}_d - \lambda \mathbf{M}_{dd} & -\lambda \mathbf{M}_{dr} \\ -\lambda \mathbf{M}_{dr}^T & \mathbf{\Lambda}_r - \lambda \mathbf{M}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_p^d \\ \boldsymbol{\eta}_p^r \end{bmatrix} = \mathbf{0} \Rightarrow \boldsymbol{\eta}_p^r = \underline{\lambda(\mathbf{\Lambda}_r - \lambda \mathbf{M}_{rr})^{-1} \mathbf{M}_{dr}^T} \boldsymbol{\eta}_p^d$$

✓ The residual term can be substituted into **only dominant term**.

5. Therefore, $\mathbf{u}_g$ can be described **with only dominant term.**

$$\mathbf{u}_g = \mathbf{T}_0^d \boldsymbol{\eta}_p^d + \mathbf{T}_0^r \underline{\boldsymbol{\eta}_p^r} \Rightarrow \mathbf{u}_g = \mathbf{T}_0^d \underline{\boldsymbol{\eta}_p^d} + \lambda \hat{\mathbf{\Psi}} \hat{\mathbf{F}}_r \hat{\mathbf{\Psi}}^T \mathbf{M}_g \mathbf{T}_0^d \underline{\boldsymbol{\eta}_p^d}$$ **:Exact form**

with $\hat{\mathbf{F}}_r = \mathbf{\Phi}_r (\mathbf{\Lambda}_r - \lambda \mathbf{M}_{rr})^{-1} \mathbf{\Phi}_r^T$ **:Residual flexibility matrix**

✓ We cannot calculate this term in direct. **Need approximation**.

6. For approximation, using $\bar{\lambda}$ instead of λ in $\hat{\mathbf{F}}_r$, the following equation is obtained

$$\Rightarrow \bar{\mathbf{F}}_r = \mathbf{\Phi}_r (\mathbf{\Lambda}_r - \bar{\lambda} \mathbf{M}_{rr})^{-1} \mathbf{\Phi}_r^T$$

Enhanced transformation matrix

7. The residual flexibility matrix $\overline{\mathbf{F}}_r$ is expanded and approximated

$$\overline{\mathbf{F}}_r = \mathbf{\Phi}_r(\mathbf{\Lambda}_r - \overline{\lambda}\mathbf{M}_{rr})^{-1}\mathbf{\Phi}_r^T = \underline{\mathbf{\Phi}_r\mathbf{\Lambda}_r^{-1}\mathbf{\Phi}_r^T} + \overline{\lambda}\mathbf{\Lambda}_r^{-1}\mathbf{M}_{rr}\mathbf{\Lambda}_r^{-1}\mathbf{\Phi}_r^T + \mathrm{O}(\overline{\lambda}^2) + \cdots$$

: Known as *Neuman series* of matrix

Considering…

$$\Rightarrow \quad \mathbf{\Phi}_r\mathbf{\Lambda}_r^{-1}\mathbf{\Phi}_r^T = \hat{\mathbf{K}}^{-1} - (\mathbf{\Phi}_d)(\mathbf{\Lambda}_d)^{-1}(\mathbf{\Phi}_d)^T = \mathbf{F}_{rs}$$

: Residual modes are expressed with the dominants term.

8. The global displacement vector is approximated as

$$\overline{\mathbf{u}}_g = \mathbf{T}_0^d\underline{\mathbf{\eta}_p^d} + \lambda\hat{\mathbf{\Psi}}\mathbf{F}_{rs}\hat{\mathbf{\Psi}}^T\mathbf{M}_g\mathbf{T}_0^d\underline{\mathbf{\eta}_p^d}$$

9. **Enhanced transformation matrix** is defined

$$\overline{\mathbf{u}}_g = \overline{\mathbf{T}}_1\mathbf{\eta}_\mathrm{p}^\mathrm{d} \quad \text{with} \quad \underline{\overline{\mathbf{T}}_1 = \overline{\mathbf{T}}_0 + \lambda\overline{\mathbf{T}}_r}, \quad \overline{\mathbf{T}}_r = \hat{\mathbf{\Psi}}\mathbf{F}_{rs}\hat{\mathbf{\Psi}}^T\mathbf{M}_g\mathbf{T}_0^d$$

: Unknown λ is another issue.

Enhanced transformation: Same size with $\overline{\mathbf{T}}_0$

Error estimator for the AMLS method

1. Mass orthonormality and stiffness orthogonality

$$\frac{1}{\lambda_i}(\boldsymbol{\varphi}_g)_i^T \mathbf{K}_g (\boldsymbol{\varphi}_g)_i = (\boldsymbol{\varphi}_g)_i^T \mathbf{M}_g (\boldsymbol{\varphi}_g)_i \Leftarrow$$

$$(\boldsymbol{\varphi}_g) \approx (\overline{\boldsymbol{\varphi}}_g)_i = \overline{\mathbf{T}}_1 (\overline{\boldsymbol{\varphi}}_p)_i = (\overline{\mathbf{T}}_0 + \lambda \overline{\mathbf{T}}_r)(\overline{\boldsymbol{\varphi}}_p)_i$$

$$\xi_i \approx (\overline{\boldsymbol{\varphi}}_p)_i^T [\underline{\bar{\lambda}_i \overline{\mathbf{T}}_0^T \mathbf{M}_g \overline{\mathbf{T}}_r} + \cancel{\bar{\lambda}_i^2 \overline{\mathbf{T}}_r^T \mathbf{M}_g \overline{\mathbf{T}}_r}](\overline{\boldsymbol{\varphi}}_p)_i \Rightarrow \mu_i = \bar{\lambda}_i (\overline{\boldsymbol{\varphi}}_p)_i^T \mathbf{A} \mathbf{F}_{rs} \mathbf{A}^T (\overline{\boldsymbol{\varphi}}_p)_i$$

$$= \mathbf{A}\mathbf{F}_{rs}\mathbf{A}^T \quad \text{with} \quad \mathbf{A} = \boldsymbol{\Phi}_d^T \widetilde{\mathbf{M}}_g$$

[Note]

Calculated during the recursive transformation procedure of the mass matrix: $\overline{\mathbf{M}} = \boldsymbol{\Phi}_d^T \widetilde{\mathbf{M}}_g \boldsymbol{\Phi}_d$

2. Symmetry of matrix $\mathbf{F}_{rs}$

$$\mathbf{F}_{rs} = \mathbf{F}_d + \mathbf{F}_u + \mathbf{F}_u^T$$

3. Finally, we can define an error estimator

$$\mu_i = \bar{\lambda}_i (\overline{\boldsymbol{\varphi}}_p)_i^T \mathbf{E} (\overline{\boldsymbol{\varphi}}_p)_i \quad \text{with} \quad \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_2^T \Rightarrow$$

$$\mathbf{E}_1 = \mathbf{A}\mathbf{F}_d\mathbf{A}^T, \quad \mathbf{E}_{i,j}^1 = \sum_{k=1}^{N_S} \mathbf{A}_{i,k} \mathbf{F}_k^d \mathbf{A}_{k,j}^T$$

$$\mathbf{E}_2 = \mathbf{A}\mathbf{F}_u\mathbf{A}^T, \quad \mathbf{E}_{i,j}^2 = \sum_{k=1}^{N_S} \mathbf{A}_{i,k} \mathbf{F}_k^u \mathbf{A}_{k,j}^T$$

