Development of Continuum Mechanics Based Beam Elements for Linear and Nonlinear Analysis

Kyungho Yoon

2014
1. Introduction to Continuum Mechanics Based Beam Elements

2. Warping Model for Discontinuously Varying Cross-section

3. Implementation for General Nonlinear Analysis

4. Improvement of Nonlinear Performance

5. Summary & Future Works
1. Introduction to Continuum Mechanics
Based Beam Elements
Applications of Beam Finite Element Analysis

- Bridge
- Offshore platform
- Fabric bundle

- Physical model: 1134 DOFs
- Solid model: 462 DOFs
- Shell model: 36 DOFs
- Beam model: 36 DOFs
Literature of Beam Element

- Search word “beam element” limit to exact keyword “finite element method”
## Various Beam Elements

### In commercial software

<table>
<thead>
<tr>
<th>Software</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABAQUS</td>
<td>B21, B31, B22, B32, B23, B33, B34</td>
</tr>
<tr>
<td>ANSYS</td>
<td>BEAM3, BEAM4, BEAM23, BEAM24, BEAM44, BEAM54, BEAM161, BEAM188, BEAM189</td>
</tr>
<tr>
<td>ADINA</td>
<td>BEAM, ISOBEAM</td>
</tr>
</tbody>
</table>

### In literature


### Degenerated beam


### Pros
- 3D geometry
- Fully coupled strain states
- Simple and Straightforward
- Modeling capability

### Cons
- Cross-sectional shape
- Nonlinear performance
Problem description #1 – Twisting Action

- Warping effect

\[ k = 26078 \]

\[ k = 81.8284 \]

Stiff: with warping

Flexible: without warping
Problem description #1 – Twisting Action

- **Mathematical theory**

<table>
<thead>
<tr>
<th></th>
<th>St. Venant torsion theory</th>
<th>Vlasov thin-walled theory</th>
<th>Benscoter theory</th>
<th>Jourawsky theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formulation</td>
<td>( u_w = \varphi(y, z) )</td>
<td>( u_w = \phi(y, z) \frac{\partial \theta_x(x)}{\partial x} )</td>
<td>( u_w = \phi(y, z) \alpha(x) )</td>
<td>( u_w = \phi(y, z) \frac{\partial \theta_x(x)}{\partial x} + \psi(y, z) )</td>
</tr>
<tr>
<td>Uniform torsion</td>
<td>O</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Non-uniform torsion</td>
<td>X</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Secondary warping</td>
<td>X</td>
<td>X</td>
<td>O</td>
<td>O</td>
</tr>
</tbody>
</table>

- **Finite element implementation**
Problem description #2 – Degenerated Element

• Degeneration

• Previous approach

• New approach
Degeneration Procedure

- Geometry interpolation of the sub-beams

\[ \tilde{x}^{(m)} = \sum_{k=1}^{q} h_k(r)^{i} \tilde{x}_k + \sum_{k=1}^{q} h_k(r) \tilde{y}_k^{(m)} v_k + \sum_{k=1}^{q} h_k(r) \tilde{z}_k^{(m)} v_z \]

- Kinematic assumption

\[ \tilde{x}_k^{(m)} = \tilde{x}_k + \tilde{y}_k^{(m)} v_y + \tilde{z}_k^{(m)} v_z \]
Degeneration Procedure

\[ t \mathbf{x}^{(m)} = \sum_{k=1}^{q} h_k(r)^t \mathbf{x}_k + \sum_{k=1}^{q} h_k(r) \mathbf{y}_k^{(m)} + \sum_{k=1}^{q} h_k(r) \mathbf{z}_k^{(m)} + \sum_{k=1}^{q} h_k(r) f_k^{(m)} (s,t)^t \alpha_k^t \mathbf{V}_k \]

\[ \mathbf{y}_k^{(m)} = \sum_{j=1}^{p} h_j(s,t) \mathbf{y}_j^{(m)} \]
\[ \mathbf{z}_k^{(m)} = \sum_{j=1}^{p} h_j(s,t) \mathbf{z}_j^{(m)} \]
\[ f_k^{(m)} (s,t) = \sum_{j=1}^{p} h_j(s,t) f_j^{(m)} \]
Calculation of Warping Function

