Ph. D. Dissertation

변형률 완화 다각형 유한요소 개발 및 변형률 완화 요소법에서의 체적 잠김 개선

Development of strain-smoothed polygonal finite elements and alleviation of volumetric locking in the strain-smoothed element method

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2023.12.15

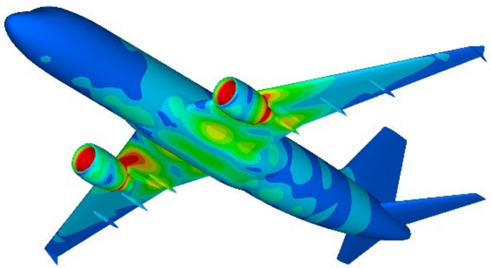
KAIST Department of Mechanical Engineering

CMSS Computational Mechanics Structural Systems

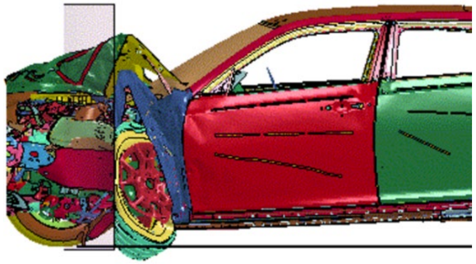
Contents

1. Introduction

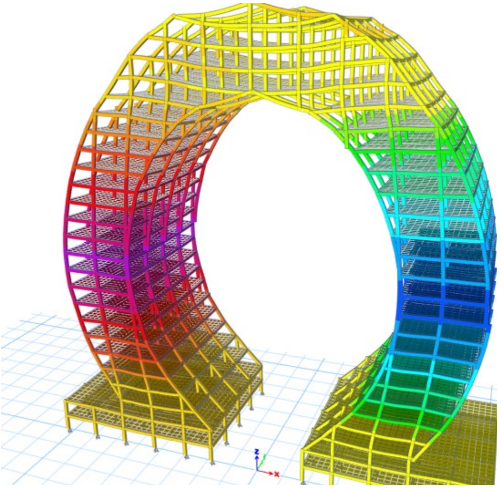
FEM in Engineering Fields



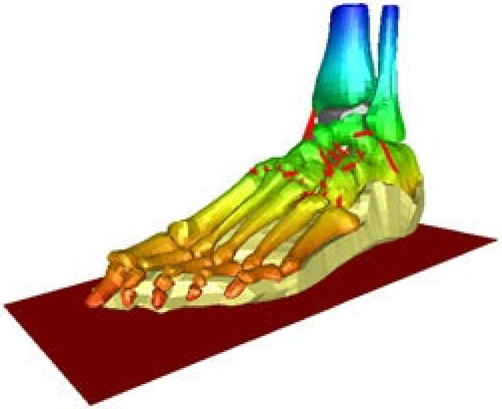
Aerospace Engineering



Automotive Engineering



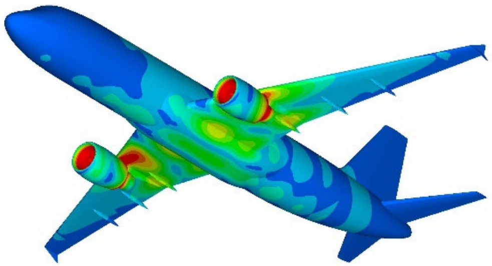
Architectural Engineering



Biomedical Engineering

Problems

Larger and
more complex models

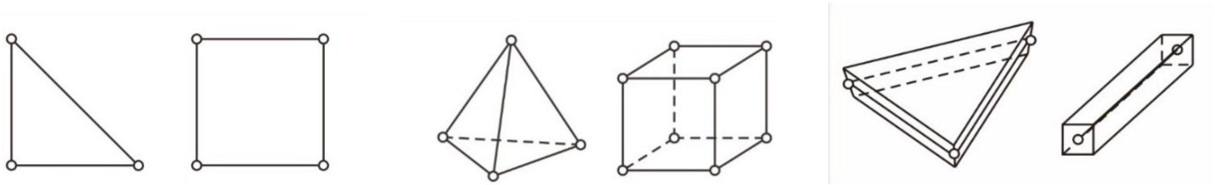


Needs

Higher **accuracy**
Lower **computational costs**

Solutions

- Advancement of computing devices
- Improvement computational methods and algorithms
- Development of finite elements and Improvement of FE solutions



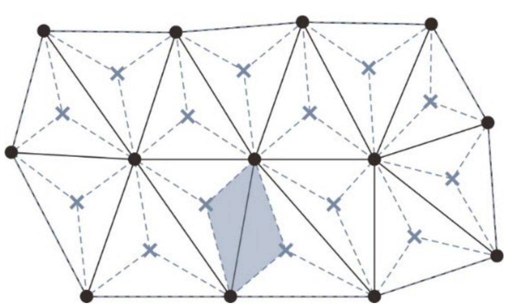
Finite Elements

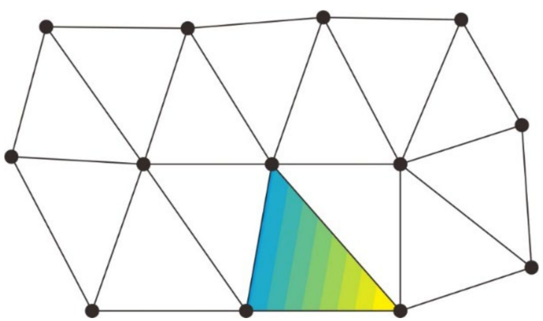
➢ For improving FE solutions

1. Reduced integrations & assumed strain methods
   - ✓ ANS (Assumed Natural Strain) and MITC (Mixed Integration of Tensorial Components).
2. Enrichment methods
   - ✓ Enriched FEM, XFEM (eXtended FEM).
3. Strain smoothing methods
   - ✓ Node, Edge, Face and Cell-based S-FEM (Smoothed FEM).

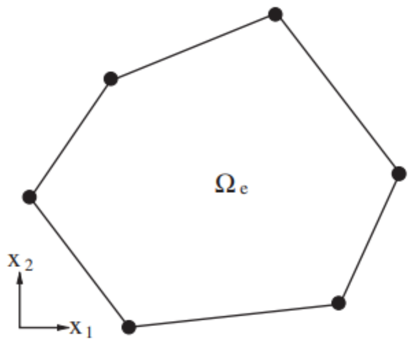
➢ Strain-smoothed element (SSE) method

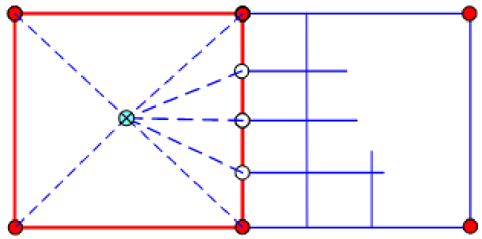
- ✓ The method requires no special smoothing domains.
- ✓ The method construct smoothed strain field within an element.


Edge-based smoothing

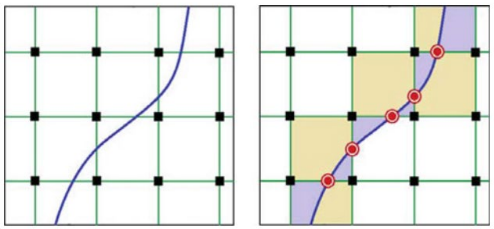

SSE method

Polygonal Elements

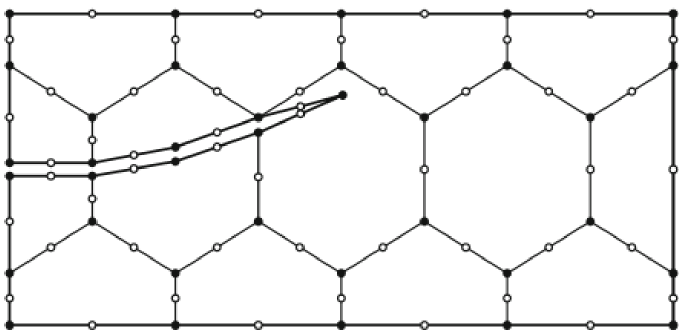




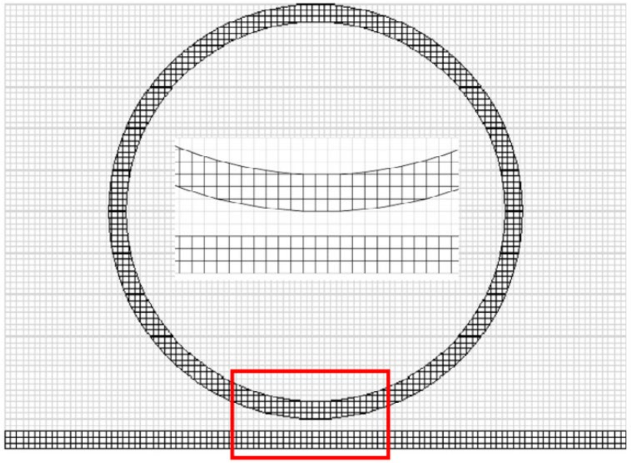
Mesh transition



Mesh modifying



Crack simulation

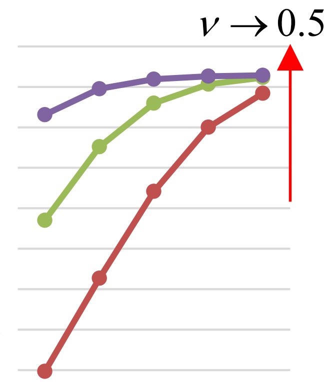


Contact simulation

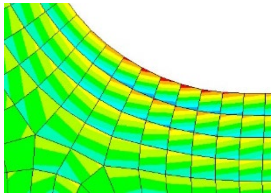
Volumetric locking

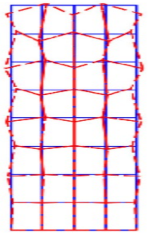
➢ Volumetric locking

- As the Poisson's ratio approaches 0.5 (nearly incompressible material), the element stiffens significantly.
- It results in much smaller displacements than expected and excessively discontinuous stress estimation.
- Reduced integration allowed for an easier way to avoid volumetric locking, but spurious zero-energy modes (or hourglass modes) may occur.



(Standard Q4 element)

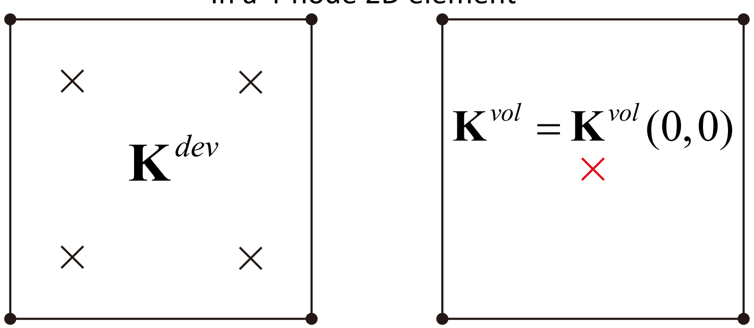




➢ Separation of volumetric term and deviatoric term

Stiffness matrix $\mathbf{K} = \mathbf{K}^{dev} + \boxed{\mathbf{K}^{vol}}$ → Related to volume change

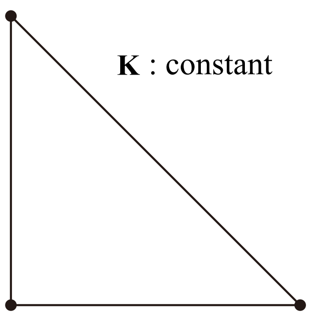
In a 4-node 2D element



Deviatoric term: (bi)linear strain field

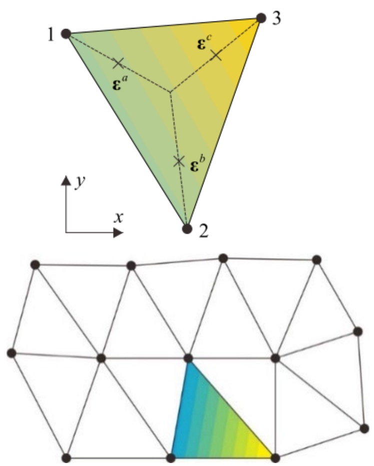
Volumetric term: Constant assumed strain

In a 3-node 2D element



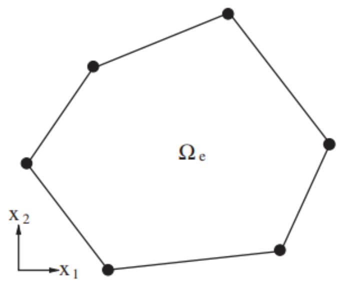
<u>Assumed strain cannot be applied</u>

Objectives



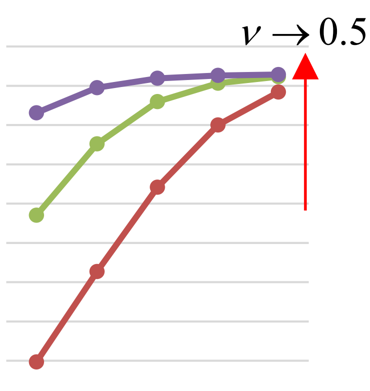
Strain-smoothed element (SSE) method

- SSE method provides further improved solutions without requiring additional degrees of freedom.
- SSE method has been successfully applied to 3-node triangular and 4-node quadrilateral 2D solid elements.



Polygonal elements

- High level of flexibility in mesh transition, and refinement.
- Further research is required to develop polygonal finite elements that provide more accurate and reliable solutions.



Volumetric locking

- The phenomenon that occurs on the material properties.
- In 3-node triangular element, there is difficulty in applying constant volumetric strain field.

2. Strain-smoothed element method

Strain smoothing

- **Finite element formulation**

$$\mathbf{K}\mathbf{U} = \mathbf{F} = \mathbf{F}_B + \mathbf{F}_S\,, \quad \mathbf{K} = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{B}^{(m)T}\mathbf{D}\mathbf{B}^{(m)}\, d\Omega\,, \;\; \mathbf{F}_B = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{H}^{(m)T}\mathbf{b}\, d\Omega\,, \;\; \mathbf{F}_S = \sum_{m=1}^{e} \int_{\Gamma^{(m)}} \mathbf{H}^{(m)T}\mathbf{t}\, d\Gamma\,.$$

Strain-displacement relation $\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\mathbf{u}^{(m)}$ where $\mathbf{B}^{(m)T} = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}.$

- **Smoothing operation**

$$\boldsymbol{\varepsilon} \xrightarrow{\displaystyle\int_{\Omega^{(k)}} \boldsymbol{\varepsilon}(\mathbf{x})\Phi_k(\mathbf{x})\,d\Omega} \tilde{\boldsymbol{\varepsilon}}_k$$

$\boldsymbol{\varepsilon}$: strain defined for finite elements.

$\tilde{\boldsymbol{\varepsilon}}_k$: smoothed strain defined for smoothing domain.

$\Omega^{(k)}$: k^{th} smoothing domain.

- **Smoothing function**

$$\Phi_k(\mathbf{x}) = \begin{cases} 1/A^{(k)}, & \mathbf{x} \in \Omega^{(k)} \\ 0, & \mathbf{x} \notin \Omega^{(k)} \end{cases}.$$

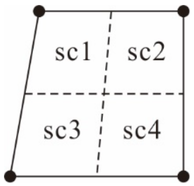
$A^{(k)}$: area of the smoothing domain $\Omega^{(k)}$.

$\Phi_k(\mathbf{x})$: smoothing function for domain $\Omega^{(k)}$.

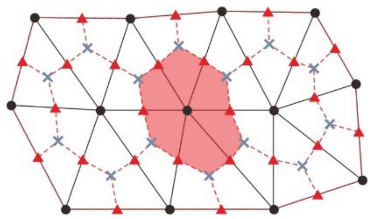
Strain smoothing

- **Chen et al. (2001)**
  - The strain smoothing method was first proposed for the Galerkin mesh-free method.

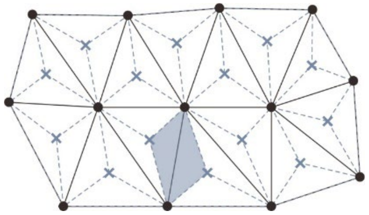
- **Liu et al. (2007)**
  - A cell-based S-FEM was proposed for 2D solid mechanics problems.
  - An element is subdivided into finite number of smoothing domains (cells).



- **Liu et al. (2009)**
  - A node-based S-FEM was proposed for 2D solid mechanics problems.
  - It gives overly soft solutions and wider bandwidth of stiffness matrix.