$\overline{N}_p \times \overline{N}_p$ matrix

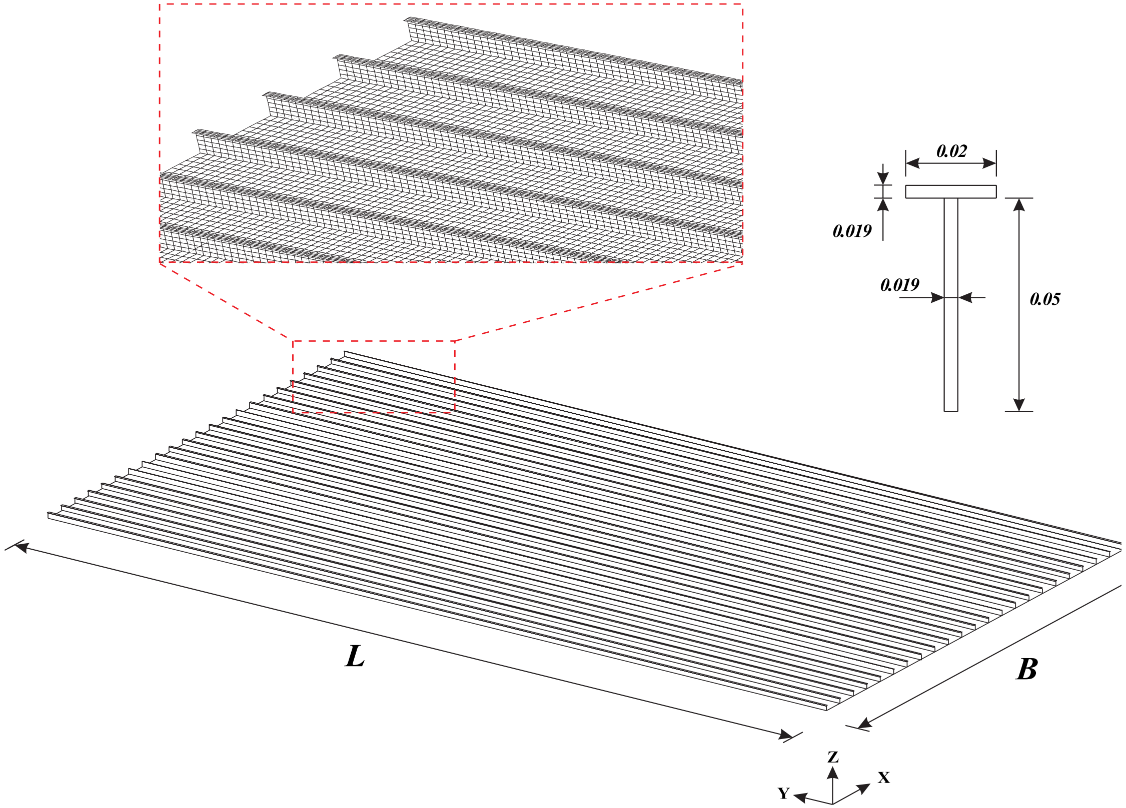
Therefore, we do not need to handle the matrices of global DOF size.

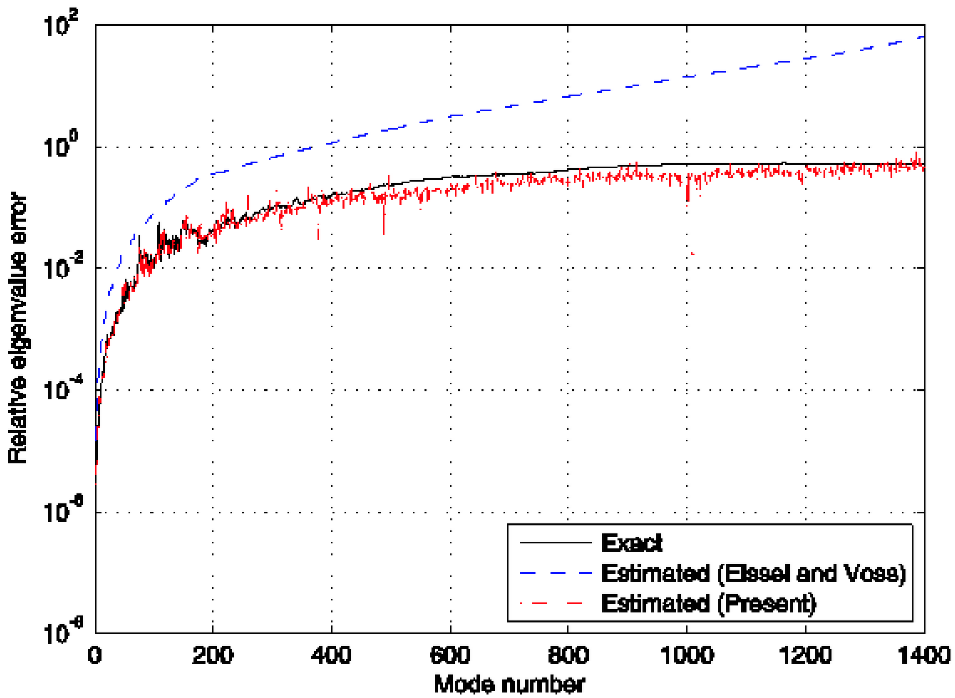
"E" matrix is simply computed with submatrix additions and multiplications.

❖ Stiffened plate structure

Dimension (L=78 m, B=44 m), <u>**1,023**</u> **substructures (Using METIS)**. Free BC is imposed.

Total DOFs = **1,004,088 →** **Retain modes = 2,200**. **Reduced system size = 5,450 (0.5 %)**.





< Relative eigenvalue errors >

❖ Stiffened plate structure

➢ Breakdowns of the computational cost

Items		Related Equations	Computation times [sec]	Ratio [%]
AMLS	Transformation procedures	13	3355.02	96.74
	Solution of the reduced eigenvalue problem	14	113.21	3.26
	Total	-	3468.23	100.00
Error estimation	Calculation of the residual flexibility matrix $\mathbf{F}_{rs}$	36	36.37	1.05
	Construction of $\mathbf{A}$ and $\mathbf{E}$ matrices	48, 51	227.31	6.55
	Calculation error estimator μ_i	50	5.83	0.17
	Total	-	269.51	7.77

The additional computation cost is about 7% compared to that of AMLS method to calculate the relative eigenvalue errors in 1,400 global modes.

❖ **"Lanczos (*eigs* in MATLAB)" : 1,400 global modes (9,500 sec). 1,500 global modes (N/A)**

Closure

1. To develop the error estimation method for the AMLS method, **an enhanced transformation matrix** was newly developed which considered the residual mode effect.

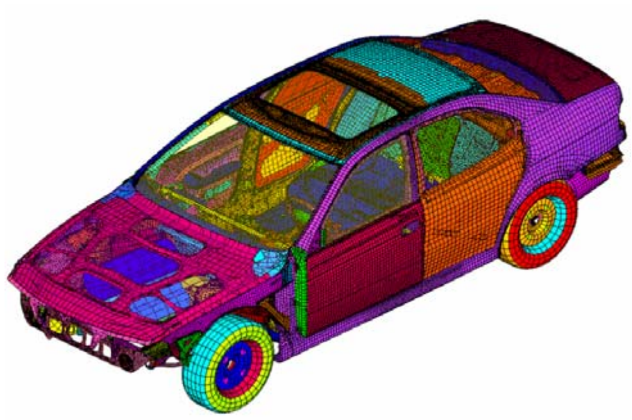
2. Using the enhanced transformation matrix, we proposed **an error estimator for the AMLS method.**

[Note]

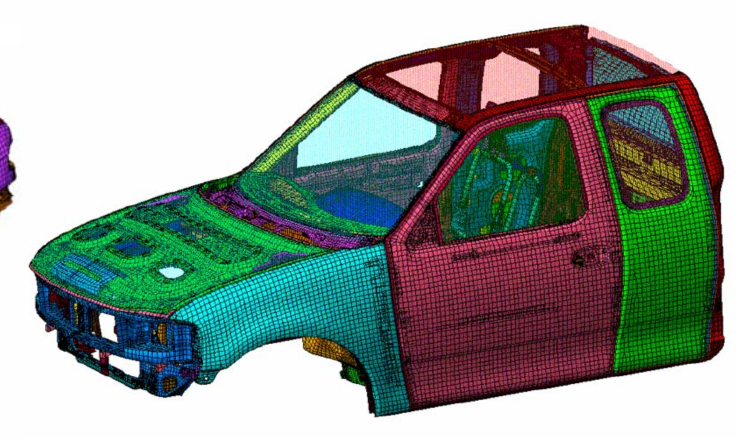
- ❖ **Error estimators** for **CB / AMLS** methods were introduced in the above.
- ❖ The success to develop the error estimator comes from "**Construction of the enhanced transformation matrix**".
- ❖ Therefore, it is possible to develop **the error estimators for any other CMS methods.**