- St. Venant equations
  \[
  \frac{\partial^2 f_k^{(m)}}{\partial y^2} + \frac{\partial^2 f_k^{(m)}}{\partial z^2} = 0 \quad \text{in} \quad \Omega
  \]
  \[
  n_y \frac{\partial f_k^{(m)}}{\partial y} + n_z \frac{\partial f_k^{(m)}}{\partial z} = n_y \bar{z} - n_z \bar{y} \quad \text{on} \quad \Gamma
  \]

- Finite element discretization
  \[
  f_k^{(m)}(s,t) = \sum_{j=1}^{p} h_j(s,t) f_k^{(j(m))}
  \]

- Multiply connected cross-section
Novel Features

◆ The formulation is simple and straightforward.

◆ The formulation can handle all complicated 3-D geometries including curved and twisted geometries, varying cross-sections, and arbitrary cross-sectional shapes (including thin/thick-walled and open/closed cross-sections).

◆ Warping effects fully coupled with bending, shearing, and stretching are automatically included.

◆ Seven degrees of freedom (only one additional degree of freedom for warping) are used at each beam node to ensure inter-elemental continuity of warping displacements.

◆ The pre-calculation of cross-sectional properties (area, second moment of area, etc.) is not required because the beam formulation is based on continuum mechanics.

◆ Analyses of short, long, and deep beams are available, and eccentricities of loadings and displacements on beam cross-sections are naturally considered.

◆ The basic formulation can be easily extended to general nonlinear analyses that consider geometrical and material nonlinearities.
Curved Beam Problem

• Problem description

- Numerical results

<table>
<thead>
<tr>
<th>D/R</th>
<th>N=4</th>
<th>N=8</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.8613E-05</td>
<td>1.8599E-05</td>
<td>1.9020E-05</td>
</tr>
<tr>
<td>0.2</td>
<td>4.0321E-06</td>
<td>4.0296E-06</td>
<td>4.2771E-06</td>
</tr>
<tr>
<td>0.5</td>
<td>6.1726E-08</td>
<td>6.1694E-08</td>
<td>6.1724E-08</td>
</tr>
</tbody>
</table>
Tapered Beam Problem

- Problem description

![Diagram of tapered beam](image1)

- Numerical results

<table>
<thead>
<tr>
<th>$\tan 0.5\theta$</th>
<th>Present study</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>1.1454E-04</td>
<td>1.16549E-04</td>
</tr>
<tr>
<td>0.095</td>
<td>5.9483E-05</td>
<td>6.01527E-05</td>
</tr>
<tr>
<td>0.225</td>
<td>2.0123E-05</td>
<td>1.91705E-05</td>
</tr>
<tr>
<td>0.475</td>
<td>8.2445E-06</td>
<td>7.11663E-06</td>
</tr>
</tbody>
</table>
Turbine Blade Problem

- Problem description
  - Multiply connected cross-section
  - Curved, twisted and tapered geometry
  - Eccentricity

- Numerical results
  - Shell: 4680 DOFs
  - Beam: 56 DOFs
A continuum mechanics based 3-D beam finite element with warping displacements and its modeling capabilities

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Abstract. In this paper, we propose a continuum mechanics based 3-D beam finite element with cross-sectional discretization allowing for warping displacements. The beam element is directly derived from the assemblage of 3-D solid elements, and this approach results in inherently advanced modeling capabilities of the beam element. In the beam formulation, warping is fully coupled with bending, shearing, and stretching. Consequently, the proposed beam elements can consider free and constrained warping conditions, eccentricities, curved geometries, varying sections, as well as arbitrary cross-sections (including thin/thick-walled, open/closed, and single/multi-cell cross-sections). We then study the modeling and predictive capabilities of the beam elements in twisting beam problems according to geometries, boundary conditions, and cross-sectional meshes. The results are compared with reference solutions obtained by analytical methods and solid and shell finite element models. Excellent modeling capabilities and solution accuracy of the proposed beam element are observed.