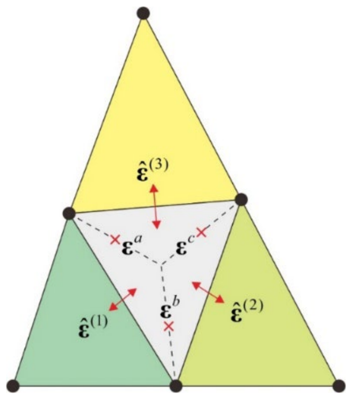
- **Liu et al. (2009)**
  - An edge-based S-FEM was proposed for 3-node triangular 2D solid element.
  - It shows the best performance among the previous strain smoothing methods.

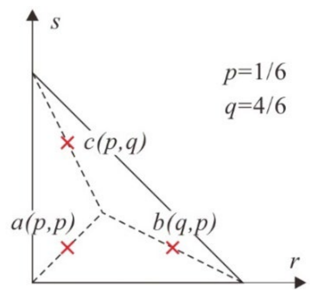


Strain-smoothed element (SSE) method

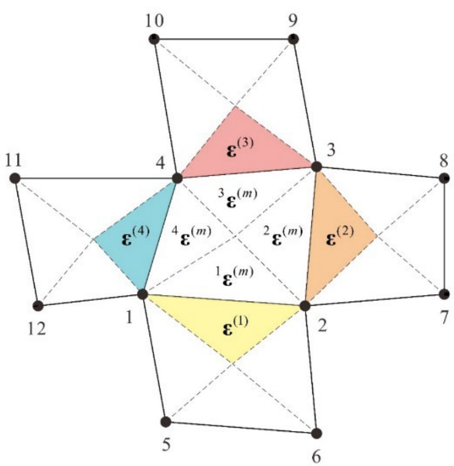
- **Lee and Lee (2018)**
  - Strain-smoothed element was first proposed for 3-node triangular 2D solid element.

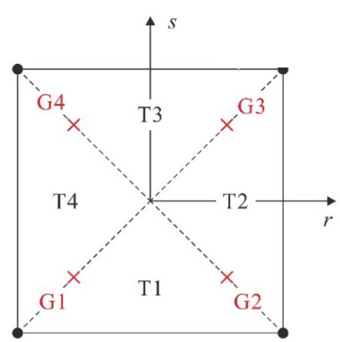
3-node element (Lee and Lee, 2018)





4-node element (Lee et al. , 2021)





In a 3-node solid element

- 1^{st} strain smoothing

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A^{(e)} + A^{(k)}}\left(A^{(e)}\boldsymbol{\varepsilon}^{(e)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right).$$

- 2^{nd} strain smoothing within elements

$$\boldsymbol{\varepsilon}^a = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(3)}),$$

$$\boldsymbol{\varepsilon}^b = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)}),$$

$$\boldsymbol{\varepsilon}^c = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)}).$$

- Smoothed strain field

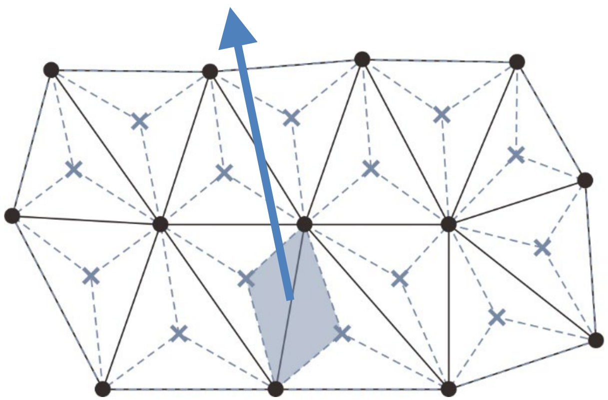
$$\overline{\boldsymbol{\varepsilon}}^{(e)} = \left[1 - \frac{1}{q-p}(r+s-2p)\right]\boldsymbol{\varepsilon}^a + \frac{r-p}{q-p}\boldsymbol{\varepsilon}^b + \frac{s-p}{q-p}\boldsymbol{\varepsilon}^c.$$

Strain-smoothed element (SSE) method

In a 3-node solid element

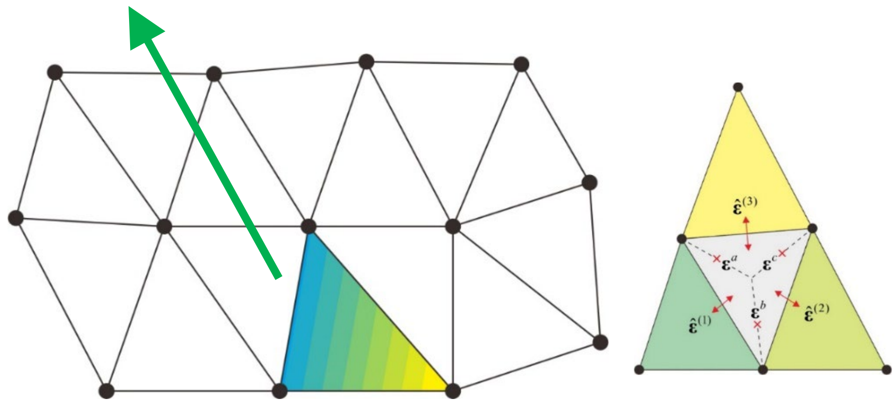
- Smoothed strain field

$$\tilde{\boldsymbol{\varepsilon}}_k = \frac{1}{A_c^{(1)} + A_c^{(2)}}\left(A_c^{(1)}\boldsymbol{\varepsilon}^{(1)} + A_c^{(2)}\boldsymbol{\varepsilon}^{(2)}\right)$$



- Strain fields in the defined smoothing domains
- Constant strain fields

$$\bar{\boldsymbol{\varepsilon}}^{(e)}(r,s) = \left[1 - \frac{1}{q-p}(r+s-2p)\right]\boldsymbol{\varepsilon}^{a} + \frac{r-p}{q-p}\boldsymbol{\varepsilon}^{b} + \frac{s-p}{q-p}\boldsymbol{\varepsilon}^{c}$$



- Strain fields in finite element domains
- (Bi-)Linear strain fields
- Improved accuracy

SSE method for various elements

- A strain-smoothed triangular and tetrahedral finite element was proposed. (2018)

- A strain-smoothed MITC3+ shell element was proposed. (2019)

- A strain-smoothed quadrilateral finite element was proposed. (2021)

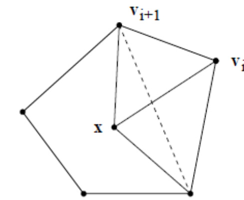
3. Strain-smoothed polygonal finite elements

Polygonal elements

▪ Various shape functions of the polygonal elements

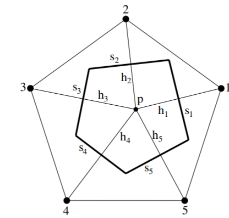
➢ **Wachspress (1975)**

- A rational basis using areas with an arbitrary point and vertices.
- It is only applied to convex polygonal elements.



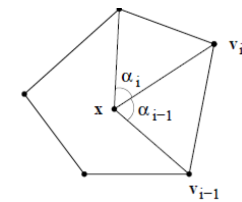
➢ **Sibson (1980)**

- Coordinates based Voronoi cells (set of bisectors) within a set of nodes was proposed.
- It has been used in mesh free methods.



➢ **Floater (2003)**

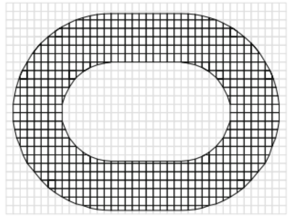
- A finite element basis using angles with an arbitrary point and vertices was proposed.
- It can be applied to concave polygonal elements.



▪ Application of polygonal elements for various problems

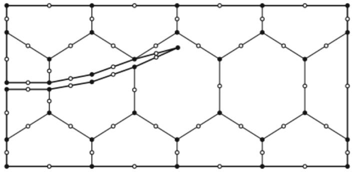
➢ **Biabanaki and Khoei (2012)**

- Polygonal elements were applied for solid mechanics problems with conformal decomposition of meshes.



➢ **Khoei et al. (2015)**

- Modeling of crack propagation with minimal remeshing was proposed using polygonal elements.

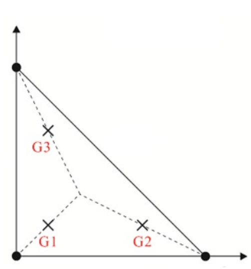


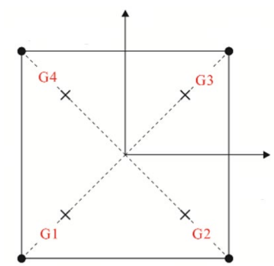
➢ **Nguyen et al. (2020)**

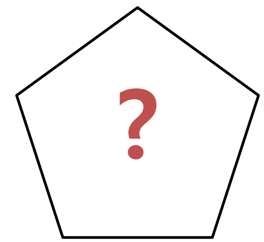
- Crack growth analysis of interfacial cracks was proposed using polygonal elements.

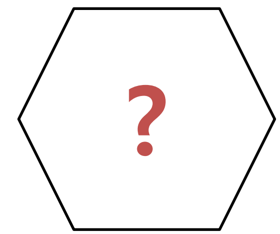
Polygonal elements

In 3-node and 4-node elements and polygonal elements



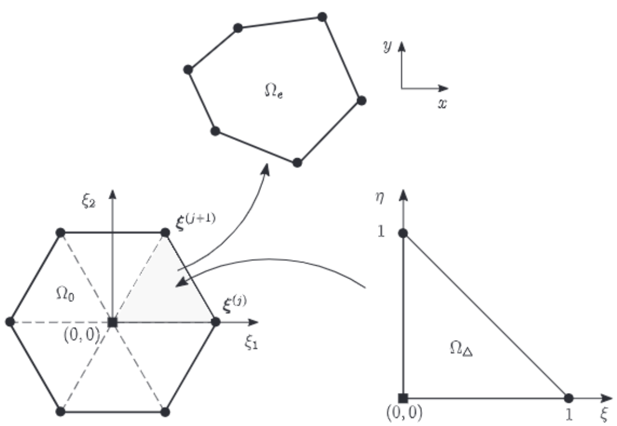




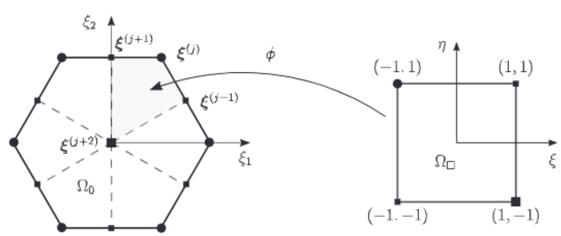


It is difficult to apply the Gauss quadrature rules to polygonal elements.

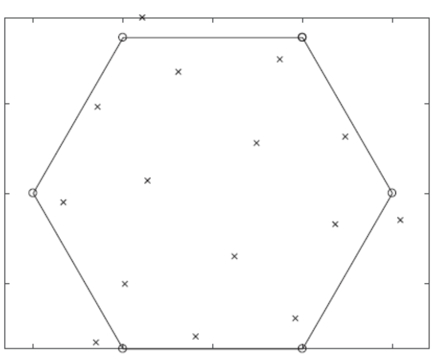
Alternatively..



Triangulation
(Sukumar and Tabarraei, 2004)



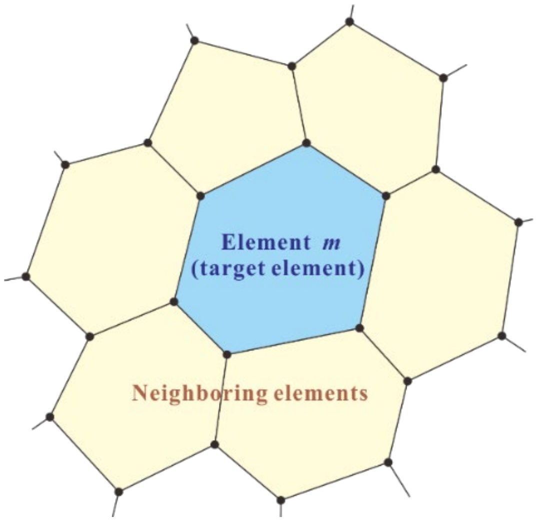
Quadrangulation
(Talischi et al, 2014)

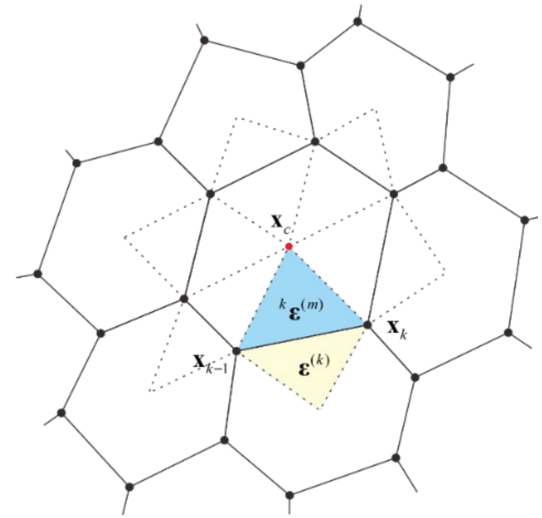


Rational polynomial
Gauss quadrature
(Thomes and Menandro, 2020)

SSE method in polygonal elements

- Sub-division of polygonal elements



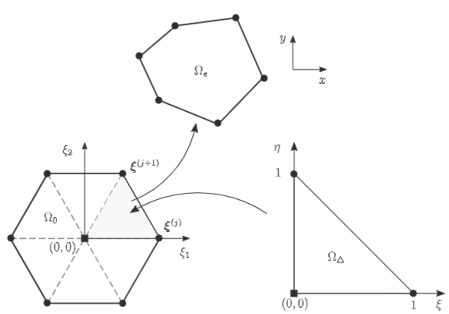


Constant strain

1. Strain smoothing

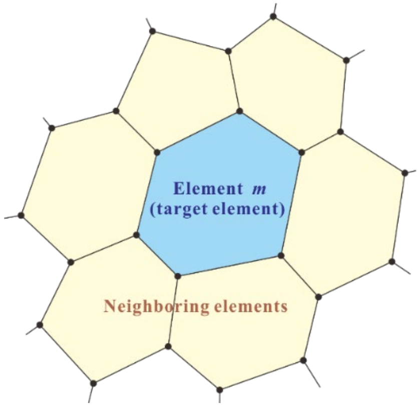
$$\int_{\Omega^{(k)}} \boldsymbol{\varepsilon}(\mathbf{x})\Phi_k(\mathbf{x})d\Omega \qquad \Phi_k(\mathbf{x}) = \begin{cases} 1/A^{(k)}, & \mathbf{x} \in \Omega^{(k)} \\ 0, & \mathbf{x} \notin \Omega^{(k)} \end{cases}.$$

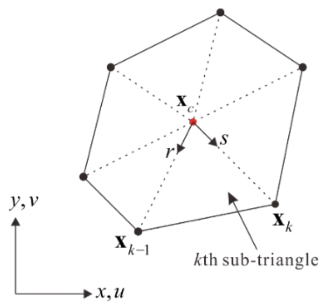
2. Numerical integration

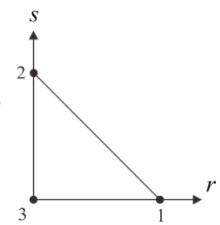


SSE method in polygonal elements

- **1st strain smoothing** with piecewise linear shape functions.