Topic 2. Enhanced AMLS method

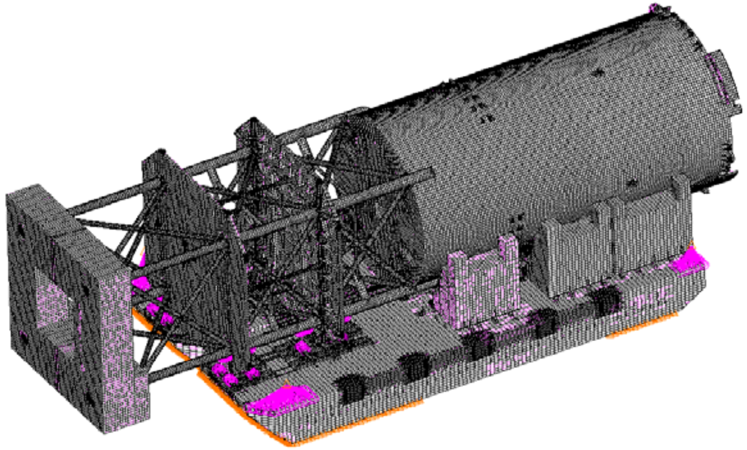
Issue



1,000,000 DOFs



2,000,000 DOFs



3,000,000 DOFs

Demands for more accurate method / large structural analyses:

- ✓ **<u>LANCZOS algorithm has a limitation</u>** to solve large DOFs problem.
- ✓ **<u>AMLS</u>** method was developed **(Alternative of LANCZOS).**
- ✓ Necessary to develop a powerful method **<u>outperforming the AMLS method</u>.**

Enhanced AMLS method

1. The enhanced transformation matrix

$$\mathbf{u}_g \approx \overline{\mathbf{u}}_g = \overline{\mathbf{T}}_1 \boldsymbol{\eta}_p^d \text{ with } \overline{\mathbf{T}}_1 = \overline{\mathbf{T}}_0 + \lambda \overline{\mathbf{T}}_r,\ \overline{\mathbf{T}}_r = \hat{\boldsymbol{\Psi}} \mathbf{F}_{rs} \hat{\boldsymbol{\Psi}}^T \mathbf{M}_g \mathbf{T}_0^d$$

Unknown

How can we handle the unknown λ ?

2. Employing O'Callahan's technique

$$\overline{\mathbf{T}}_0 \Rightarrow \overline{\mathbf{M}}_p \ddot{\overline{\boldsymbol{\eta}}}_p + \overline{\mathbf{K}}_p \overline{\boldsymbol{\eta}}_p = \mathbf{0} \Rightarrow -\lambda \overline{\mathbf{M}}_p \overline{\boldsymbol{\eta}}_p + \overline{\mathbf{K}}_p \overline{\boldsymbol{\eta}}_p = \mathbf{0} \Rightarrow \lambda \overline{\boldsymbol{\eta}}_p = \overline{\mathbf{M}}_p^{-1} \overline{\mathbf{K}}_p \overline{\boldsymbol{\eta}}_p$$

Reduced model of original AMLS method

$$\Rightarrow \overline{\mathbf{T}}_1 = \overline{\mathbf{T}}_0 + \lambda \overline{\mathbf{T}}_r = \overline{\mathbf{T}}_0 + \hat{\boldsymbol{\Psi}} \mathbf{F}_{rs} \hat{\boldsymbol{\Psi}}^T \mathbf{M}_g \mathbf{T}_0^d \overline{\mathbf{M}}_p^{-1} \overline{\mathbf{K}}_p$$

New transformation matrix.

3. Reduced mass and stiffness matrices

$$\tilde{\mathbf{M}}_p = \overline{\mathbf{T}}_1^T \mathbf{M}_g \overline{\mathbf{T}}_1 = \underline{\overline{\mathbf{M}}_p} + \overline{\mathbf{T}}_r^T \mathbf{M}_g \overline{\mathbf{T}}_0 + \overline{\mathbf{T}}_0^T \mathbf{M}_g \overline{\mathbf{T}}_r + \overline{\mathbf{T}}_r^T \mathbf{M}_g \overline{\mathbf{T}}_r$$

$$\tilde{\mathbf{K}}_p = \overline{\mathbf{T}}_1^T \mathbf{K}_g \overline{\mathbf{T}}_1 = \underline{\overline{\mathbf{K}}_p} + \overline{\mathbf{T}}_r^T \mathbf{K}_g \overline{\mathbf{T}}_0 + \overline{\mathbf{T}}_0^T \mathbf{K}_g \overline{\mathbf{T}}_r + \overline{\mathbf{T}}_r^T \mathbf{K}_g \overline{\mathbf{T}}_r$$