Keywords: beams; finite elements; torsion; twisting; warping; arbitrary cross-sections
2. Warping Model for Discontinuously Varying Cross-section
Motivation

• Previous models
  • Free warping
  • Constrained warping
  • Varying & curved geometry

• Challenge
  • Twisting center
  • Warping displacement shape
New Warping Displacement Model

- Previous model

\[ u^{(m)}_w(r,s,t) = \sum_{k=1}^{q} h_k(r) f^{(m)}_k(s,t) \alpha_k V_r^k \]

- New warping displacement

\[ u^{(m)}_w(r,s,t) = \sum_{k=1}^{q} h_k(r) \left[ f^{(m)}_k(s,t) \alpha_k + f^{(m)}_L(s,t) \beta_L^k + f^{(m)}_R(s,t) \beta_R^k \right] V_r^k \]

(a) cross-section (A)  cross-section (B)

(b) beam region (A)  beam region (B)

(c) 

Nodal warping DOFs

- \( \beta^{(4)}_L \) + \( \alpha^{(3)}_x \)
- \( \alpha^{(4)}_x \)
- \( \beta^{(4)}_x = \beta^{(3)}_L \)

\[ \beta^{(4)}_L \quad \alpha^{(4)}_x \quad \beta^{(4)}_x = \beta^{(3)}_L \quad \alpha^{(3)}_x \quad \beta^{(3)}_x \]

Interface warpings

Free warping
Free Warping Function

• **Existing method**
  1. Solve St. Venant equation for global coordinates
  2. Calculate twisting center
  3. Calculate warping function for twisting center

  3 step procedure

• **Proposed new method**
  ✓ Simultaneously calculate the **free warping function**
    and the corresponding **twisting center**

  Coordinate transformed St. Venant equation
  + Zero bending moment condition

\[
\begin{bmatrix}
GK & GN_y & -GN_z \\
GH_y & 0 & 0 \\
GH_z & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
F \\
\lambda_y \\
\lambda_z \\
\end{bmatrix}
= 
\begin{bmatrix}
GB \\
0 \\
0 \\
\end{bmatrix}
\]
Interface Warping Function

Coordinate transformed St. Venant equation with twisting center multiplier
+ Zero bending moment condition
+ Constraint condition for interconnected domain

\[
\begin{bmatrix}
G_1K^{(I)} & 0 & G_1N^{(I)}_{\bar{y}} & -G_1N^{(I)}_{z} & L^{(I)} \\
0 & G_2K^{(II)} & G_2N^{(II)}_{\bar{y}} & -G_2N^{(II)}_{z} & -L^{(II)} \\
G_1H^{(I)}_{\bar{y}} & G_2H^{(II)}_{\bar{y}} & 0 & 0 & 0 \\
G_1H^{(I)}_{z} & G_2H^{(II)}_{z} & 0 & 0 & 0 \\
L^{(I)T} & -L^{(II)T} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
F^{(I)} \\
F^{(II)} \\
\lambda_{\bar{y}} \\
\lambda_{y} \\
\lambda
\end{bmatrix}
=
\begin{bmatrix}
G_1B^{(I)} \\
G_2B^{(II)}
\end{bmatrix}
\]
Problem description

Discontinuously Varying Thin-walled Cross-section

Numerical results

107 DOFs

230 DOFs

Bending induced by torsion
Partially Constrained Warping Problem

- Problem description

\[ M_x = 1.0 \text{N} \cdot \text{m} \]

\[ x = 0 \quad 10 \quad x = 10 \]

(a)

(b) 8 DOFs used

- Numerical results

<table>
<thead>
<tr>
<th>Solid element model</th>
<th>Present beam element model</th>
<th>Beam element model with the displacement model in Eq. (8)</th>
<th>Fully constrained warping</th>
<th>Free warping</th>
</tr>
</thead>
</table>

(c)

Image captions and figures: (a) Schematic of the warping problem, (b) DOFs used, (c) Another view of the warping problem.