Geometry of the kth sub-triangle

$$\mathbf{x} = h_1\mathbf{x}_{k-1} + h_2\mathbf{x}_k + h_3\mathbf{x}_c \qquad \text{(where } \mathbf{x}_c = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \text{)}$$

Displacement interpolation

$$\mathbf{u} = h_1\mathbf{u}_{k-1} + h_2\mathbf{u}_k + h_3\mathbf{u}_c \qquad \text{(where } \mathbf{u}_c = \frac{1}{n}\sum_{i=1}^{n}\mathbf{u}_i \text{)}$$

$$(h_1 = r,\ h_2 = s,\ h_3 = 1 - r - s)$$

→ Strain field within the kth sub-triangle (of the target element m)

$$^{k}\boldsymbol{\varepsilon}^{(m)} = [\varepsilon_{11} \quad \varepsilon_{22} \quad 2\varepsilon_{12}]^T = {}^{k}\mathbf{B}^{(m)}\mathbf{u}^{(m)}$$

where

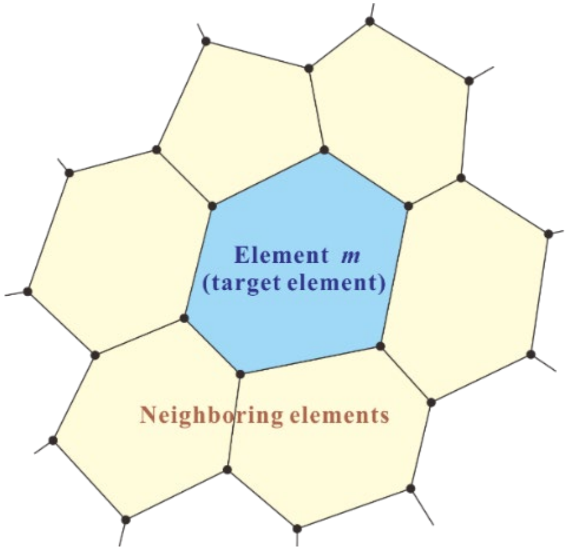
$$^{k}\mathbf{B}^{(m)} = \left[{}^{k}\mathbf{B}_1 \quad {}^{k}\mathbf{B}_2 \quad \cdots \quad {}^{k}\mathbf{B}_n\right], \qquad \mathbf{u}^{(m)} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]^T,$$

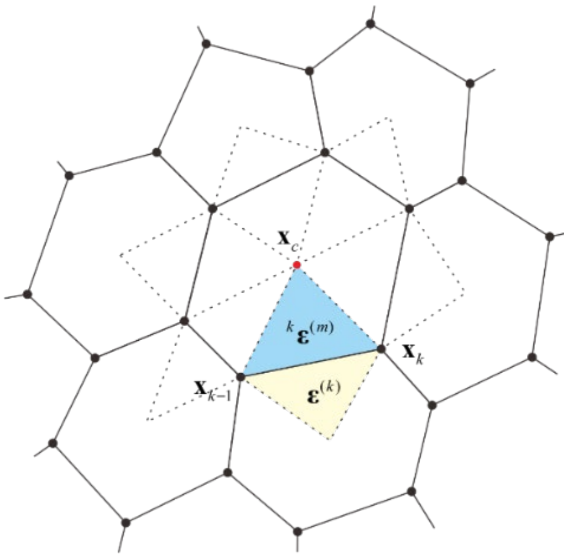
$$^{k}\mathbf{B}_i = \begin{bmatrix} \delta_{i(k-1)}h_{1,x} + \delta_{ik}h_{2,x} + \frac{1}{n}h_{3,x} & 0 & \delta_{i(k-1)}h_{1,y} + \delta_{ik}h_{2,y} + \frac{1}{n}h_{3,y} \\ 0 & \delta_{i(k-1)}h_{1,y} + \delta_{ik}h_{2,y} + \frac{1}{n}h_{3,y} & \delta_{i(k-1)}h_{1,x} + \delta_{ik}h_{2,x} + \frac{1}{n}h_{3,x} \end{bmatrix}^T$$

: the strain-displacement matrix corresponding to node i

SSE method in polygonal elements

- **1st strain smoothing** with piecewise linear shape functions.





${}^k\boldsymbol{\varepsilon}^{(m)}$, $A_k^{(m)}$: (constant) strain and area of the <u>kth sub-triangle of the target element m</u>

$\boldsymbol{\varepsilon}^{(k)}$, $A^{(k)}$: the strain and area of <u>its neighboring sub-triangle</u>

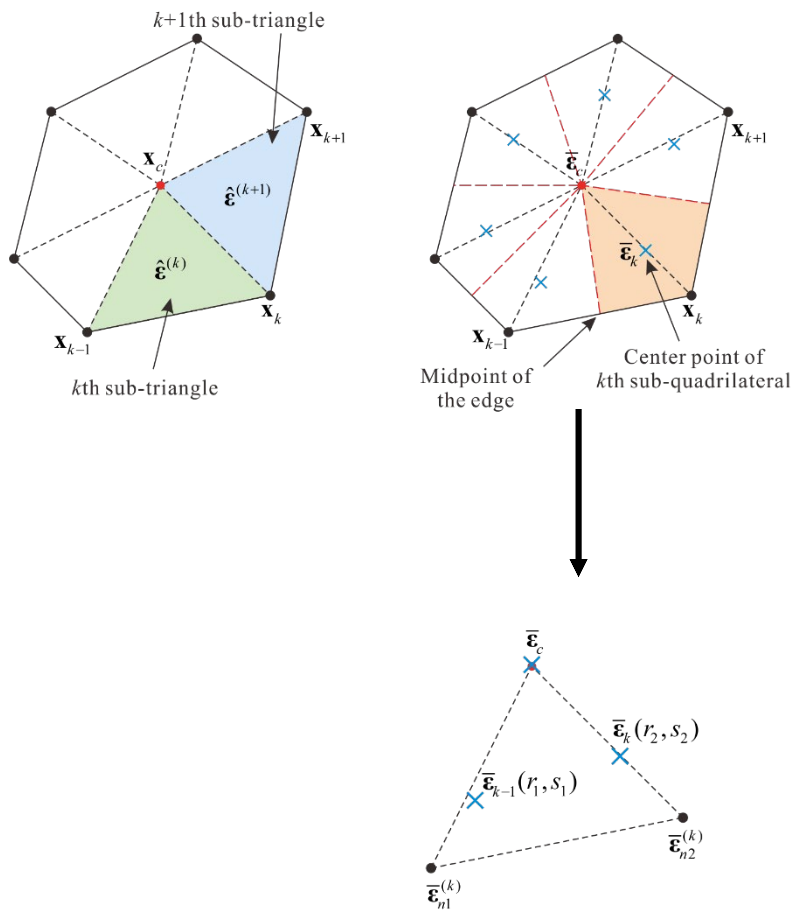
→ The smoothed strain between the ${}^k\boldsymbol{\varepsilon}^{(m)}$ and $\boldsymbol{\varepsilon}^{(k)}$

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A_k^{(m)} + A^{(k)}}\left(A_k^{(m)}\,{}^k\boldsymbol{\varepsilon}^{(m)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right)$$

(If there is no neighboring element, $\hat{\boldsymbol{\varepsilon}}^{(k)} = {}^k\boldsymbol{\varepsilon}^{(m)}$.)

SSE method in polygonal elements

- **2nd strain smoothing** by assigning the strains to the center point of sub-quadrilateral.



The smoothed strain between two neighboring sub-triangles

$$\bar{\boldsymbol{\varepsilon}}_k = \frac{1}{A_k^{(m)} + A_{k+1}^{(m)}}\left(A_k^{(m)}\hat{\boldsymbol{\varepsilon}}^{(k)} + A_{k+1}^{(m)}\hat{\boldsymbol{\varepsilon}}^{(k+1)}\right)$$

The strain at the center point (center strain)

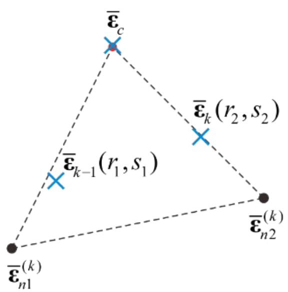
$$\bar{\boldsymbol{\varepsilon}}_c = \frac{\sum_{k=1}^{n} A_k^{(m)}\bar{\boldsymbol{\varepsilon}}_k}{\sum_{k=1}^{n} A_k^{(m)}}$$

→ The nodal strains
(using the natural coordinates of the sub-triangle)

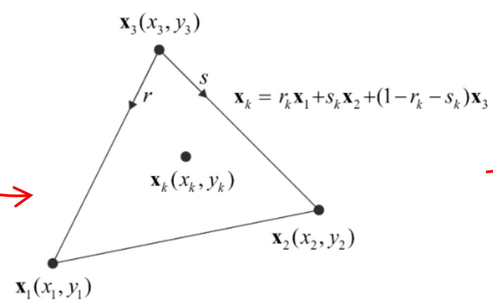
$$\begin{bmatrix} \bar{\varepsilon}_{n1}^{(k)} \\ \bar{\varepsilon}_{n2}^{(k)} \end{bmatrix} = \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\varepsilon}_{k-1} - \bar{\varepsilon}_c(1 - r_1 - s_1) \\ \bar{\varepsilon}_k - \bar{\varepsilon}_c(1 - r_2 - s_2) \end{bmatrix}$$

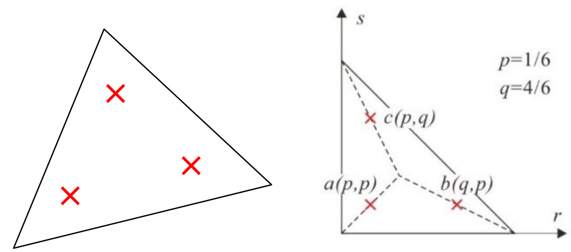
SSE method in polygonal elements

- **Interpolated smoothed strains** are assigned to Gauss points of the sub-triangles.



Linear interpolation

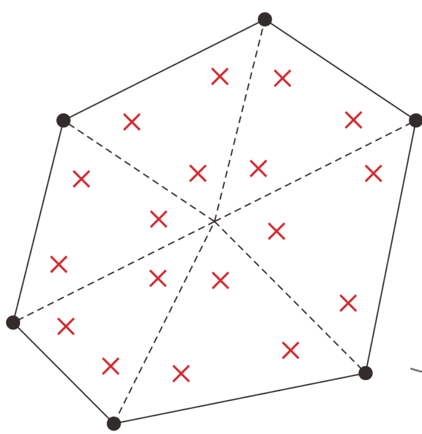


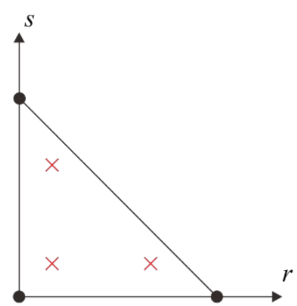


Strains at Gauss points of the sub-triangle

$$\bar{\boldsymbol{\varepsilon}}^{(m)} = \bar{\mathbf{B}}^{(m)}\bar{\mathbf{u}}^{(m)}$$

Numerical integration over the polygonal element



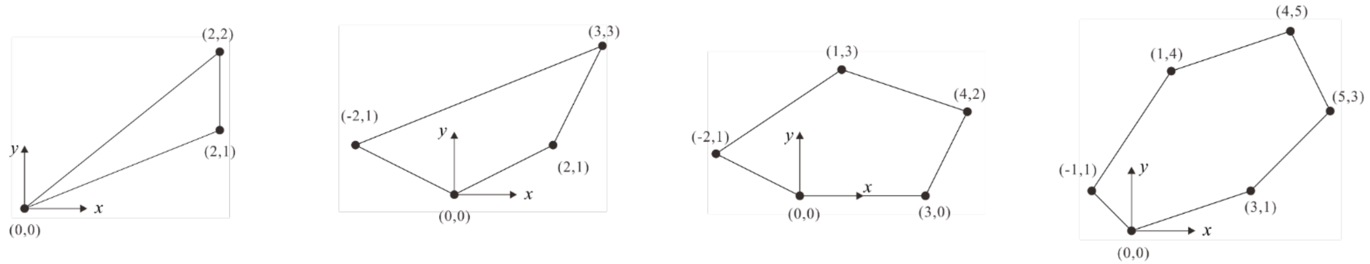


$$\mathbf{K}^{(m)} = \int_{V^{(m)}} \bar{\mathbf{B}}^{(m)T}\mathbf{C}^{(m)}\bar{\mathbf{B}}^{(m)}\,dV^{(m)}$$

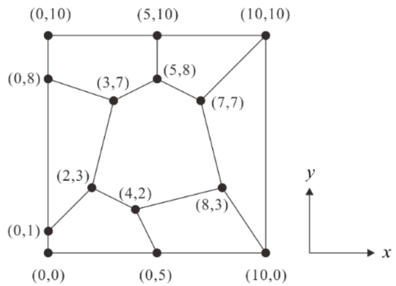
Basic numerical tests

- **Isotropic element test**
  - The proposed elements yield the same results regardless of the element node numbering sequences.

- **Zero energy mode test**
  - The stiffness matrix of the 2D element must contain only three zero-energy modes.
  - The zero-energy mode tests are performed using the polygons from triangle to hexagon.



- **Patch tests**
  - The minimum number of DOFs is constrained to prevent rigid body motions.
  - Appropriate loadings are applied to obtain a constant stress field.
  - <u>The same stress value should be obtained at all points</u> on the elements to pass the patch tests.

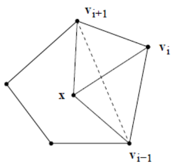


- The proposed polygonal elements practically pass the patch tests.

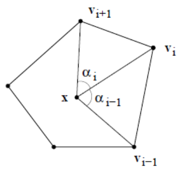
Numerical examples

➢ Finite elements considered

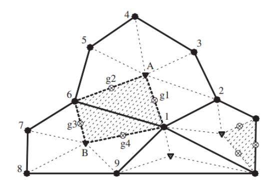
- **Wachspress** : Polygonal elements based on Wachspress coordinate



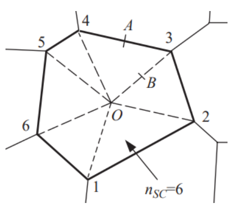
- **Mean value** : Polygonal elements based on mean value coordinate

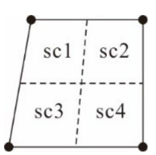


- **ES-FEM** : Edge-based smoothed polygonal finite element (Nguyen-Thoi et al. , 2011)



- **CS-FEM** : Cell-based smoothed polygonal finite element (Dai et al. , 2007)





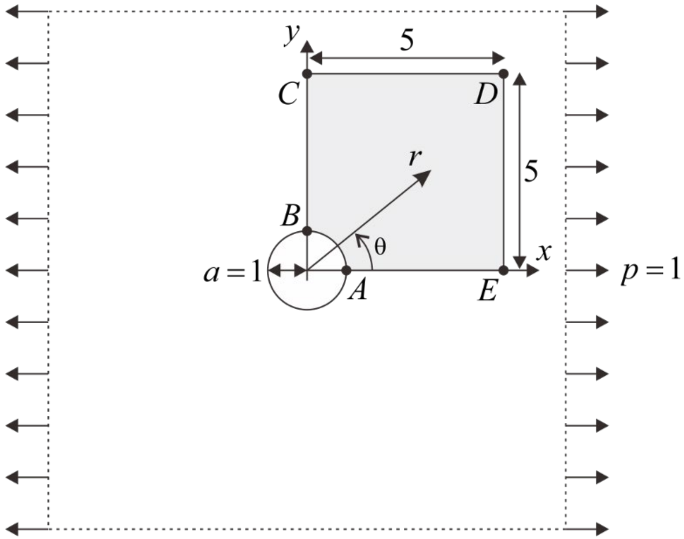
- **SSE (proposed)** : Strain-smoothed polygonal finite elements

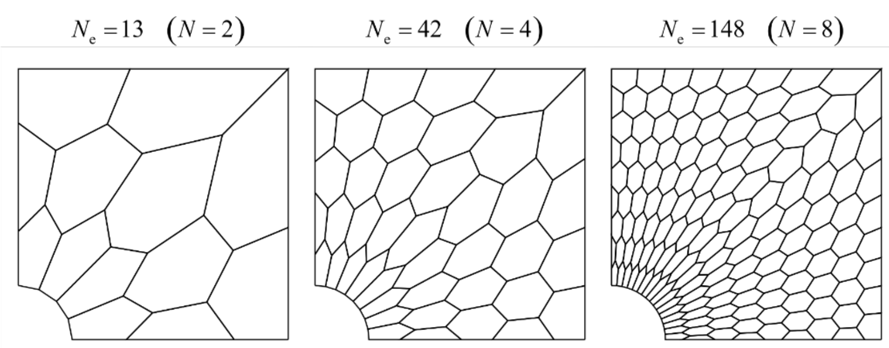
➢ Reference solution

- Reference solutions are calculated using a <u>64×64</u> regular mesh of 9-node 2D solid elements.