✓ **Reduced matrices of original AMLS method**

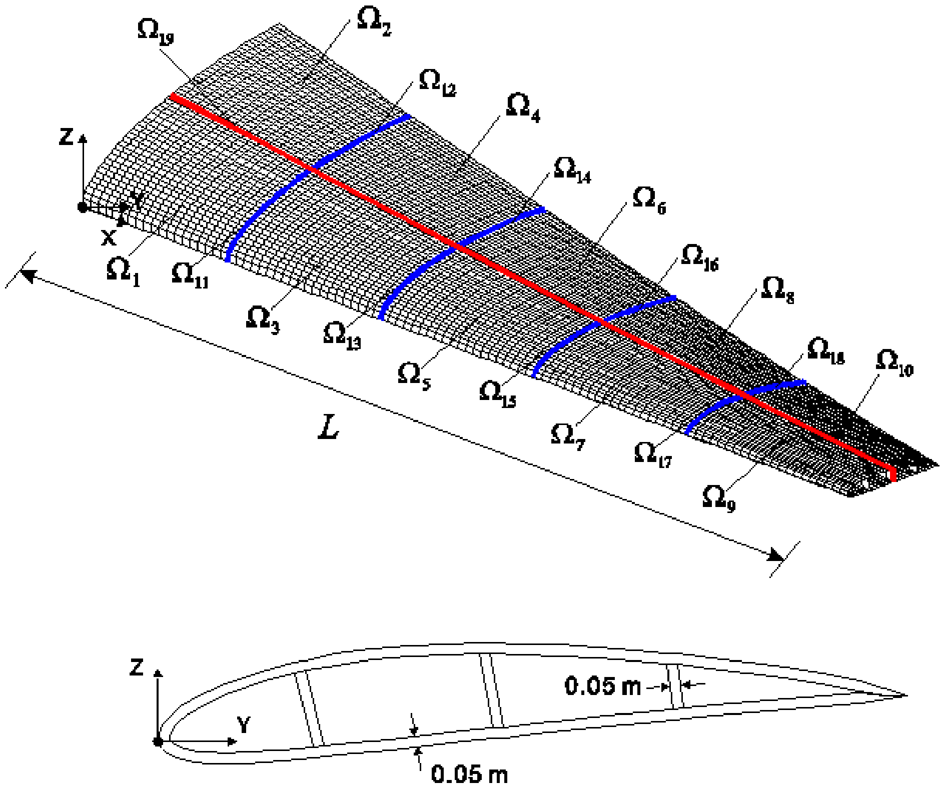
4. Reduced eigenvalue problem

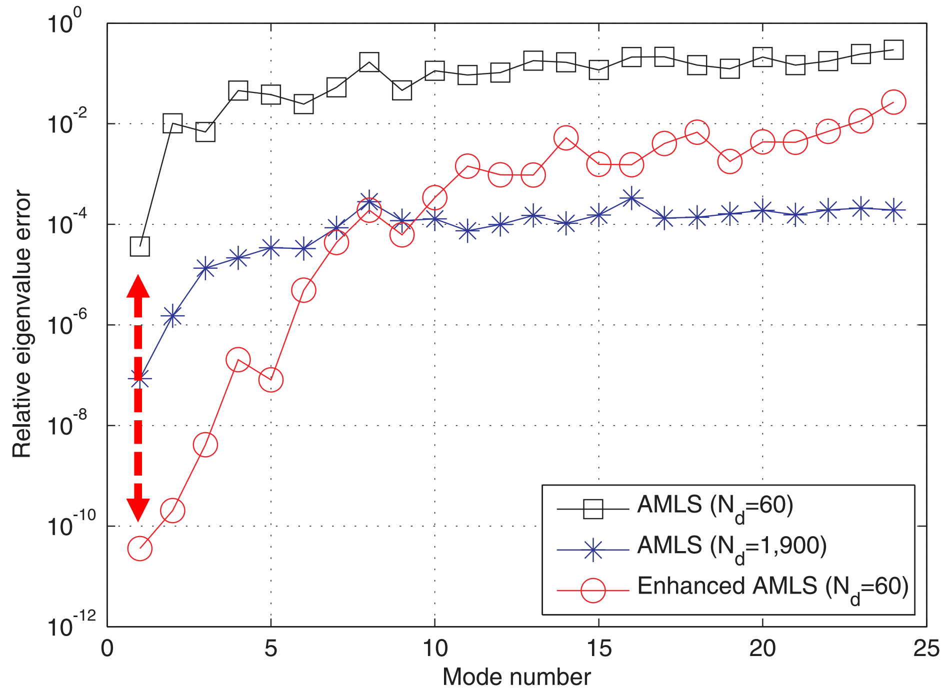
$$\tilde{\mathbf{K}}_p (\overline{\boldsymbol{\varphi}})_i = \overline{\lambda}_i \tilde{\mathbf{M}}_p (\overline{\boldsymbol{\varphi}})_i \Rightarrow (\boldsymbol{\varphi}_g)_i \approx (\overline{\boldsymbol{\varphi}}_g)_i = \overline{\mathbf{T}}_1 (\overline{\boldsymbol{\varphi}}_p)_i$$

❖ Turbine blade structure

Dimension (L=35 m, Thickness=0.05 m), **28 substructures**. **Fixed BC at $x = 0$.**

Total DOFs 51,308 → Retain modes 60. **Reduced system size 1,260 (2.45 %)**.





Accurate 10^5 times for 1^{st} mode.

Closure

1. The **enhanced transformation matrix** considering the residual mode effect is derived.

2. Using the enhanced transformation matrix, **the enhanced AMLS method** was presented.

❖ However, **there are limitations** to use **as an eigenvalue solver**.

- ✓ Does not have capacity to solve large DOFs problems **over 50,000 DOFs**.
- ✓ **Very complicated formulation** of the enhanced transformation matrix

$$\overline{\mathbf{T}}_1 = \overline{\mathbf{T}}_0 + \underline{\hat{\mathbf{\Psi}}\,\mathbf{F}_{rs}\,\hat{\mathbf{\Psi}}^T\,\mathbf{M}_g\,\mathbf{T}_0^d}\,\overline{\mathbf{M}}_p^{-1}\overline{\mathbf{K}}_p \Rightarrow$$

  - **Global matrix computation.**
  - **Fully populated matrix.**
  - **Induce lack of memory and huge costs.**

- ✓ Need to improve its **computational efficiency**.

  1) Formulation modification in the **submatrix computing level**.

  2) **An optimized algorithm** for computer programming.

Topic 3. Algebraic dynamic condensation method

Introduction

Improved Reduced System (IRS) method (O'Callahan, *1989*)

1. **Master / Slave DOFs selection, and reduction.**

$$\mathbf{u}_g = \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_m \end{bmatrix} \approx \bar{\mathbf{u}}_g = \mathbf{T}_1 \mathbf{u}_m \quad \text{with} \quad \mathbf{T}_1 = \mathbf{T}_G + \mathbf{T}_a \mathbf{H}_G \quad (= \mathbf{T}_G + \lambda \mathbf{T}_a)$$

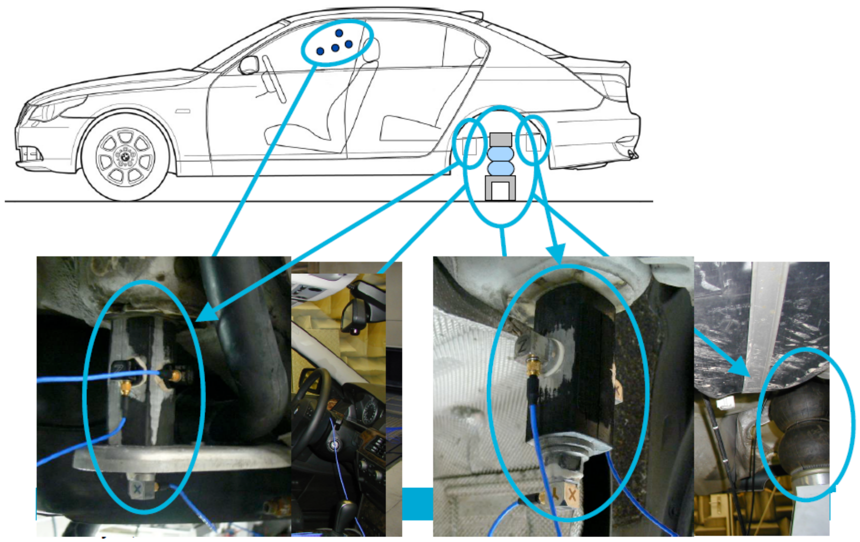
$$\underbrace{\mathbf{T}_G = \begin{bmatrix} -\mathbf{K}_s^{-1}\mathbf{K}_{sm} \\ \mathbf{I}_m \end{bmatrix}}_{\text{Guyan's transformation matrix}}, \quad \mathbf{T}_a = \begin{bmatrix} \mathbf{K}_s^{-1}(\mathbf{M}_{sm} - \mathbf{M}_s \mathbf{K}_s^{-1}\mathbf{K}_{sm}) \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{H}_G = \overline{\mathbf{M}}_G^{-1}\overline{\mathbf{K}}_G$$

2. **Considering inertial effect of slave DOFs**

   → Good accuracy.

3. **Very simple formulation.**

$$\overline{\mathbf{M}}_1 = \mathbf{T}_1^T \mathbf{M}_g \mathbf{T}_1 \ , \quad \overline{\mathbf{K}}_1 = \mathbf{T}_1^T \mathbf{K}_g \mathbf{T}_1$$



4. **No need to sub- eigenvalue analyses.**

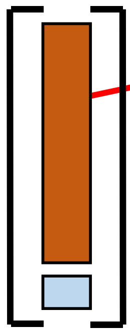
Issue

What is the drawback / limitation?

✓ Cannot apply to large FE model (over 100,000 DOFs).