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Curved beam problem

• Problem description

- (Load case - I)
  - $F_x = 1.0N$
  - $M_x = 1.0N \cdot m$

- (Load case - II)

- $\phi = 45^\circ$
  - $R = 50$

- DOFs used

• Numerical results

- Solid element model
  - Present beam element model

- (Load case - I)

- (Load case - II)
Modeling the warping displacements for discontinuously varying arbitrary cross-section beams

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SCImago Journal Rank (SJR): 1.919

Computers and Structures
3. Implementation for General Nonlinear Analysis
Implementation for nonlinear analysis

- Priniple of virtual work

\[
\int_0^{t+\Delta t} \delta \epsilon_{ij} \delta \epsilon_{ij} d^0V = \int_0^{t+\Delta t} \mathcal{R}
\]

with parameter \( \mathbf{U}_k = [u_k \ v_k \ w_k \ \theta_x \ \theta_y \ \theta_z \ \alpha_k] \)

<table>
<thead>
<tr>
<th>Geometric nonlinearity</th>
<th>Material nonlinearity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strain-displacement relation</td>
<td>Finite rotation</td>
</tr>
</tbody>
</table>
| \( \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) | \( \mathbf{R} = \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix} \) | \( S_{ij} = C_{ijkl} \epsilon_{kl} \)
| \( \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \) | \( \mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \) | \( \epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p \) | \( S_{ij} = C_{ijkl} \epsilon_{kl}^e \)
| Green-Lagrange strain | Rodrigues formulation | Implicit return mapping scheme with beam state projection |

- Incremental equilibrium equation

\[
\int_0^V \mathbf{\tilde{C}}_{ijrs} \delta \epsilon_{ij} \delta \epsilon_{rs} d^0V + \int_0^V \mathbf{\tilde{S}}_{ij} \delta \mathbf{\tilde{S}}_{ij} d^0V + \int_0^V \mathbf{\tilde{S}}_{ij} \delta \mathbf{\tilde{S}}_{ij} d^0V = \int_0^{t+\Delta t} \mathcal{R} - \int_0^V \mathbf{\tilde{S}}_{ij} \delta \mathbf{\tilde{S}}_{ij} d^0V
\]
Implementation for nonlinear analysis

- **Geometry interpolation**

\[
\begin{align*}
{t}^{(m)}x_k &= \sum_{k=1}^{q} h_k(r) {t}^{(m)}x_k + \sum_{k=1}^{q} h_k(r) y^{(m)}_{k} V_{y}^k + \sum_{k=1}^{q} h_k(r) z^{(m)}_{k} V_{z}^k + \sum_{k=1}^{q} h_k(r) f_{k}^{(m)}(s,t) \alpha_k {t}^k V_{x}^k \\
\end{align*}
\]

- **Displacement interpolation**

\[
\begin{align*}
{0}^{(m)}u_k &= \sum_{k=1}^{q} h_k(r) {0}^{(m)}u_k + \sum_{k=1}^{q} h_k(r) y^{(m)}_{k} (t+\Delta t) V_{y}^k - \sum_{k=1}^{q} h_k(r) z^{(m)}_{k} (t+\Delta t) V_{z}^k + \sum_{k=1}^{q} h_k(r) f_{k}^{(m)}(s,t) (t+\Delta t) \alpha_k {t}^k V_{x}^k \\
\end{align*}
\]

- **Finite rotation parameterization**

\[
R(0 \theta^k) = I + \sin_0 \theta^k \hat{R}(0 \theta^k) + \frac{1 - \cos_0 \theta^k}{\theta^k} \hat{R}(0 \theta^k)^2
\]

with

\[
0 \theta^k = \begin{bmatrix}
\theta_x^k & \theta_y^k & \theta_z^k
\end{bmatrix}^T
\]

\[
0 \theta^k = \sqrt{\theta_x^k + \theta_y^k + \theta_z^k}
\]

\[
\hat{R}(0 \theta^k) = \begin{bmatrix}
0 & -\theta_z^k & \theta_y^k \\
\theta_z^k & 0 & -\theta_x^k \\
-\theta_y^k & \theta_x^k & 0
\end{bmatrix}
\]