Numerical examples

1) Infinite plate with a circular hole





- **Force**

  analytical solutions of infinite plate with a hole

$$\sigma_{xx}(r,\theta) = p\left(1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3a^4}{2r^4}\cos 4\theta\right)$$

$$\sigma_{yy}(r,\theta) = p\left(-\frac{a^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3a^4}{2r^4}\cos 4\theta\right)$$

$$\sigma_{xy}(r,\theta) = p\left(-\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) + \frac{3a^4}{2r^4}\sin 4\theta\right)$$

- **Boundary condition**

  u=0 along ***BC*** and *v*=0 along ***AE***

- **Material property (plane strain condition)**

  $E = 3\times 10^7, \quad \nu = 0.3,$

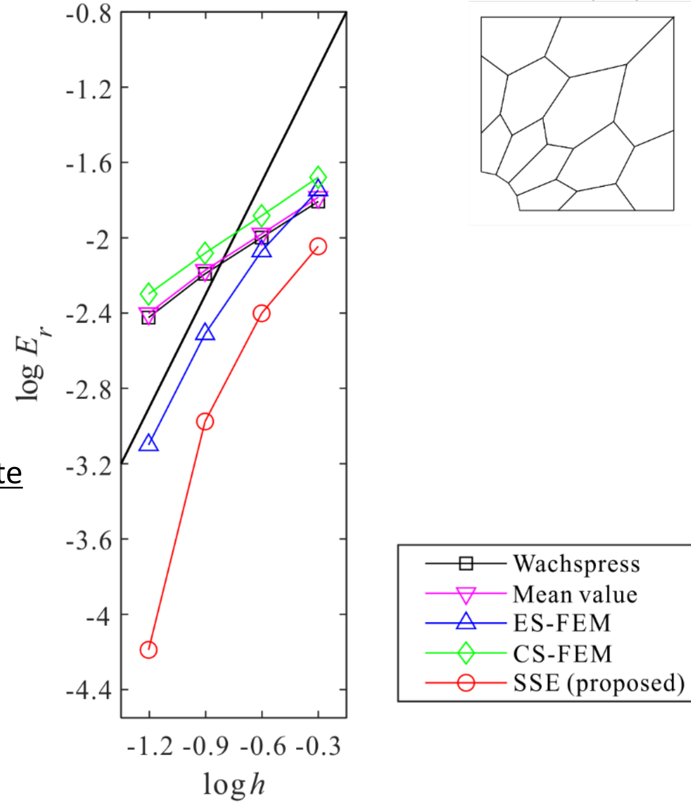
- **Meshes** (N elements along the upper edge)

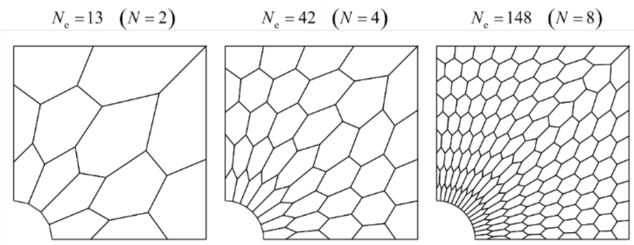
  N =2, 4, 8, 16.

Numerical examples

1) Infinite plate with a circular hole

- **Convergence curves**





More accurate

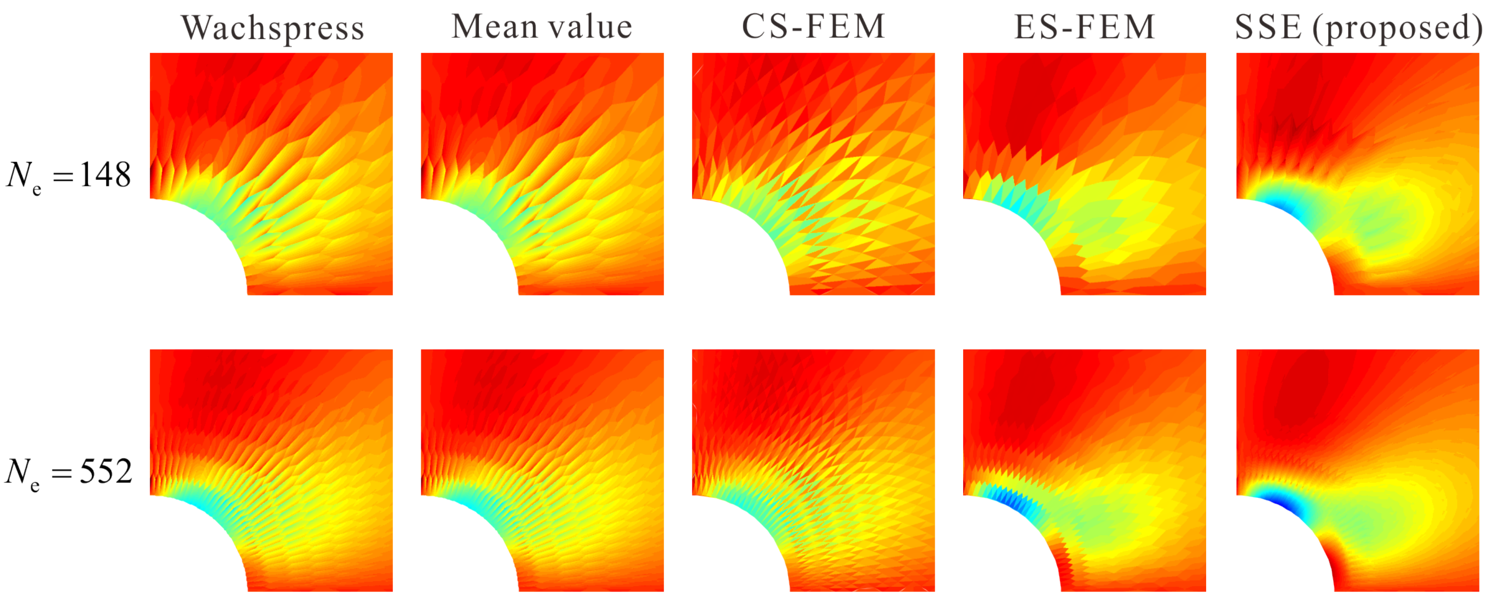
More elements

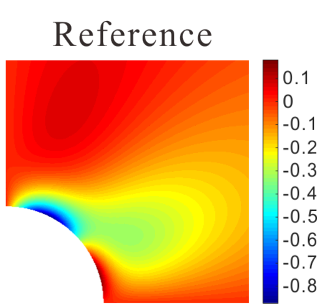
h=1/N

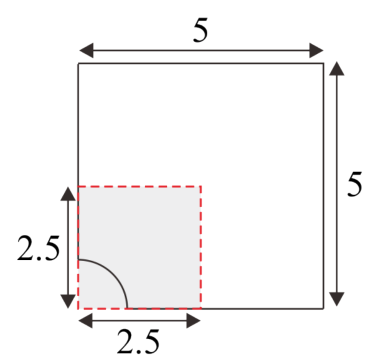
Numerical examples

1) Infinite plate with a circular hole

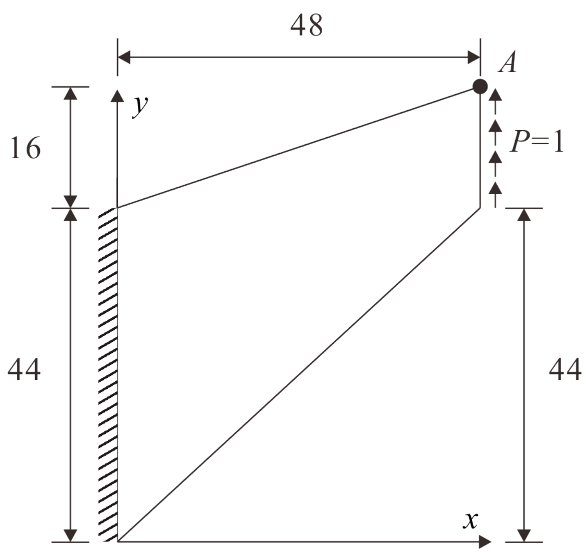
- **Shear stress distributions (in 2.5x2.5 area)**







2) Cook's skew beam



- **Force**

  Distributed shearing force $P = 1$

- **Boundary condition**

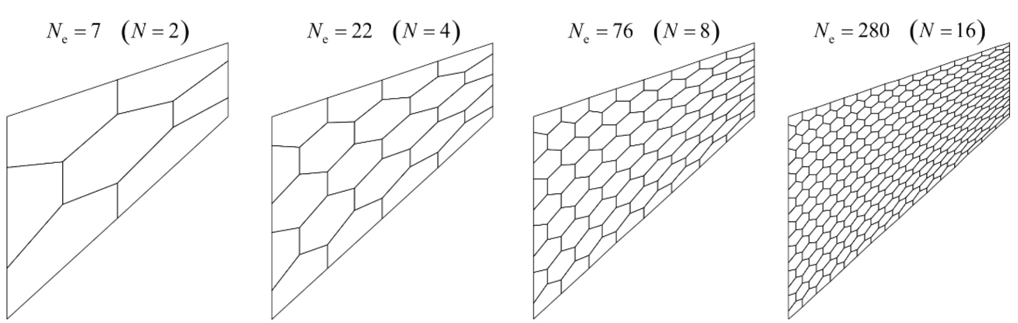
  Left edge is clamped

- **Material property (plane stress condition)**

  $E = 3\times10^7, \quad \nu = 0.3$

- **Meshes** (N elements along the upper edge)

  N =2, 4, 8, 16

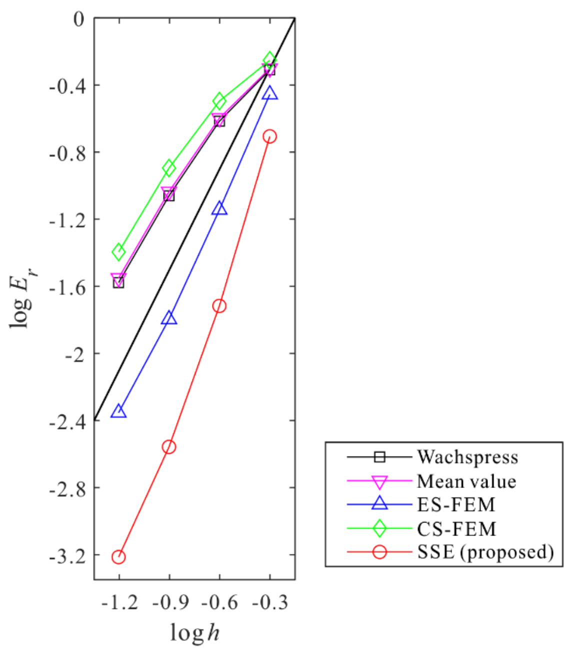


Numerical examples

2) Cook's skew beam

- **Convergence curves**

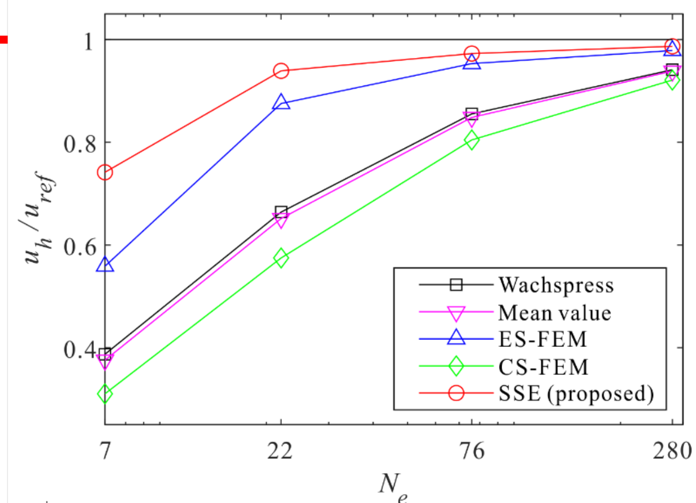
More accurate

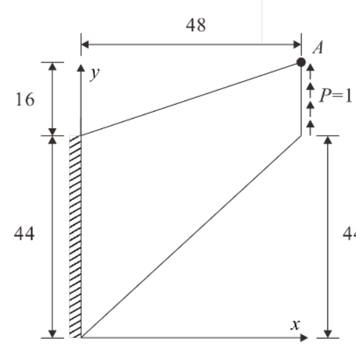


More elements

- **Normalized horizontal displacements (u_h / u_{ref}) at point *A***

More accurate





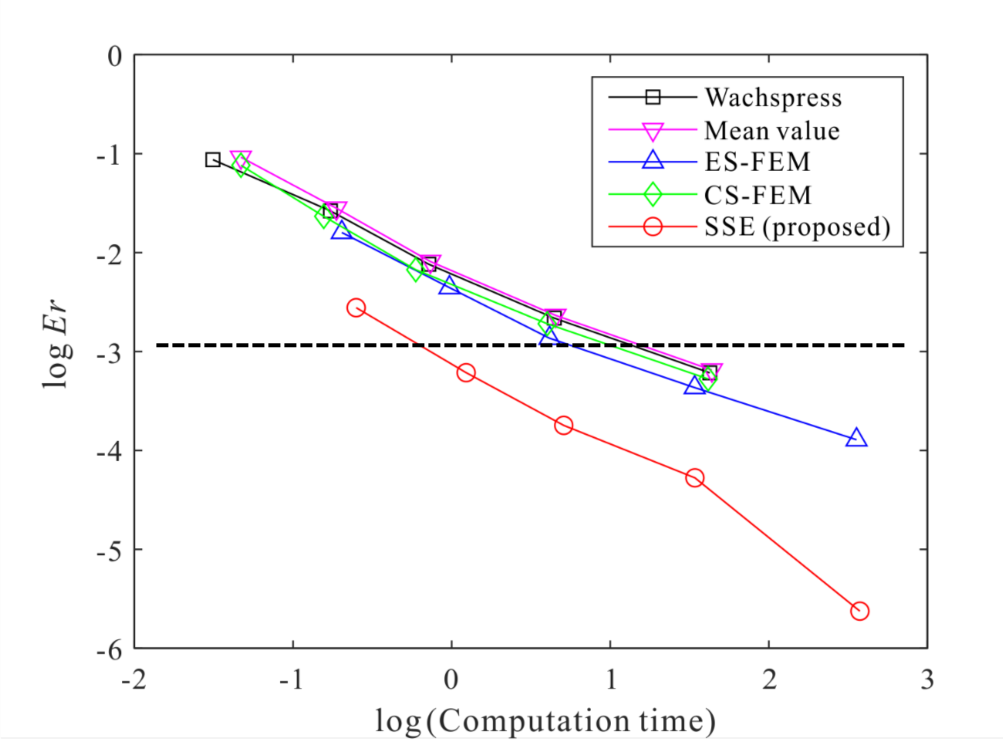
More elements

Numerical examples

2) Cook's skew beam

- **Computational costs**

More accurate



More computation time

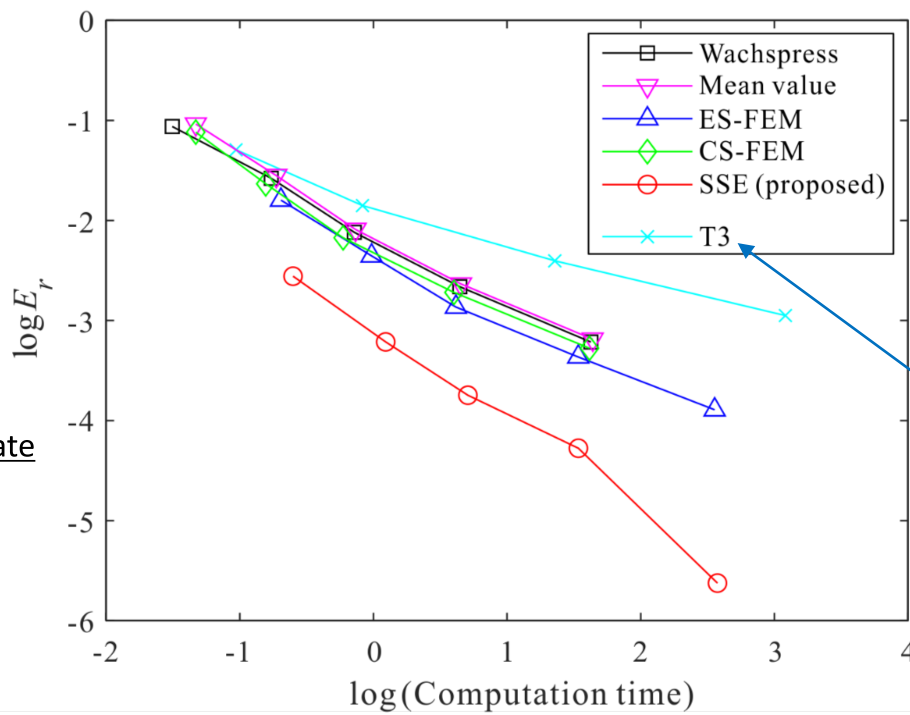
* Computations are performed in a PC with Intel Core i7-4790, 3.60GHz CPU, and 8 GB RAM.

Skyline solver is used for solving linear equations.