: IRS transformation matrix

$$\mathbf{T}_1 = \begin{bmatrix} -\mathbf{K}_s^{-1}\mathbf{K}_{sm} + \mathbf{K}_s^{-1}(\mathbf{M}_{sm} - \mathbf{M}_s\mathbf{K}_s^{-1}\mathbf{K}_{sm}) \\ \mathbf{I}_m \end{bmatrix}, \quad \mathbf{T}_1 =$$

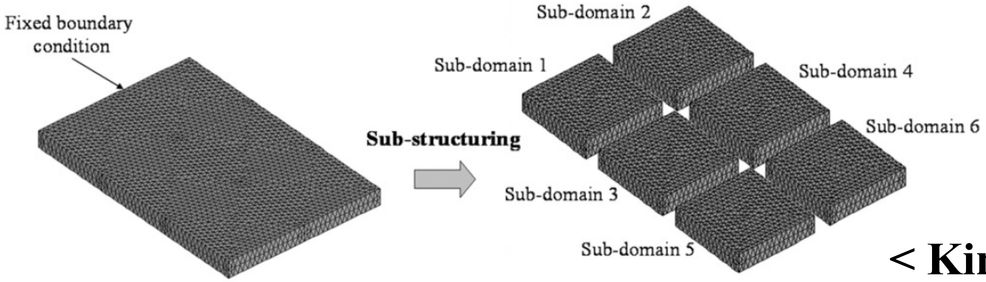


Fully populated part

→ Induce large costs.

Require large memory.

✓ "Domain based partitioning" is not suitable



Substructures: Still large DOFs

→ Large matrix inversion $\mathbf{K}_s^{-1}$

< Kim and Cho, *IJNME*, 2007 >

✓ Rayleigh-Ritz procedure using transformation matrix

$$\overline{\mathbf{M}} = \mathbf{T}_1^T\mathbf{M}_g\mathbf{T}_1, \quad \overline{\mathbf{K}} = \mathbf{T}_1^T\mathbf{K}_g\mathbf{T}_1$$: Expensive global matrix operations.

Design focus

❖ How do we improve the **reduction efficiency** of the IRS method ?

1. Design focus ?

✓ Improving "**computational efficiency**" of the IRS method.

→ **Avoiding** expensive **global matrix operations** and **matrix populations**.

2. Key ideas ?

✓ Employing "**algebraic substructuring**" algorithm.

:Using "**METIS**" adopted in the AMLS method.

✓ **Submatrix** operations.

✓ Transformation **without the global transformation matrix**.

3. What is the "**Strengths**", which is expected ?

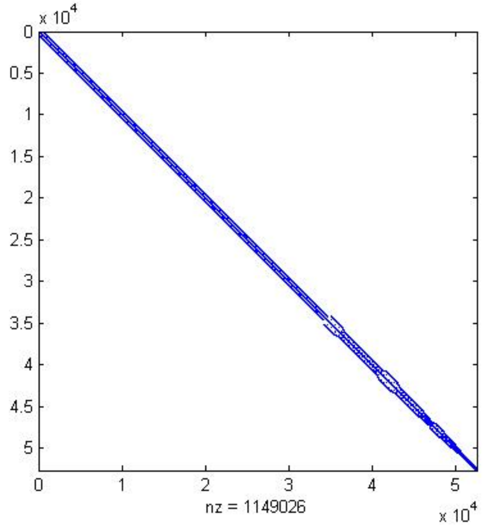
✓ Formulations would be **expressed in submatrix form**: Computationally efficient.

✓ We can **handle** practical engineering problems with **large DOFs in PC**.

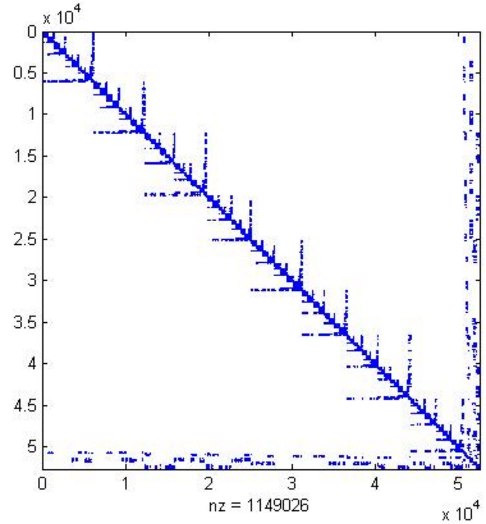
Algebraic dynamic condensation

Step 1. Algebraic substructuring

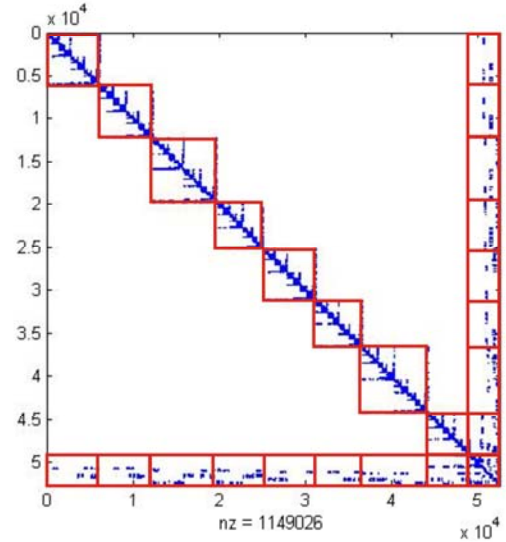
(1) A sparse matrix



(2) Matrix permutation



(3) Partitioned into submatrices



$$\mathbf{M}_g = \left[\begin{array}{ccc|c} \mathbf{M}_1 & & \mathbf{0} & \mathbf{M}_1^c \\ & \ddots & & \vdots \\ \mathbf{0} & & \mathbf{M}_n & \mathbf{M}_n^c \\ \hline (\mathbf{M}_1^c)^T & \cdots & (\mathbf{M}_n^c)^T & \mathbf{M}_b \end{array}\right], \quad \mathbf{K}_g = \left[\begin{array}{ccc|c} \mathbf{K}_1 & & \mathbf{0} & \mathbf{K}_1^c \\ & \ddots & & \vdots \\ \mathbf{0} & & \mathbf{K}_n & \mathbf{K}_n^c \\ \hline (\mathbf{K}_1^c)^T & \cdots & (\mathbf{K}_n^c)^T & \mathbf{K}_b \end{array}\right]$$

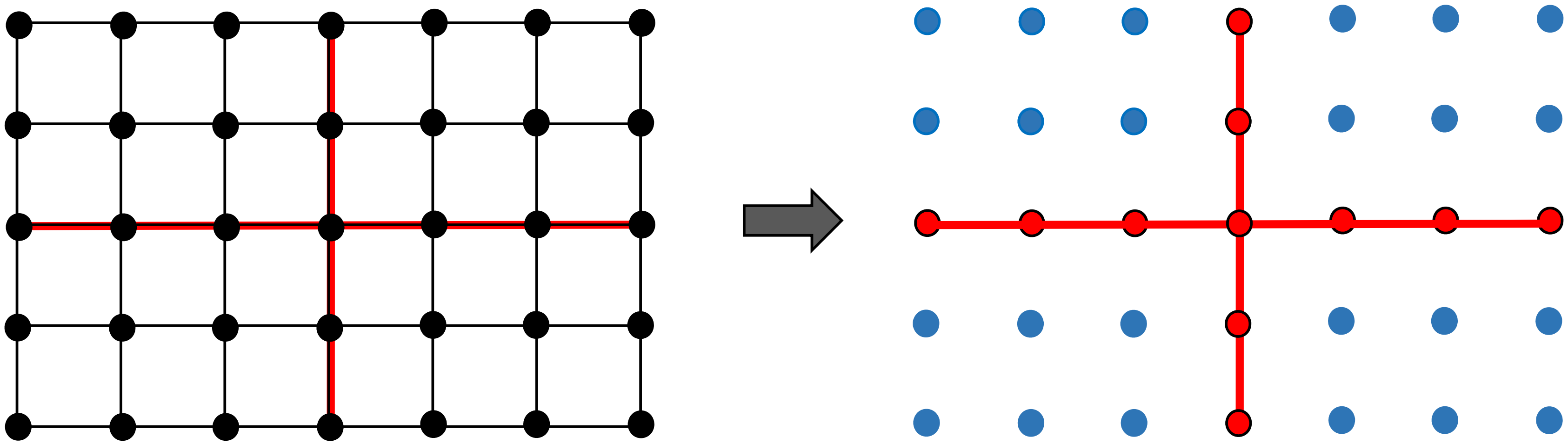
[Note]

Matrix permutation = Renumbering nodes in FE model.