- **Finite rotation of director**

\[
\begin{align*}
(t+\Delta t) V_{y}^k &= R(0 \theta^k) (t) V_{y}^k \\
(t+\Delta t) V_{z}^k &= R(0 \theta^k) (t) V_{z}^k
\end{align*}
\]
Wagner Strain


In the models based on 'geometrically exact beam theory', however, the Wagner effects *viz.* the Wagner term within the finite strains due to the second-order of twist and the corresponding Wagner moment stress resultant [3,4] do not appear to be considered or included. The Wagner effects also appear to have been ignored in a number of the finite beam elements that are based on the weak form of the continuum equilibrium equations, and in some FE packages such as ABAQUS [1] and ANSYS [2]. It has been found [3–5] that the Wagner term in the finite strains and the corresponding Wagner moment play an important part.

Additionally introduce Wagner moment and Wagner strain

Explicitly consideration


In the above studies, nonlinear terms such as flexural-torsional coupling or shortening effects are usually ignored. Many other studies have been devoted to nonlinear behaviour of these structures in bending and torsion and proved that in large torsion, the shortening effect is important and

Additionally introduce three Wagner’s coefficients

Explicitly consideration

✔ To consider fully coupled nonlinear strain terms is very important
Wagner strain

- Geometry interpolation

\[
\begin{align*}
0\mathbf{x}^{(m)} &= \begin{bmatrix}
0x \\
\bar{y}^{(m)} \\
\bar{z}^{(m)}
\end{bmatrix}, \\
t\mathbf{x}^{(m)} &= \begin{bmatrix}
(t x + f_k^{(m)}t \alpha) \\
\bar{y}^{(m)} \cos \theta_x - \bar{z}^{(m)} \sin \theta_x \\
\bar{y}^{(m)} \sin \theta_x + \bar{z}^{(m)} \cos \theta_x
\end{bmatrix}
\end{align*}
\]

- Covariant vectors

\[
\begin{align*}
0\mathbf{g}_1^{(m)} &= \begin{bmatrix}
\partial^0 x / \partial r \\
0 \\
0
\end{bmatrix}, \\
t\mathbf{g}_1^{(m)} &= \begin{bmatrix}
\partial^t x / \partial r + f_k^{(m)} \partial^t \alpha / \partial r \\
\partial \theta_x / \partial r (-\bar{y}^{(m)} \sin \theta_x - \bar{z}^{(m)} \cos \theta_x) \\
\partial \theta_x / \partial r (\bar{y}^{(m)} \cos \theta_x - \bar{z}^{(m)} \sin \theta_x)
\end{bmatrix}
\end{align*}
\]

- Covariant Green-Lagrange strain

\[
\begin{align*}
0\varepsilon_{11}^{(m)} &= \frac{\partial^0 x}{\partial r} f_k^{(m)} \frac{\partial^t \alpha}{\partial r} + \frac{1}{2} \left(f_k^{(m)} \frac{\partial^t \alpha}{\partial r}\right)^2 + \frac{1}{2} \left(\bar{y}^{(m)} \cos \theta_x - \bar{z}^{(m)} \sin \theta_x\right)^2 \left(\frac{\partial \theta_x}{\partial r}\right)^2
\end{align*}
\]

- Local Green-Lagrange strain

\[
\begin{align*}
0\varepsilon_{11} &= f_k^{(m)} \frac{\partial^t \alpha}{\partial x} + \frac{1}{2} \left(f_k^{(m)} \frac{\partial^t \alpha}{\partial x}\right)^2 + \frac{1}{2} \left(\bar{y}^{(m)} \cos \theta_x - \bar{z}^{(m)} \sin \theta_x\right)^2 \left(\frac{\partial \theta_x}{\partial x}\right)^2
\end{align*}
\]
Cross-Shaped Beam Problem