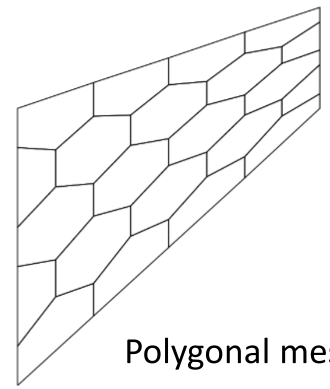
2) Cook's skew beam

- **Computational costs**

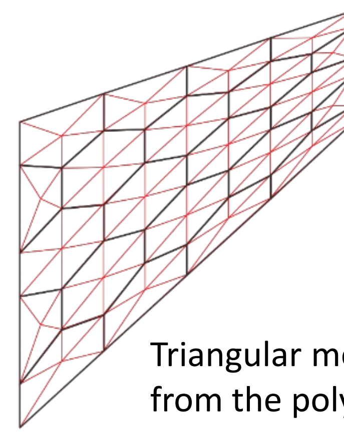
More accurate



More computation time



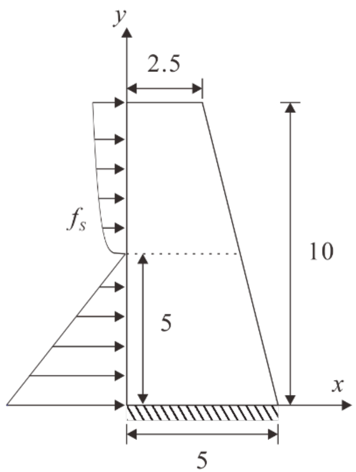
Polygonal mesh



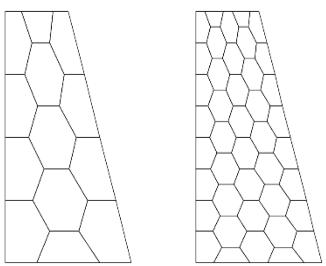
Triangular mesh
from the polygons

Numerical examples

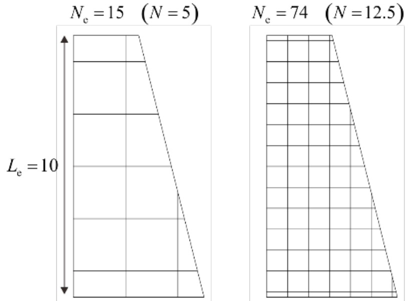
3) Dam problem



$N_e = 13 \quad (N = 4)$ $N_e = 42 \quad (N = 8)$

1 

$N_e = 15 \quad (N = 5)$ $N_e = 74 \quad (N = 12.5)$

2 

- **Force**

  Compressive surface force

  $$f_S = \begin{cases} 5 - y & 0 \le y \le 5 \\ (y-5)^{1/5} & 5 \le y \le 10 \end{cases}$$

- **Boundary condition**

  The block is clamped along its bottom

- **Material property (plane strain condition)**

  $E = 3 \times 10^{10}, \quad \nu = 0.2$

- **Meshes**

  1 - N elements along the left edge

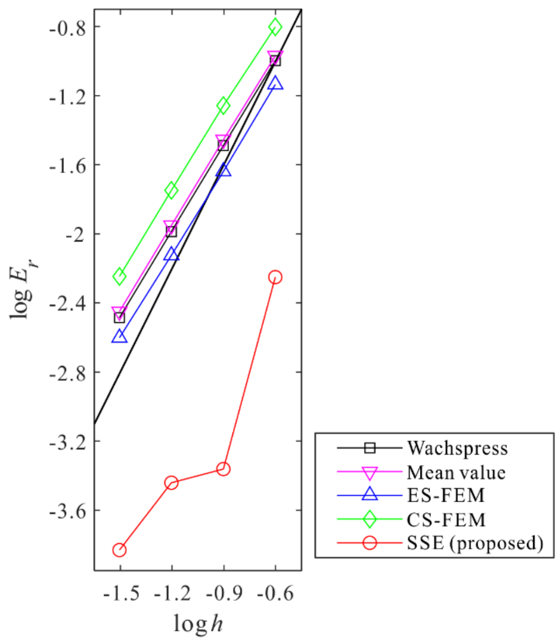
  $N =$2, 4, 8, 16

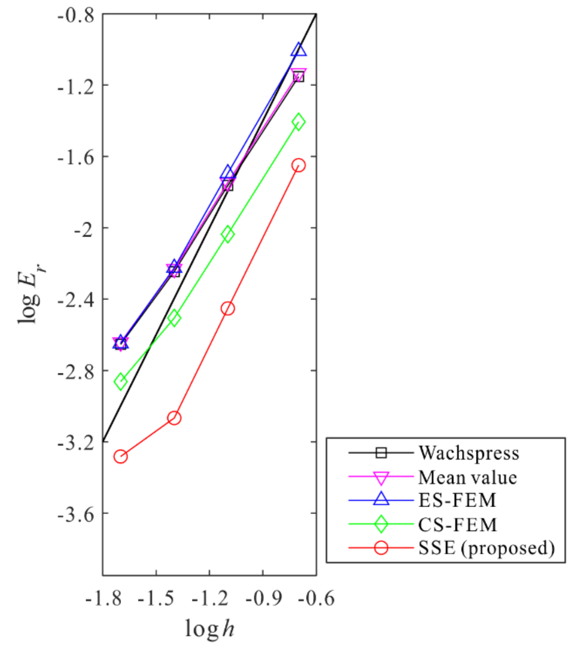
  2 - meshes constructed using the paving and cutting algorithm

  $N =$5, 12.5, 25, 50 ($N = L_e / h_{\text{grid}} = 10 /$ (grid size))

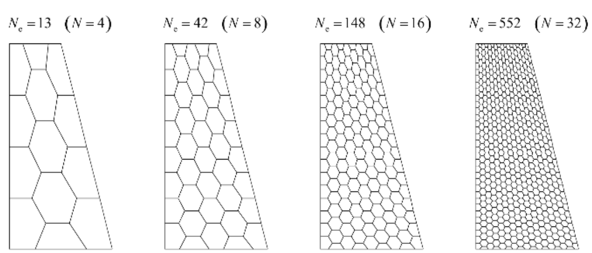
3) Dam problem

- **Convergence curves**

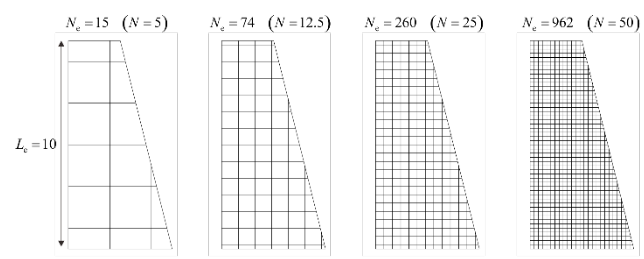




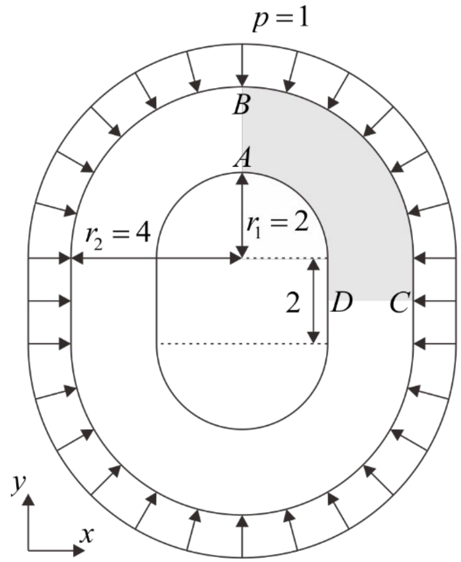
Meshes 1



Meshes 2



4) Ring problem



- **Force**

  Normal surface force $p=1$

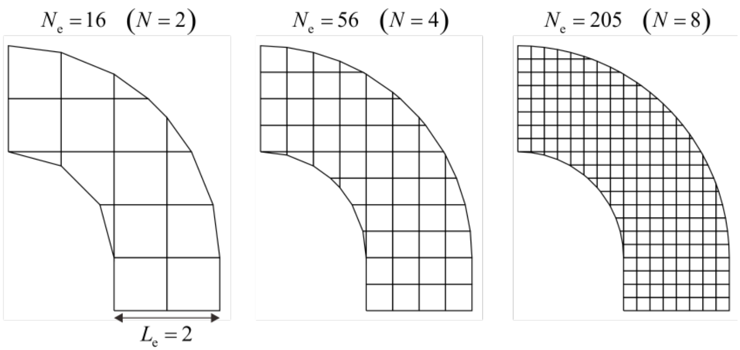
- **Boundary condition**

  Symmetric problem :

  u=0 along ***AB*** and *v*=0 along ***CD***

- **Material property (plane stress condition)**

  $E=3\times10^3, \quad \nu=0.3$



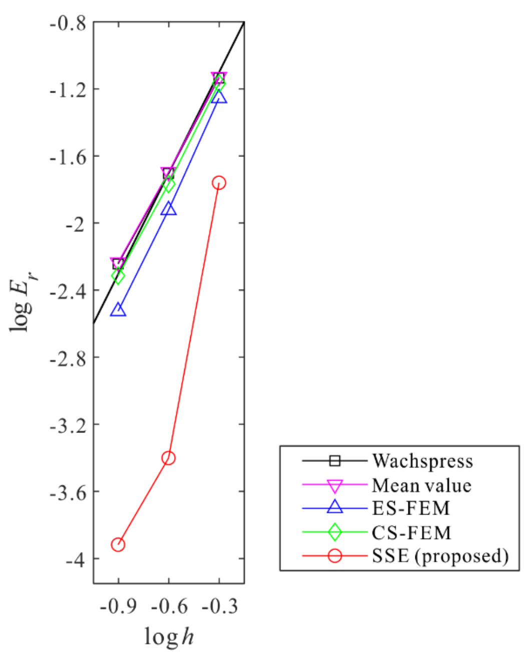
- **Meshes**

  constructed using the paving and cutting algorithm

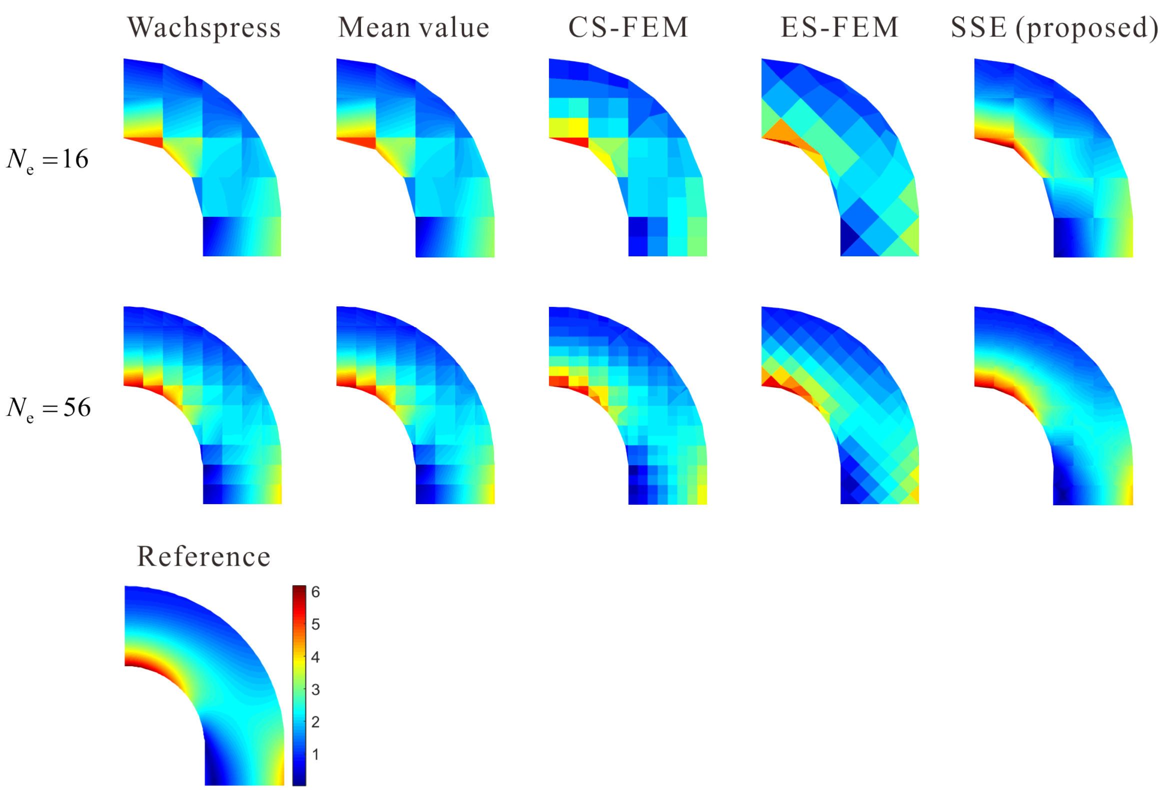
  N = 2, 4, 8 ($N=L_e/h_{grid}=2/$(grid size))

4) Ring problem

- **Convergence curves**



- **Von Mises stress distributions**



Comments on proposal

유승화 교수님

- 해당 요소를 활용 가능한 상황에 대한 조건
- 다각형 요소를 잘 활용할 수 있는 실용적 예제

윤정환 교수님

- 다각형 메쉬와 기존 삼각형 메쉬를 이용한 솔리드 요소의 비교
- 다각형 요소의 도형적 특성 (Jacobian / Edge ratio)에 따른 제한 조건

장대준 교수님

- 다각형 메쉬와 기존 삼각형/사각형 메쉬를 이용한 솔리드 요소와의 차별점
- 다각형 메쉬의 품질 (메쉬 크기의 균일함/다양한 각의 다각형의 혼합도)에 따른 해석 정도

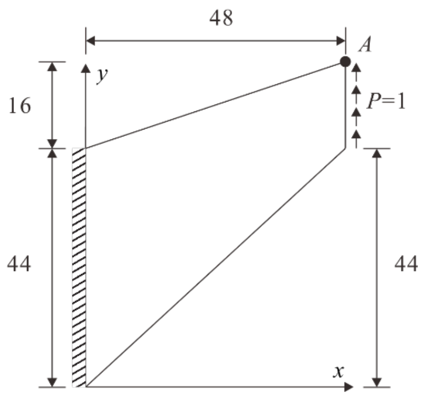
전형민 교수님

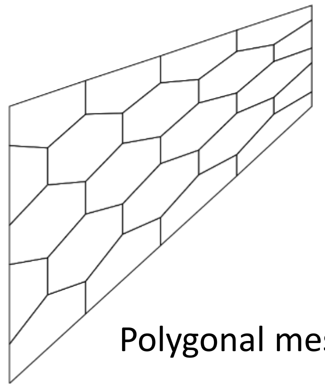
- 다각형 메쉬와 기존 삼각형 메쉬를 이용한 SSE와의 비교
- 다각형 요소를 잘 활용할 수 있는 실용적 예제

Comments on proposal

- **Comparison between polygonal elements and triangular elements** (윤정환/장대준/전형민 교수님)

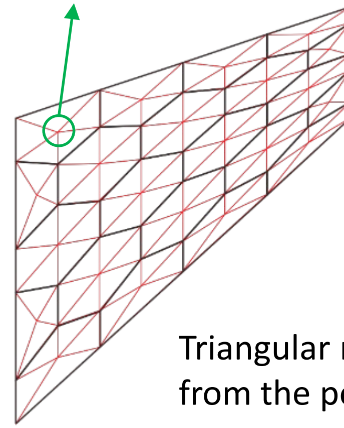
Cook's skew beam





Polygonal mesh

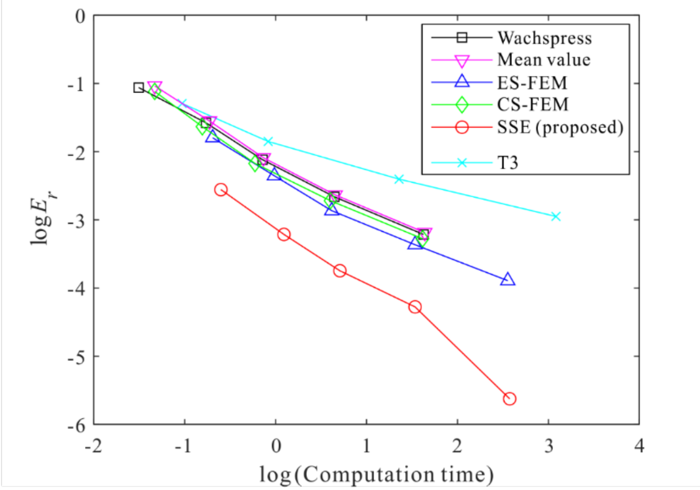
Additional node (DOFs) at the center of polygon