→ Does not alter the physical characteristic of original FE model.

Algebraic dynamic condensation

Step 2. Substructural stiffness condensation



1. Equations of motion

$$\begin{bmatrix} \mathbf{K}_s & \mathbf{K}_c \\ \mathbf{K}_c^T & \mathbf{K}_b \end{bmatrix}\begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_b \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{M}_s & \mathbf{M}_c \\ \mathbf{M}_c^T & \mathbf{M}_b \end{bmatrix}\begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_b \end{bmatrix}$$

2. Expanding the 1st row linear equation

$$\mathbf{u}_s = (\mathbf{K}_s - \lambda \mathbf{M}_s)^{-1}(\mathbf{K}_c - \lambda \mathbf{M}_c)\mathbf{u}_b = [\underline{\boldsymbol{\Psi}_c} + \lambda \mathbf{K}_s^{-1}\hat{\mathbf{M}}_c + o(\lambda^2) + \cdots]\,\mathbf{u}_b \,,$$

Considering…

Static condensation matrix

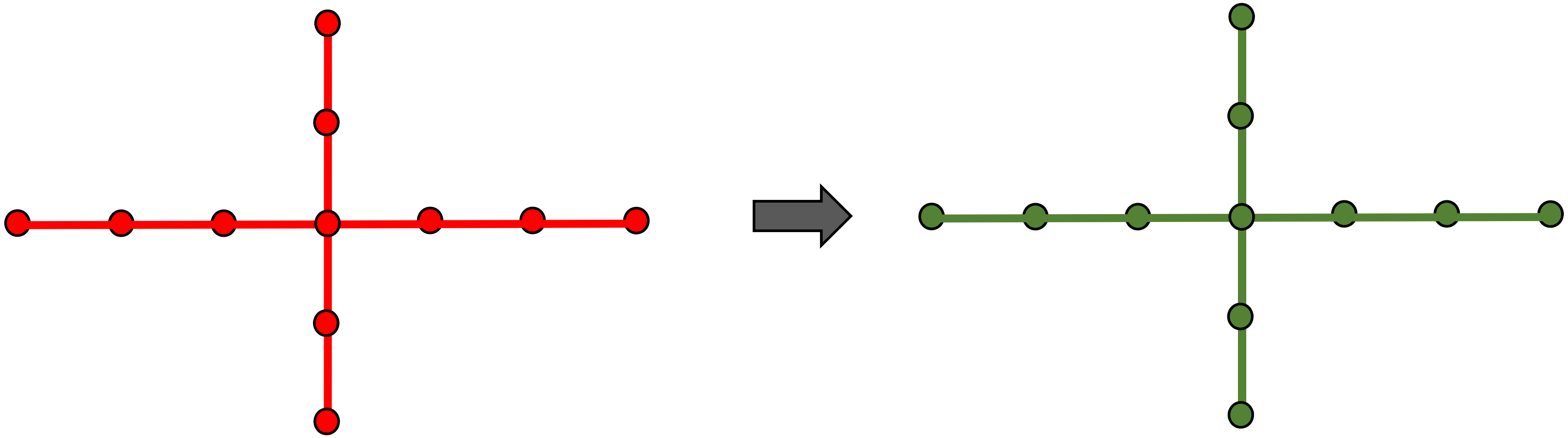
$$\boxed{\mathbf{u}_g \approx \bar{\mathbf{u}}_g = \begin{bmatrix} \bar{\mathbf{u}}_s \\ \mathbf{u}_b \end{bmatrix} = \boldsymbol{\Psi}\mathbf{u}_b \;\; \text{with} \;\; \boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_c \\ \mathbf{I}_b \end{bmatrix}}$$

3. Static condensation procedure

$$\hat{\mathbf{M}}_b = \boldsymbol{\Psi}^T \mathbf{M}_g \boldsymbol{\Psi} \;,\;\; \hat{\mathbf{K}}_b = \boldsymbol{\Psi}^T \mathbf{K}_g \boldsymbol{\Psi} \;\; \Rightarrow \;\; \boxed{\hat{\mathbf{M}}_b = \mathbf{M}_b + \sum_{i=1}^{n}(\mathbf{M}_i^c)^T \boldsymbol{\Psi}_i^c + \sum_{i=1}^{n}(\boldsymbol{\Psi}_i^c)^T \hat{\mathbf{M}}_i^c, \;\; \hat{\mathbf{K}}_b = \mathbf{K}_b + \sum_{i=1}^{n}(\mathbf{K}_i^c)^T \boldsymbol{\Psi}_i^c}$$

Algebraic dynamic condensation

Step 3. Interface boundary reduction



$N_b \times N_b$ matrix

: Still large size

$\overline{N}_b \times \overline{N}_b$ matrix

$\overline{N}_b$ is the number of the dominant interface normal modes.

1. Eigenvalue problem for interface boundary

$$\hat{\mathbf{K}}_b \mathbf{u}_b = \overline{\lambda}\, \hat{\mathbf{M}}_b \mathbf{u}_b \;\Rightarrow\; \mathbf{u}_b = \mathbf{\Phi}_b \mathbf{q}_b = [\underline{\mathbf{\Phi}_b^d} \;\; \mathbf{\Phi}_b^r] \begin{bmatrix} \mathbf{q}_b^d \\ \mathbf{q}_b^r \end{bmatrix} \;\Rightarrow\; \mathbf{u}_b \approx \overline{\mathbf{u}}_b = \mathbf{\Phi}_b^d \mathbf{q}_b^d$$

Considering…

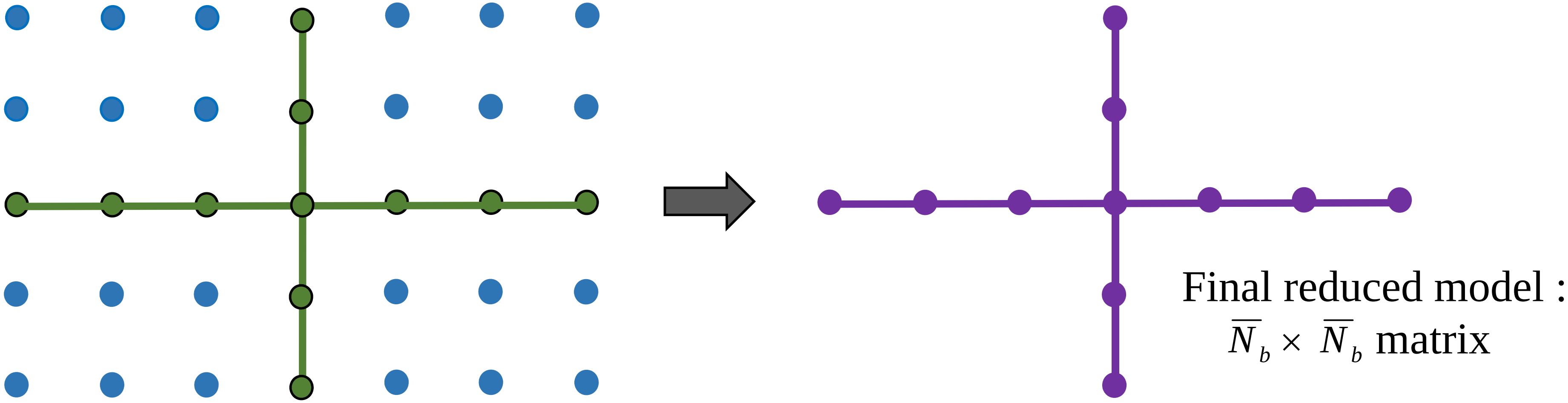
2. Reduce interface boundary

$$\overline{\mathbf{M}}_b = (\mathbf{\Phi}_b^d)^T \hat{\mathbf{M}}_b (\mathbf{\Phi}_b^d) \;, \quad \overline{\mathbf{K}}_b = (\mathbf{\Phi}_b^d)^T \hat{\mathbf{K}}_b (\mathbf{\Phi}_b^d)$$