- Problem description

42 DOFs

Elasto perfectly plastic material model

\[ E = 2.0 \times 10^{11} N / m^2 \]
\[ \nu = 0 \]
\[ H = 0 \]
Cross-Shaped Beam Problem

- Numerical results

---

![Graph showing numerical results](image)

<table>
<thead>
<tr>
<th>Beam problems</th>
<th>Solid element model (ADINA)</th>
<th>Shell element model (ADINA)</th>
<th>Beam element models</th>
<th>Solid element</th>
<th>Present beam element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-shaped beam</td>
<td>○</td>
<td>X</td>
<td>ADINA BEAM (ADINA)</td>
<td>ADINA BEAM</td>
<td>Present beam</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>BEAM188 (ANSYS)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Present beam</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>DOFs</td>
<td>111894</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Load step</td>
<td>800</td>
<td>1</td>
</tr>
</tbody>
</table>
Twisted Cantilever Beam Problem

- Problem description

Elastic material model

\[ E = 2.0 \times 10^{11} \text{ N/m}^2 \]

\[ v = 0 \]
Twisted Cantilever Beam Problem

- Numerical results

<table>
<thead>
<tr>
<th>Beam problems</th>
<th>Solid element model (ADINA)</th>
<th>Shell element model (ADINA)</th>
<th>Beam element models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twisted cantilever beam problem</td>
<td>○</td>
<td>○</td>
<td>×</td>
</tr>
<tr>
<td>(Load Case I)</td>
<td>ADINA BEAM (ADINA)</td>
<td>BEAM188 (ANSYS)</td>
<td>Present beam</td>
</tr>
<tr>
<td>Twisted cantilever beam problem</td>
<td>○</td>
<td>○</td>
<td>×</td>
</tr>
<tr>
<td>(Load Case II)</td>
<td>ADINA BEAM (ADINA)</td>
<td>BEAM188 (ANSYS)</td>
<td>Present beam</td>
</tr>
</tbody>
</table>
Lateral post buckling problem

- Problem description

(a) 

Load case - I

Load case - II

Load case - III

Load case - IV

(b) 

(c)
Lateral post buckling problem

- Numerical results

(a) [Graph showing load vs. displacement for different models]

(b) [Graph showing load vs. displacement for different models]
Nonlinear performance of continuum mechanics based beam elements focusing on large twisting behaviors

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7. Theoretical and Computational Fluid Dynamics
8. Journal of Turbulence
10. International Journal for Multiscale Computational Engineering
4. Improvement of Nonlinear Performance
Problem Description

- Problems in nonlinear analysis
  1. Upper bound
  2. Slow convergence
  3. Solution reliability

\[ \int_{0V} \bar{C}_{ijrs} \delta_{0} \bar{\epsilon}_{ij} dV + \int_{0V} \bar{S}_{ij} \delta_{0} \bar{\eta}_{ij} dV = \int_{0V} \bar{S}_{ij} \delta_{0} \bar{\kappa}_{ij} dV = r + \Delta r - \int_{0V} \bar{S}_{ij} \delta_{0} \bar{\epsilon}_{ij} dV \]

1. Upper bound means accumulated error.
2. There is no upper bound when simulating with linear stiffness matrix only.

The error is accumulated in nonlinear stiffness matrix term.
Problem Description


  “Under rotations including a significant rigid-body component, many elements produce over-stiff solutions due to ‘self-straining’.”

  “It was shown that the finite element interpolation of all of these rotation variables spoils the objectivity of the adopted strain measures with respect to a rigid rotation. The formulations based on the interpolation of iterative and incremental rotations were additionally proved to be dependent on the history of deformation. Interestingly, the co-rotational technique does not necessarily suffer from these drawbacks.”