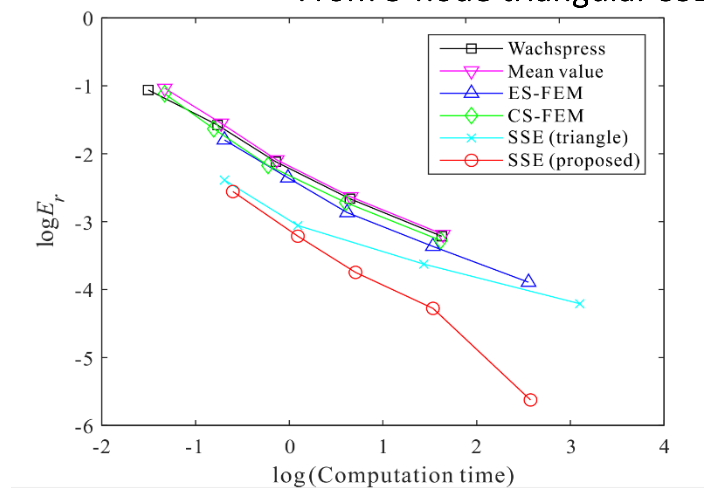
Triangular mesh
from the polygonal mesh

Computational cost

From standard T3



From 3-node triangular SSE



Comments on proposal

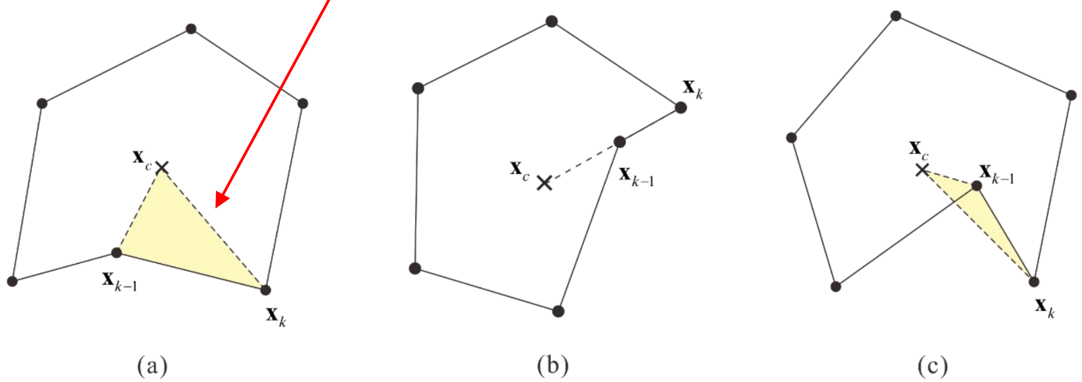
- **Conditions of applicability of Polygonal SSE** (유승화 교수님)

For signed areas of all sub-triangles,

convex and weakly concave polygonal : $A_k^{(m)} = \frac{1}{2}(x_c y_{k-1} - x_{k-1} y_c + x_{k-1} y_k - x_k y_{k-1} + x_k y_c - x_c y_k) > 0$

by Lien S, Kajiya JT (1984)

Signed area of sub-triangle, $A_k^{(m)}$



Signed area : $A_k^{(m)} > 0$ $\quad$ $A_k^{(m)} = 0$ $\quad$ $A_k^{(m)} < 0$

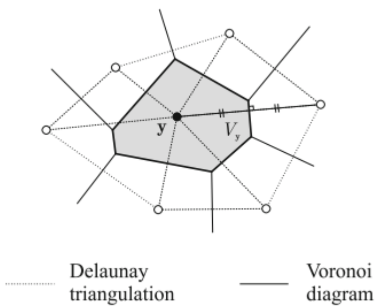
Convex
or weakly concave
(allowed)

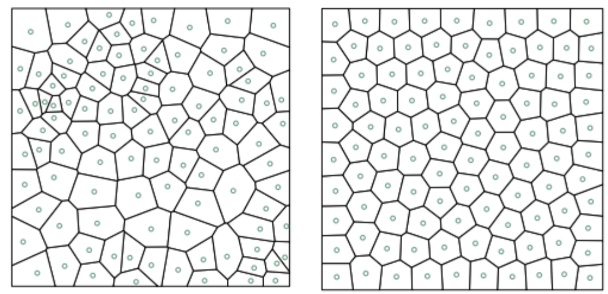
Concave
(not allowed)

Comments on proposal

- **Condition of mesh quality of polygonal mesh** (윤정환/장대준 교수님)

Automatic meshing based on Voronoi diagram

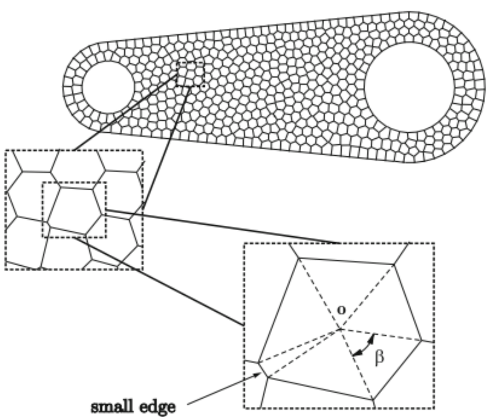




The iteration is performed to ensure uniform distribution of center points of the polygons.

↓

While the mesh size becomes uniform, issues may arise with the sizes of edges.



(C. talischi *et al*, 2012)

For a *n*-side polygon,

Angle of sub-triangle : $\beta < \epsilon\left(\frac{2\pi}{n}\right)$

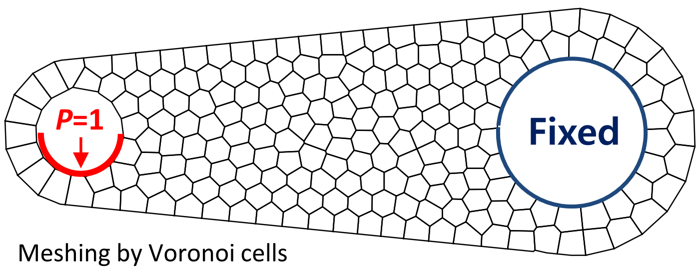
ϵ : user-defined tolerance (Generally, 0.1)

The edge can be collapsed into a single node.

Comments on proposal

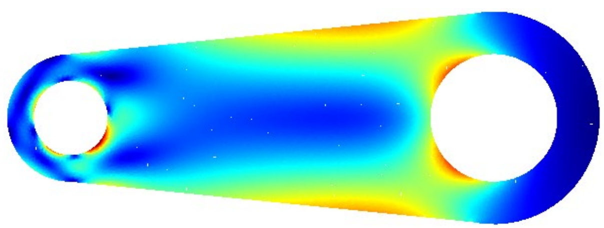
- **Practical example utilizing polygonal elements** (유승화/전형민 교수님)

$N_e = 400$

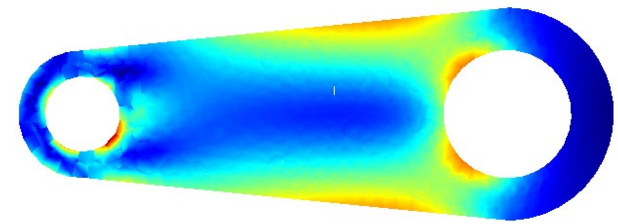


Meshing by Voronoi cells

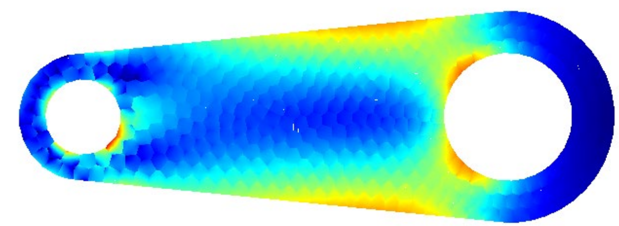
Reference (Von-Mises stress)



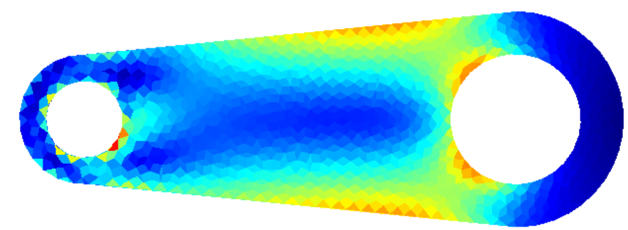
Proposed



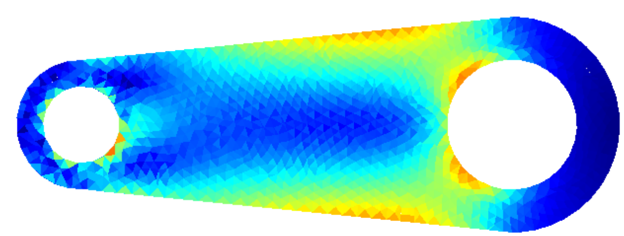
Wachspress



nES-FEM



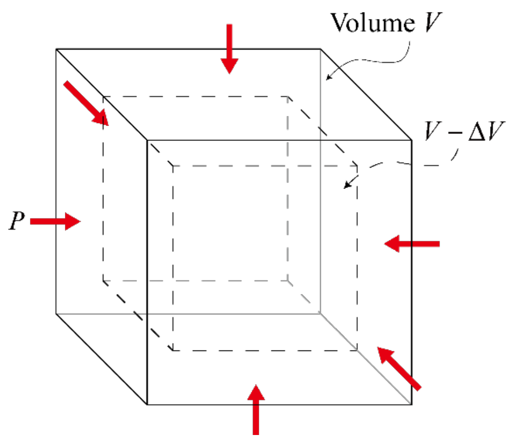
nCS-FEM



4. Treatment of volumetric locking in the strain-smoothed element method

Volumetric locking

- Bulk modulus in (nearly) incompressible material



Bulk modulus κ : ratio of the pressure change to the volume change

$$\kappa = -V\frac{\Delta P}{\Delta V} \longleftrightarrow \Delta P = -\kappa\frac{\Delta V}{V}$$

$$\rightarrow \quad p = -\kappa\varepsilon_v$$

p : hydrostatic stress

ε_v : volumetric strain

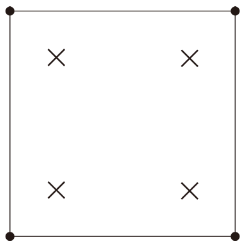
Volumetric locking occurs in situations where the volumetric strain is very small.

$$p = -\underset{\uparrow}{\kappa}\ \underset{\downarrow}{\varepsilon_v}$$

Extremely large material property   Very small volumetric strain

In the volumetric locking situations,

- The interpolation function fails to reflect the small strain.
- Almost same volume change across all integration points. (over-constraint)

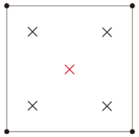


Volumetric locking

Treatments for volumetric locking

- **Mixed formulations for displacement and pressure**
  - The degree of freedom of pressure due to volumetric strain is considered.
  - In the 3-node triangular element, it is known that the order of the pressure term is not sufficient.

- **(Selective) reduced integration**

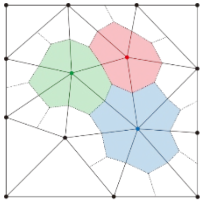


  - Reduced integration resolves locking but can lead to spurious zero-energy modes; to address this, reduced integration is performed only over the volumetric term.
  - It is the most widely known method, but it is impossible to perform for the 3-node triangular element.

- **Mixed and enhanced assumed strain (Simo et al. 1992, Braess 1998)**
  - Enriched strains are assumed from displacement and pressure fields.
  - These methods generally require additional internal degrees of freedom.

Volumetric locking in strain smoothing methods

- **Node-based smoothed(NS) FEM**



  - Strain smoothing through the node-based smoothing domain is performed.
  - It is known to be immune to volumetric locking.
  - It shows somewhat lower performance due to an overly soft analysis result.

- **Selective ES/NS FEM**
  - To enhance the performance of the element, edge-based and node-based FEM are combined.
  - Depending on the combination, there might be some performance differences.

Volumetric locking

- Separation of volumetric and deviatoric strain

(In the plane strain condition)

Volumetric strain : $$\varepsilon^{vol} = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} = \mathbf{B}^{vol}\mathbf{u}$$

Deviatoric strain : $$\varepsilon_{ij}^{dev} = \varepsilon_{ij} - \frac{1}{2}\varepsilon^{vol}\delta_{ij} = \mathbf{B}_{ij}^{dev}\mathbf{u}$$

$$(C_{11}^{dev} = C_{22}^{dev} = 2G, C_{12}^{dev} = G)$$

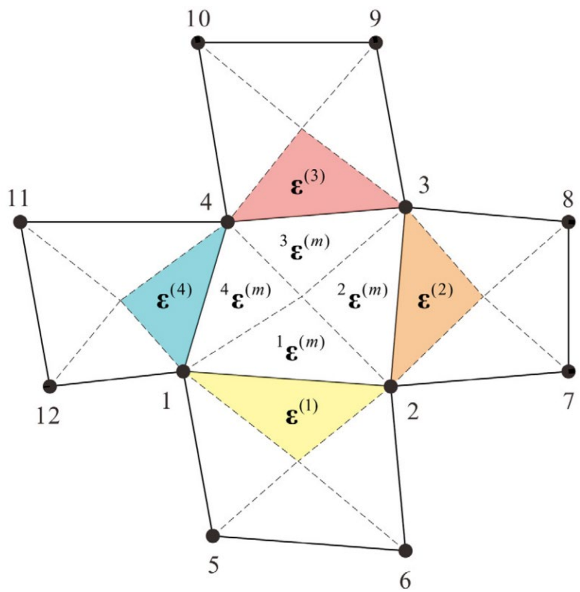
Stiffness matrix : $$\mathbf{K} = \underbrace{\int_V \boxed{\left(\kappa + \frac{\eta}{3}G\right)} \left(\mathbf{B}^{vol}\right)^T \mathbf{B}^{vol} dV}_{\text{Volumetric term}} + \underbrace{\int_V B_{ij}^{dev} C_{ij}^{dev} B_{ij}^{dev} dV}_{\text{Deviatoric term}} = \mathbf{K}^{vol} + \mathbf{K}^{dev}$$

$$\kappa = \frac{E}{3(1-2\nu)}; \quad \nu \to 0.5,\ \mathbf{K}^{vol} \to \infty$$

To avoid excessive stiffening of the volumetric term, additional treatment is required.

Volumetric locking in SSE

- In the case of 4-node quadrilateral finite element (C. Lee *et al*, 2021)

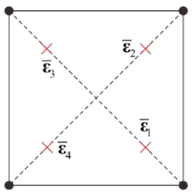


Smoothed strain between the sub-triangle and its neighboring sub-triangle

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A_k^{(m)} + A^{(k)}}\left(A_k^{(m)}\,{}^{k}\boldsymbol{\varepsilon}^{(m)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right)$$

Smoothed strains in integration points of the element

$$\overline{\boldsymbol{\varepsilon}}_k = \frac{1}{A_{k-1}^{(m)} + A_k^{(m)}}\left(A_{k-1}^{(m)}\hat{\boldsymbol{\varepsilon}}^{(k-1)} + A_k^{(m)}\hat{\boldsymbol{\varepsilon}}^{(k)}\right) \quad \text{with } k=1,2,3,4$$



Constant volumetric strain

$${}^{vol}\overline{\varepsilon}^{(m)} = \frac{1}{A^{(m)}}\sum_{k=1}^{4} A_k^{(m)}\left(\hat{\varepsilon}_{11}^{(k)} + \hat{\varepsilon}_{22}^{(k)}\right)$$

(constant value in an element)

Deviatoric strain

$${}^{dev}\overline{\varepsilon}_{ij}^{(m)} = \overline{\varepsilon}_{ij}^{(m)} - \frac{1}{2}\,{}^{vol}\overline{\varepsilon}^{(m)}\delta_{ij}$$

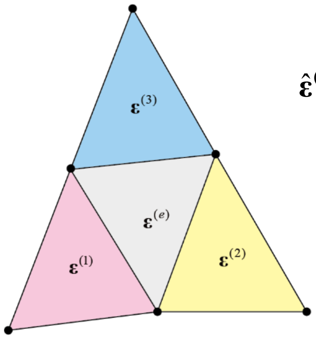
(values in 4 integration points)

* This procedure cannot be directly applied to 3-node triangular element.