Algebraic dynamic condensation

Step 4. Substructural inertial effect condensation

Inertial effect



1. Expanded the 1st row linear equation

$$\mathbf{u}_s = (\mathbf{K}_s - \lambda \mathbf{M}_s)^{-1}(\mathbf{K}_c - \lambda \mathbf{M}_c)\mathbf{u}_b = [\underbrace{\boldsymbol{\Psi}_c + \lambda \mathbf{K}_s^{-1}\hat{\mathbf{M}}_c}_{\text{Considering…}} + o(\lambda^2) + \cdots]\,\mathbf{u}_b$$

$$\Rightarrow \quad \bar{\mathbf{u}}_s = (\boldsymbol{\Psi}_c + \lambda \mathbf{K}_s^{-1}\hat{\mathbf{M}}_c)\mathbf{u}_b, \qquad \bar{\mathbf{u}}_b = \boldsymbol{\Phi}_b^d\, \mathbf{q}_b^d$$

Inertial effect condensation matrix

$$\Rightarrow \quad \boxed{\bar{\mathbf{u}}_g = \begin{bmatrix} \bar{\mathbf{u}}_s \\ \bar{\mathbf{u}}_b \end{bmatrix} = \boldsymbol{\Psi}_1 \mathbf{q}_b^d \quad \text{with} \quad \boldsymbol{\Psi}_1 = \bar{\boldsymbol{\Psi}} + \lambda\, \boldsymbol{\Psi}_a}$$

2. Handling unknown λ

$$\hat{\mathbf{K}}_b\,\mathbf{u}_b = \bar{\lambda}\,\hat{\mathbf{M}}_b\,\mathbf{u}_b \quad \Rightarrow \quad \bar{\lambda}\,\mathbf{q}_b^d = \mathbf{H}_b \mathbf{q}_b^d \quad \text{with} \quad \mathbf{H}_b = \bar{\mathbf{M}}_b^{-1}\,\bar{\mathbf{K}}_b$$

Algebraic dynamic condensation

3. Inertial effect condensation procedure

No transformation matrix.

$$\tilde{\mathbf{M}}_b = \mathbf{\Psi}_1^T \mathbf{M}_g \mathbf{\Psi}_1$$

$$\tilde{\mathbf{K}}_b = \mathbf{\Psi}_1^T \mathbf{K}_g \mathbf{\Psi}_1$$

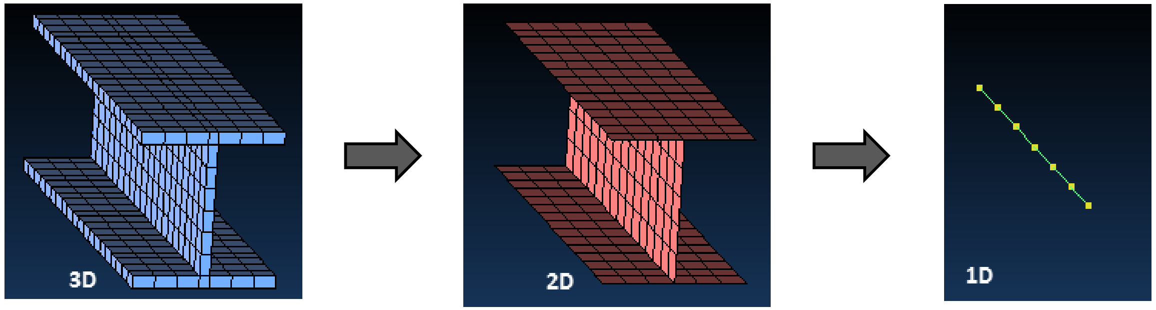
$$\tilde{\mathbf{M}}_b = \overline{\mathbf{M}}_b + \mathbf{R}_1 + \mathbf{R}_1^T + \mathbf{R}_2, \quad \tilde{\mathbf{K}}_b = \overline{\mathbf{K}}_b + \mathbf{H}_b^T \mathbf{R}_1$$

with

$$\mathbf{R}_1 = \sum_{i=1}^{n} (\mathbf{\Phi}_b^d)^T (\hat{\mathbf{M}}_i^c)^T \mathbf{A}_i \mathbf{H}_b \;, \quad \mathbf{R}_2 = \sum_{i=1}^{n} \mathbf{H}_b^T \mathbf{A}_i^T \mathbf{M}_i \mathbf{A}_i \mathbf{H}_b \;, \quad \mathbf{A}_i = \mathbf{K}_i^{-1} \hat{\mathbf{M}}_i^c (\mathbf{\Phi}_b^d)$$

✓ **Reduced system : Simple matrix summation and multiplications at a submatrix level.**

❖ Similar with "FEM degeneration" :



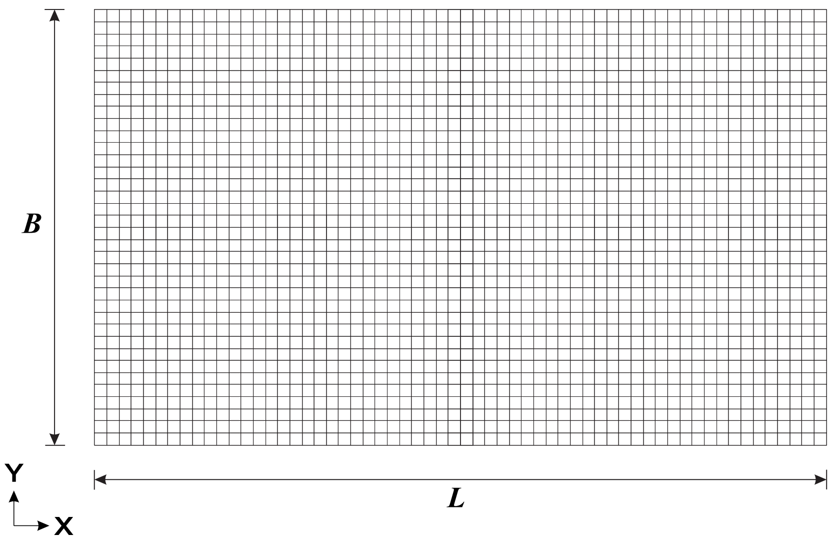
Step 5. Approximated global eigenvectors

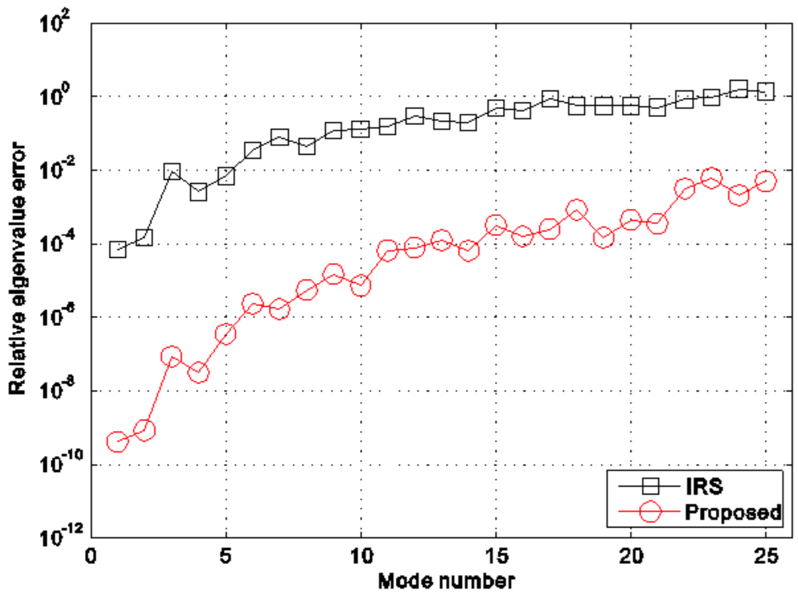
Transformation matrix: $$\mathbf{\Psi}_1 = \begin{bmatrix} \mathbf{\Psi}_1^1 \\ \mathbf{\Psi}_2^1 \\ \vdots \\ \mathbf{\Psi}_n^1 \\ \hline \mathbf{\Phi}_b^d \end{bmatrix} \quad \text{with} \quad \mathbf{\Psi}_i^1 = \mathbf{\Psi}_i^c \mathbf{\Phi}_b^d + \mathbf{K}_i^{-1} \hat{\mathbf{M}}_i^c \mathbf{\Phi}_b^d \mathbf{H}_b \quad \Rightarrow \quad (\overline{\boldsymbol{\varphi}}_g)_i = \mathbf{\Psi}_1 \overline{\boldsymbol{\varphi}}_i$$

❖ Rectangular plate problem

Dimension (L=20 m, B=12 m, Thickness=0.025 m), 16 substructures. No B.C.