‘Self-straining’ of the rigid body rotation

- Stiffening of rigid body rotation
- Deficiency of shear eigenmode
**Stiffening of rigid body rotation**

- **Principle of virtual work**

\[
\delta \Pi = \int_{0}^{V} \epsilon_{11} C_{ijkl} \delta \epsilon_{kl} d \Omega + \int_{0}^{V} \epsilon_{ij} S_{ij} \delta \eta_{ij} d \Omega
\]

\[
\delta \Pi_{\text{linear}} = \int_{0}^{V} \epsilon_{11} E_{0} \delta \epsilon_{11} + \epsilon_{22} \delta \epsilon_{22} + 4G_{0} \epsilon_{12} \delta \epsilon_{12} d \Omega
\]

\[
\delta \Pi_{\text{nonlinear}} = \int_{0}^{V} \epsilon_{11} S_{11} \delta \eta_{11} + \epsilon_{22} S_{22} \delta \eta_{22} + 2 \epsilon_{12} S_{12} \delta \eta_{12} d \Omega
\]

- **Eigen deformation**

\[
\varphi_{r} = [\varphi_{r_{1}} \varphi_{r_{2}}]^{T} = \frac{\sqrt{2}}{4} [-y \ x]^{T}
\]

- **Strain values**

\[
\begin{align*}
0 e_{11} &= \delta \epsilon_{11} = \frac{\partial \varphi_{r_{1}}}{\partial x} = 0 \\
0 e_{22} &= \delta \epsilon_{22} = \frac{\partial \varphi_{r_{2}}}{\partial y} = 0 \\
0 e_{12} &= \delta \epsilon_{12} = \frac{1}{2} \left( \frac{\partial \varphi_{r_{1}}}{\partial y} + \frac{\partial \varphi_{r_{2}}}{\partial x} \right) = 0
\end{align*}
\]

\[
\begin{align*}
\delta o \eta_{11} &= \frac{\partial \varphi_{r_{1}}}{\partial x} \frac{\partial \varphi_{r_{1}}}{\partial x} + \frac{\partial \varphi_{r_{2}}}{\partial y} \frac{\partial \varphi_{r_{2}}}{\partial x} = \frac{1}{8} \\
\delta o \eta_{22} &= \frac{\partial \varphi_{r_{1}}}{\partial y} \frac{\partial \varphi_{r_{1}}}{\partial y} + \frac{\partial \varphi_{r_{2}}}{\partial x} \frac{\partial \varphi_{r_{2}}}{\partial y} = \frac{1}{8} \\
\delta o \eta_{12} &= \frac{1}{2} \left( \frac{\partial \varphi_{r_{1}}}{\partial x} \frac{\partial \varphi_{r_{1}}}{\partial y} + \frac{\partial \varphi_{r_{1}}}{\partial x} \frac{\partial \varphi_{r_{2}}}{\partial y} + \frac{\partial \varphi_{r_{2}}}{\partial x} \frac{\partial \varphi_{r_{1}}}{\partial y} + \frac{\partial \varphi_{r_{2}}}{\partial x} \frac{\partial \varphi_{r_{2}}}{\partial y} \right) = 0
\end{align*}
\]

- **Internal virtual work**

\[
\delta \Pi_{\text{linear}} |_{u=\varphi_{r}} = 0 \quad \delta \Pi_{\text{nonlinear}} |_{u=\varphi_{r}} = \frac{1}{8} \int_{0}^{V} (\epsilon_{11} S_{11} + \epsilon_{22} S_{22}) d \Omega
\]
Deficiency of shear eigenmode

- Eigen deformation

\[ \phi_A = \begin{bmatrix} \phi_{A1} \\ \phi_{A2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \]

\[ \phi_B = \phi_A - \varphi_r = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \]

\[ \phi_C = \phi_A + \varphi_r = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \]