Treatment of volumetric locking in 3-node triangular element

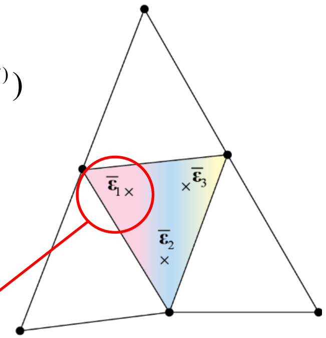
- SSE method procedure in 3-node triangular finite element

Strains in target and neighboring element



$$\hat{\boldsymbol{\varepsilon}}^{(i)} = \frac{1}{A^{(e)} + A^{(i)}}\left(A^{(e)}\boldsymbol{\varepsilon}^{(e)} + A^{(i)}\boldsymbol{\varepsilon}^{(i)}\right)$$

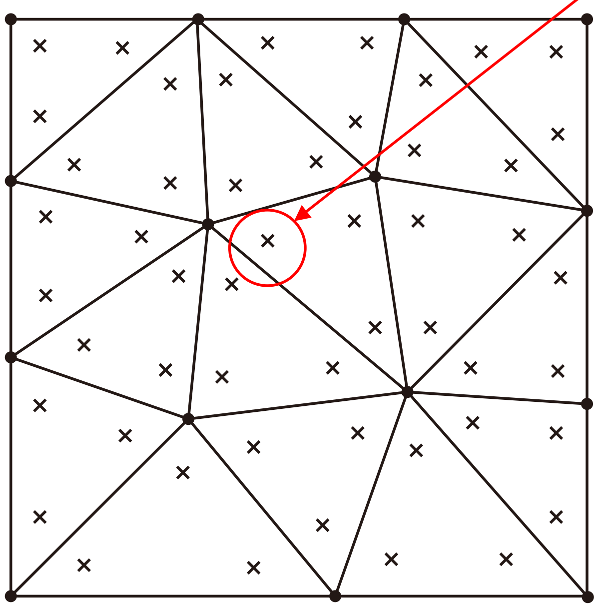
Smoothed strains in integration points



$$\bar{\boldsymbol{\varepsilon}}_1 = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(3)})$$

$$\bar{\boldsymbol{\varepsilon}}_2 = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)})$$

$$\bar{\boldsymbol{\varepsilon}}_3 = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)})$$

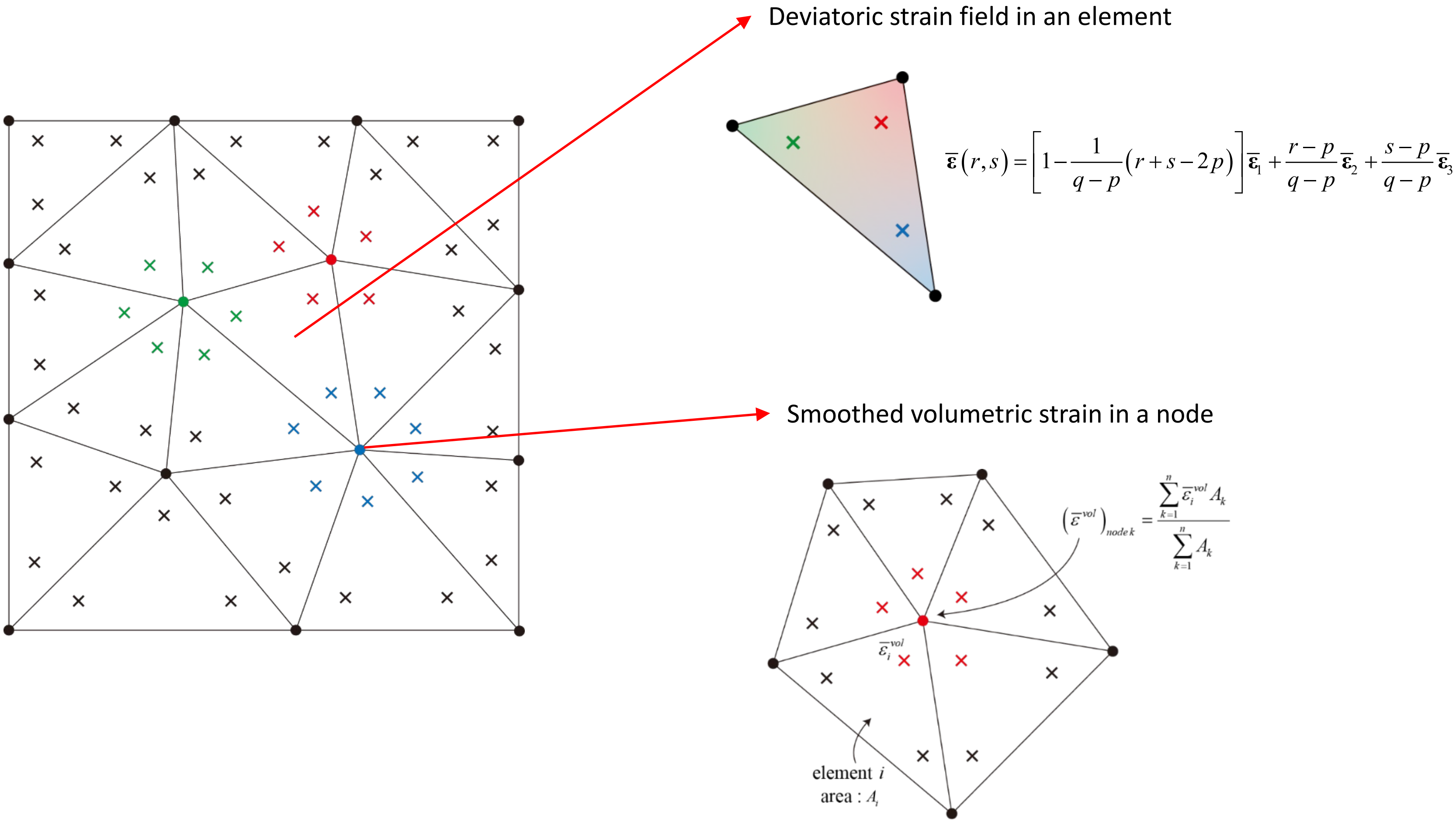


The smoothed strains obtained by SSE are assigned to integration points, and a linear strain field within the element is constructed.

$$\bar{\boldsymbol{\varepsilon}}(r,s) = \left[1 - \frac{1}{q-p}(r+s-2p)\right]\bar{\boldsymbol{\varepsilon}}_1 + \frac{r-p}{q-p}\bar{\boldsymbol{\varepsilon}}_2 + \frac{s-p}{q-p}\bar{\boldsymbol{\varepsilon}}_3$$

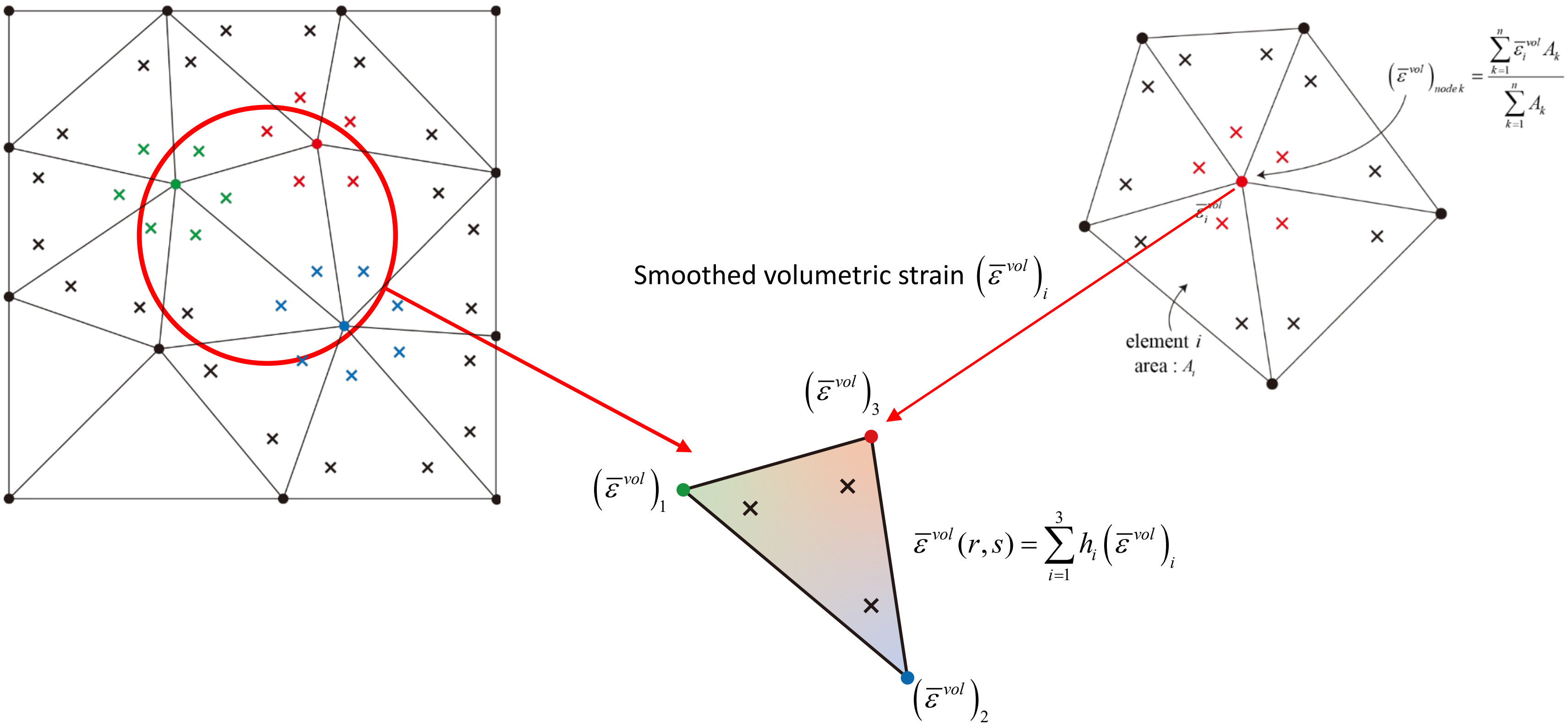
Treatment of volumetric locking in 3-node triangular element

- Smoothed volumetric strains in a node by strain-smoothed element method



Treatment of volumetric locking in 3-node triangular element

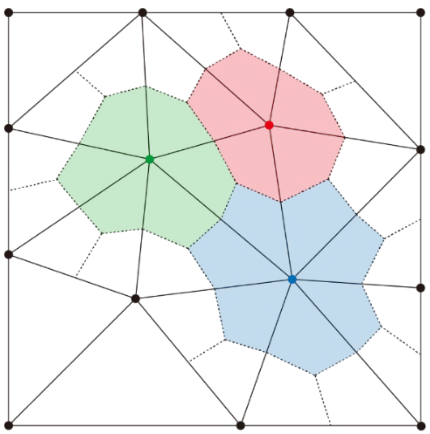
- Smoothed volumetric strains and a strain field within an element



Treatment of volumetric locking in 3-node triangular element

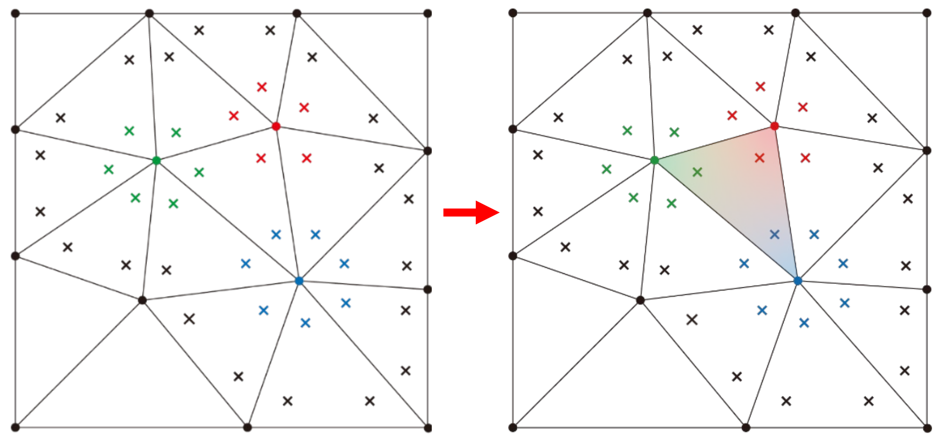
- Node-based smoothed element and nodal strain from SSE

Smoothed FEM



Node-based smoothed FEM

Volumetric strain from SSE



Smoothed volumetric strain

Strain field in an element

Requires a special smoothing domain

Strain field is defined within an element

Basic numerical tests

Isotropic element test

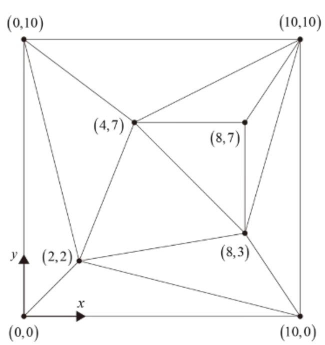
- The proposed elements yield the same results regardless of the element node numbering sequences.

Zero energy mode test

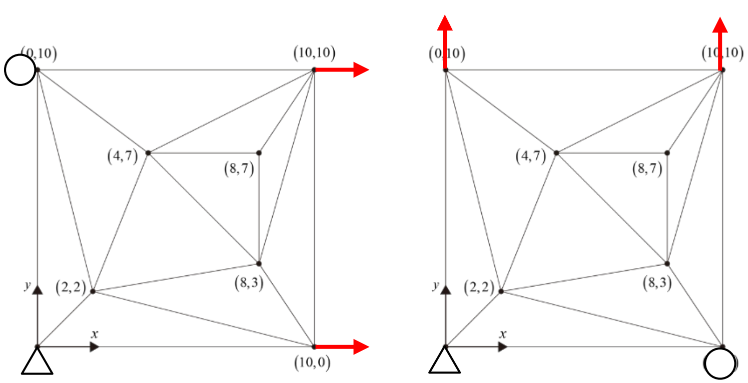
- The number of zero-energy modes of the stiffness matrix in rigid body condition is counted.
- The stiffness matrix of the 2D element must contain only three zero-energy modes.

Patch tests

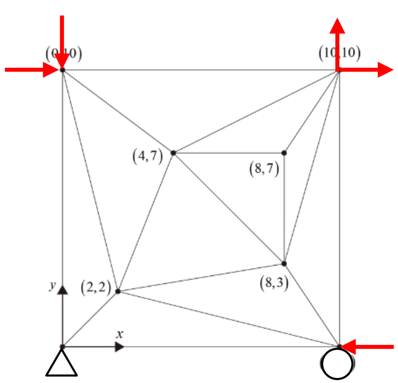
- The minimum number of DOFs is constrained to prevent rigid body motions.
- Appropriate loadings are applied to obtain a constant stress field.
- <u>The same stress value should be obtained at all points</u> on the elements to pass the patch tests.



Mesh for patch test



Normal stress condition

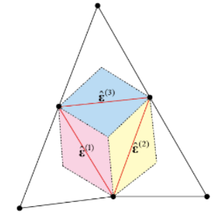


Shear stress condition

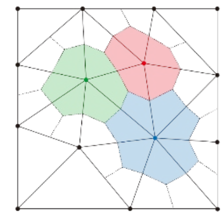
Numerical examples

➢ Finite elements considered

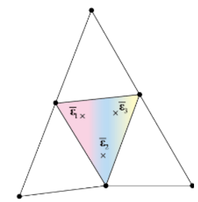
- **ES-FEM** Edge-based smoothed finite element (Nguyen-Thoi et al. , 2011)



- **NS-FEM** : Node-based smoothed finite element (Liu et al. , 2009)



- **SSE** : Strain-smoothed 3-node triangular finite elements without volumetric locking treatment



- **Proposed** : Strain-smoothed 3-node triangular finite elements with volumetric smoothed strains

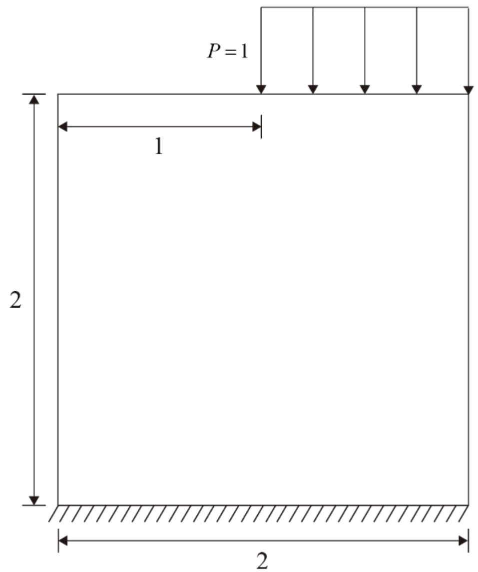
➢ Evaluation method

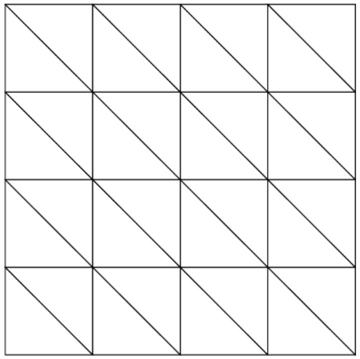
- Convergence curves obtained using the s-norm $E_h = \dfrac{\left\|\mathbf{u}_{ref} - \mathbf{u}_h\right\|_S^2}{\left\|\mathbf{u}_{ref}\right\|_S^2}$ with $\left\|\mathbf{u}_{ref} - \mathbf{u}_h\right\|_S^2 = \int_{\Omega_{ref}} \Delta\boldsymbol{\varepsilon}^T \Delta\boldsymbol{\tau} d\Omega_{ref}.$

➢ Reference solution

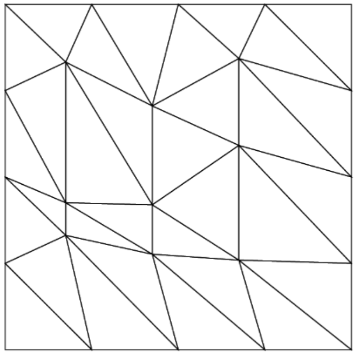
- Reference solutions are calculated using a $\underline{N=96}$ regular mesh of <u>9-node 2D solid elements</u> with reduced integration for the volumetric term.