Total DOFs = **11,285 → Retain interface modes = 100**. **Reduced system size = 100 (0.88 %)**.





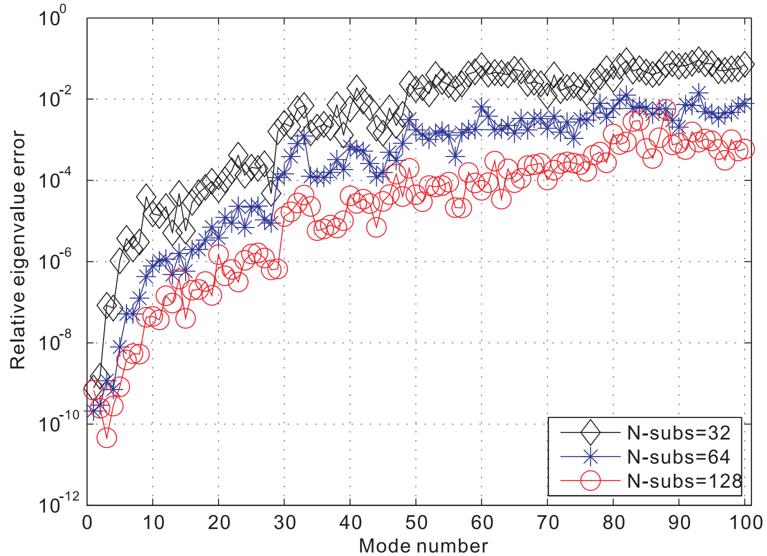
	Items	Related Equations	Computation times [sec]	Ratio [%]
IRS	Transformation procedure	8, 13	205.70	99.95
	Reduced eigenvalue problem	14	0.10	0.05
	Total	-	205.80	100.00
Proposed	Algebraic substructuring	16	0.05	0.02
	Calculation of $\hat{\mathbf{M}}_b$ and $\hat{\mathbf{K}}_b$ matrices	25	0.61	0.30
	Eigenvalue problem for the interface boundary	22	0.49	0.24
	Calculation of $\overline{\mathbf{M}}_b$, $\overline{\mathbf{K}}_b$ and $\mathbf{H}_b$ matrices	28, 33	0.03	0.01
	Calculation of $\tilde{\mathbf{M}}_b$ and $\tilde{\mathbf{K}}_b$ matrices	36	0.70	0.34
	Reduced eigenvalue problem	37	0.10	0.05
	Total	-	1.98	0.96

Compared to the computation time required for the IRS method: 100 times faster

❖ Truss structure

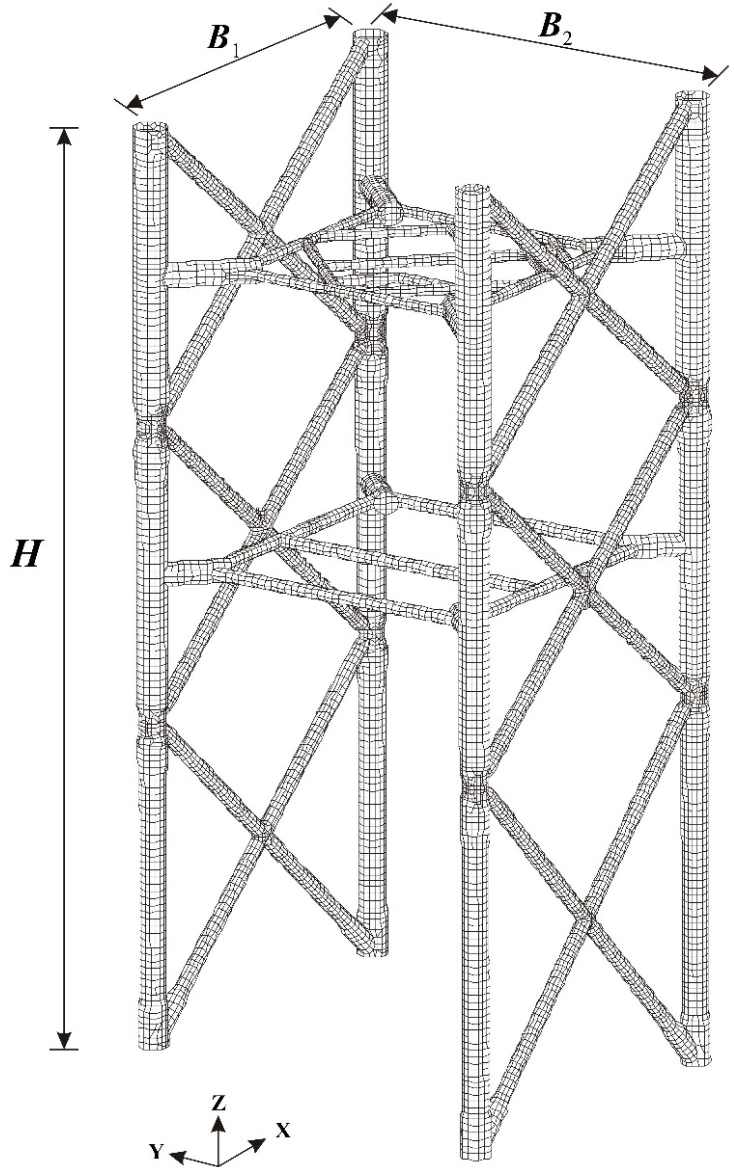
Dimension (L=98 m, B_1=45 m, B_2=48 m), No B.C.

Total DOFs = **155,766 → Retain interface modes = 150**. **Reduced system size = 150 (0.09 %)**.



Items	Number of substructures	Computation times [sec]
IIRS	-	N/A
Proposed	32	339.5
	64	168.6
	128	101.9

Faster



✓ **As the number of substructures increases,**

both accuracy and computation cost improve.

Closure

1. We develop a new efficient reduced-order modeling technique, named **algebraic dynamic condensation**.

2. The formulation can be simply **expressed using multiplications and summations of submatrices**, and thus it presents **excellent computational efficiency**.

3. The proposed method also provides **better accuracy** than the IRS method, and it can **handle relatively large FE models**, for which the IRS method fails to work.

4. Future works

Future works

Issue 1. Solution "Reliability"

- ✓ Error estimator applicable to the CB method **using the interface reduction technique**.
- ✓ **Iterative mode selection algorithms** to construct accurate reduced-order models.
- ✓ **More simplified error estimator** considering FE models with more than several millions of DOFs.

Issue 2. Solution "Accuracy"

- ✓ **A new eigenvalue solver** using the enhanced AMLS method

  → Increasing computational efficiency. Optimized algorithm for implementation.

Issue 3. Solution "Efficiency"

- ✓ **An effective iterative scheme** could also be developed.
- ✓ Employing the **multi-level algebraic substructuring** to solve FE models with more than several millions of DOFs.

Q & A