- Internal virtual work

\[
\begin{cases}
\delta \Pi_{\text{linear}} \bigg|_{u=\phi_A} = 2G, & \delta \Pi_{\text{nonlinear}} \bigg|_{u=\phi_A} = \frac{1}{8} \int_V (\dot{S}_{11} + \dot{S}_{22}) d^0V \\
\delta \Pi_{\text{linear}} \bigg|_{u=\phi_B} = 2G, & \delta \Pi_{\text{nonlinear}} \bigg|_{u=\phi_B} = \frac{1}{2} \int_V \dot{S}_{22} d^0V \\
\delta \Pi_{\text{linear}} \bigg|_{u=\phi_C} = 2G, & \delta \Pi_{\text{nonlinear}} \bigg|_{u=\phi_C} = \frac{1}{2} \int_V \dot{S}_{11} d^0V
\end{cases}
\]
Total Lagrangian:
\[
\int_{0}^{t} \delta \epsilon_{ij} \left( 0 C_{ijrs} e_{rs} d^0 V + \int_{0}^{t} \delta \eta_{ij} S_{ij} d^0 V \right) = R - \int_{0}^{t} \delta e_{ij} S_{ij} d^0 V
\]

\[K_L = \sum_{i=1}^{8} \phi_i \lambda_i \phi_i^T\]

\[K_{NL} = \phi_{11} \lambda_A \phi_A^{11^T} + \phi_{11} \lambda_B \phi_B^{11^T} + \phi_{11} \lambda_C \phi_C^{11^T} + \phi_{11} \lambda_D \phi_D^{11^T}
\]

\[
\phi_{11} \lambda_A \phi_A^{11^T} = \phi_3 \lambda_D \phi_3^T + \phi_6 \lambda_D \phi_6^T + \phi_3 \lambda_D \phi_3^T + \phi_6 \lambda_D \phi_6^T
\]
Eigen Recomposition

Total Lagrangian: \[
\int_0^V \delta e_{ij} 0 C^{ijrs} e_{rs} d^0V + \int_0^V \delta \eta_{ij} ^t S_j d^0V = \int_0^V \delta e_{ij} ^t S_j d^0V
\]

\[\begin{align*}
K_L & \quad \text{K}_{NL}
\end{align*}\]

Eigen decomposition: \[K_{NL} = \sum_i \lambda_i \phi_i \phi_i^T\]

Estimation of eigenvalue: \[\lambda_i = \phi_i^T K_{NL} \phi_i \cong \bar{\phi}_i^T K_{NL} \bar{\phi}_i \quad (\bar{\phi}_i: \text{assumed eigenvectors})\]

Eigen recomposition: \[\bar{K}_{NL} = \sum_i \lambda_i \bar{\phi}_i \bar{\phi}_i^T\]

♦ Assumed Eigenvector for 2D beam
Straight cantilever problem

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### Straight cantilever problem

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19% improved
Right angle frame problem
Offshore jacket problem

20 increment steps without recomposition

40 increment steps without recomposition

Single increment step with recomposition

Two increment steps with recomposition
Summary & Future works
● The continuum mechanics based beam finite elements allowing twisting behavior are proposed.

● The new modeling method to construct continuous warping displacement fields for discontinuously varying arbitrary cross-sections is presented.

● The nonlinear formulation of the continuum mechanics based beam elements is presented.

● A novel numerical method to improve nonlinear performance is proposed.
Future works

- Continuum mechanics based beam element
  - Composite (warping, zigzag, slip …)
  - Distortional mode
  - Jourasky type warping
  - Implementation for dynamic analysis

- Warping for discontinuously varying cross-section
  - Implementation for nonlinear analysis
  - Buckling analysis

- Nonlinear implementation
  - Buckling analysis (Inelastic, twisted structure)
  - Characteristics of twisting actions (poynting and swift effect)

- Improvement of nonlinear performance
  - Spurious energy mode (locking treatment)
  - Extension of eigen recomposition method
  - Recursive load step, problem dependent tangent stiffness matrix
Curriculum Vitae

• **Education**
  - KAIST, Ph. D. Program, Division of Ocean System Engineering
  - KAIST, Master of Science, Division of Ocean System Engineering
  - KAIST, Bachelor of Science, Department of Mechanical Engineering

• **Publications**


• **Presentations**

Q & A