1) Square block problem





Regular mesh



Distorted mesh

$(N=4)$

- **Force**

  Compression pressure of total magnitude $P=1$ at the right half top of the structure

- **Boundary condition**

  The block is clamped along its bottom

- **Material property (plane strain condition)**

  $E=3\times10^7$, $\nu=0.3, 0.49, 0.499$ and 0.4999

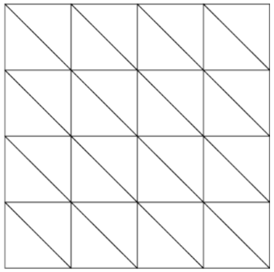
- **Regular and distorted Meshes**

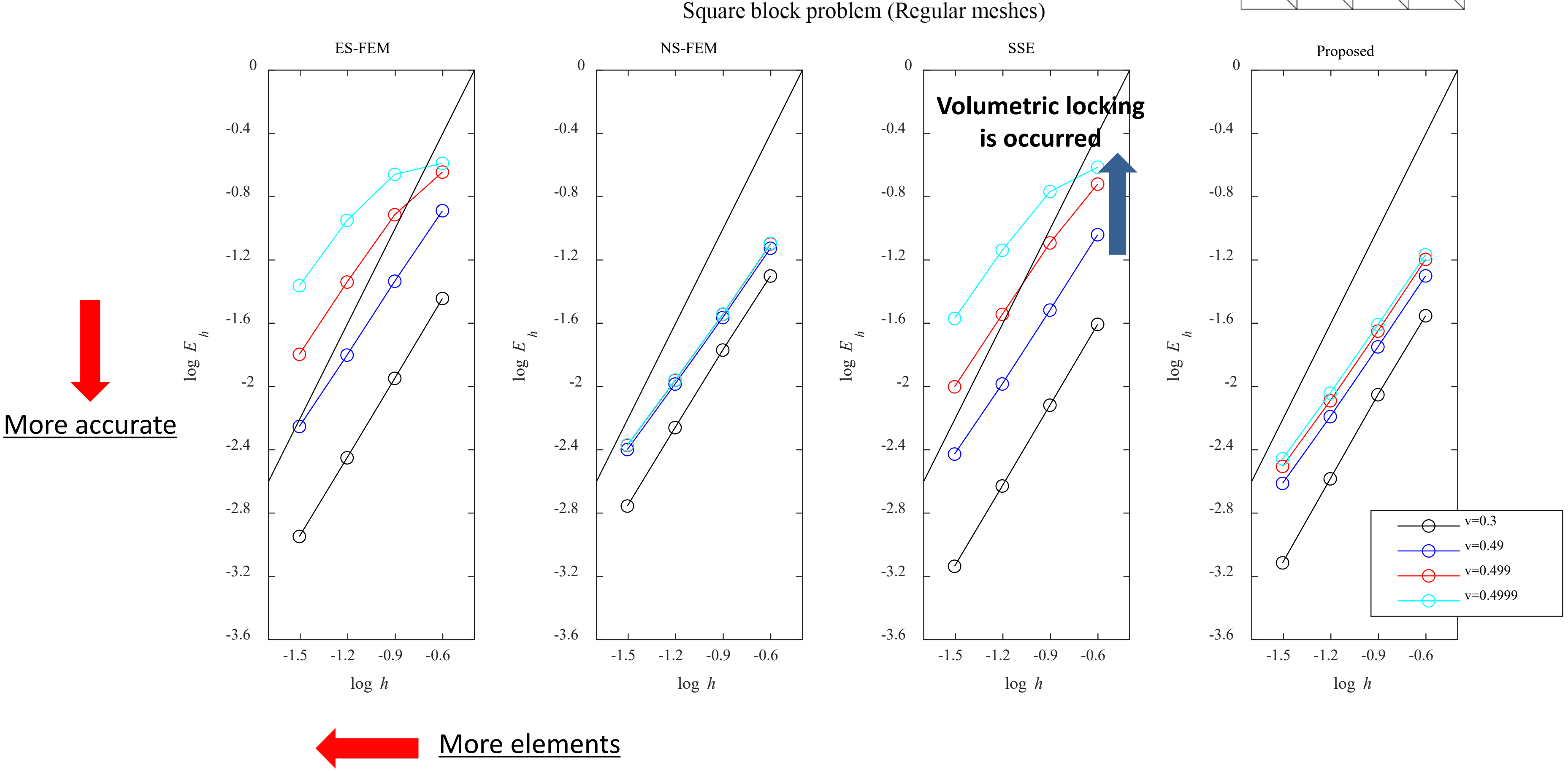
  $N\times N$ elements; $N=4$, 8, 16 and 32.

Numerical examples

1) Square block problem

- **Convergence curves (Regular meshes)**



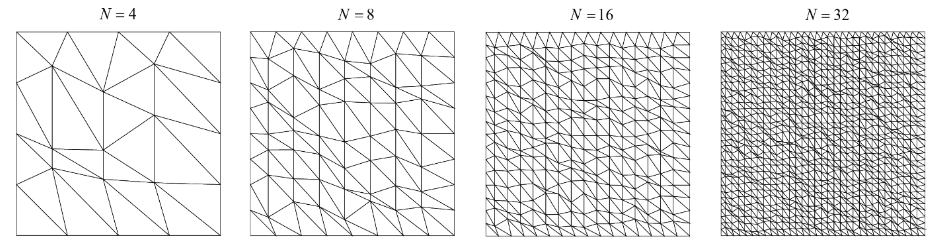


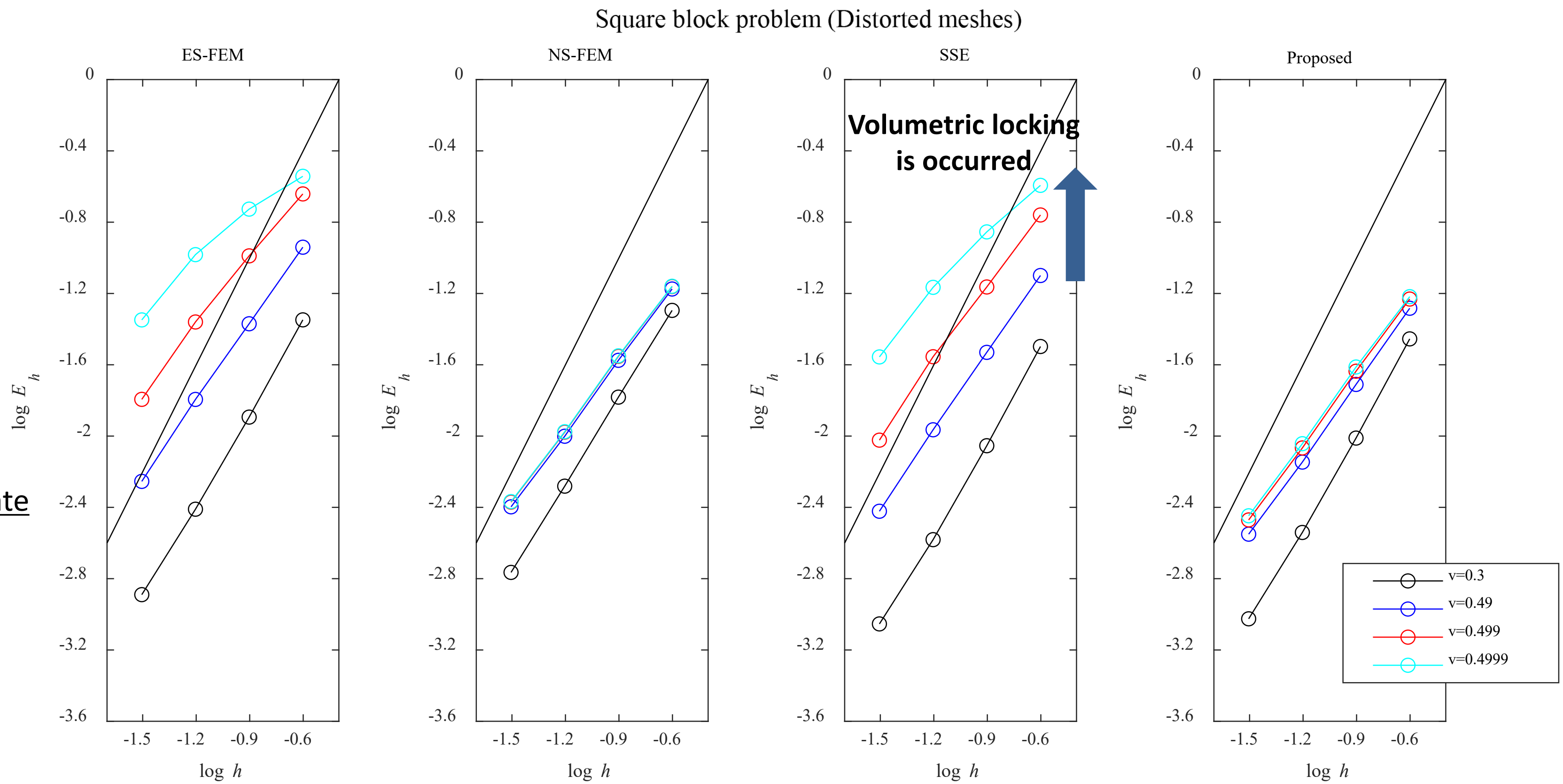
h=1/N

Numerical examples

1) Square block problem

- **Convergence curves (Distorted meshes)**



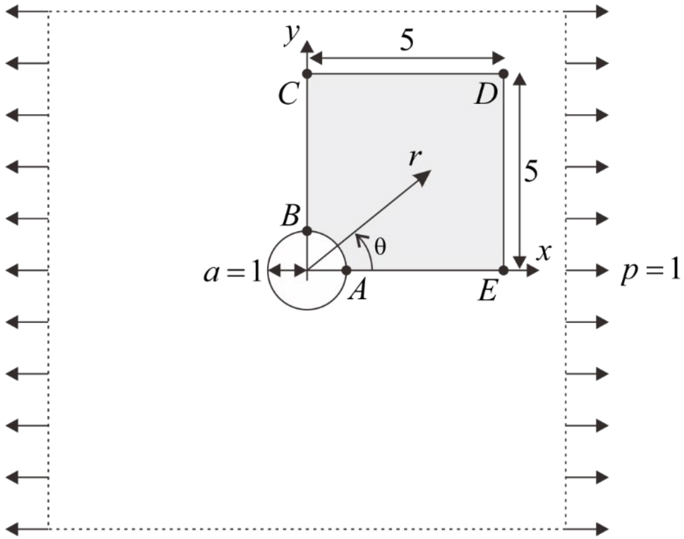


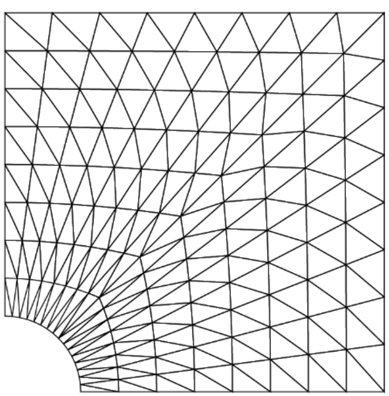
More accurate

More elements

h=1/N

2) Infinite plate with a circular hole





$(N = 8)$

- **Force**

  analytical solutions of infinite plate with a hole

  $$\sigma_{xx}(r,\theta) = p\left(1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3a^4}{2r^4}\cos 4\theta\right)$$

  $$\sigma_{yy}(r,\theta) = p\left(-\frac{a^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3a^4}{2r^4}\cos 4\theta\right)$$

  $$\sigma_{xy}(r,\theta) = p\left(-\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) + \frac{3a^4}{2r^4}\sin 4\theta\right)$$

- **Boundary condition**

  u=0 along ***BC*** and *v*=0 along ***AE***

- **Material property (plane strain condition)**

  $E = 3\times 10^7$, $\nu = 0.3, 0.49, 0.499$ and 0.4999

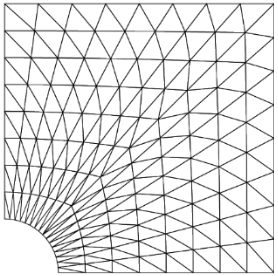
- **Meshes** (Two sets of $N\times N$ elements that are symmetric about the diagonal)

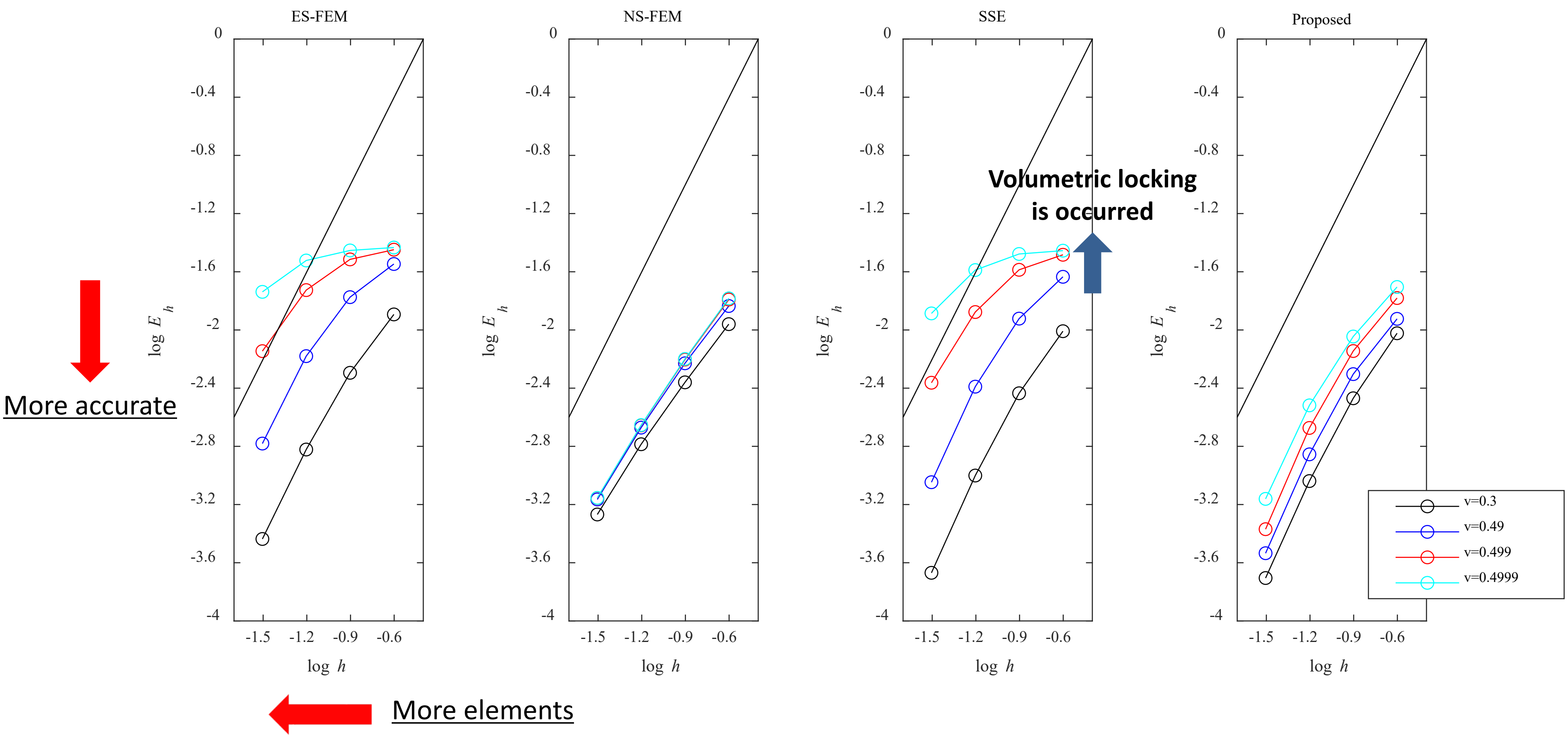
  $N =$ 4, 8, 16 and 32.

Numerical examples

2) Infinite plate with a circular hole

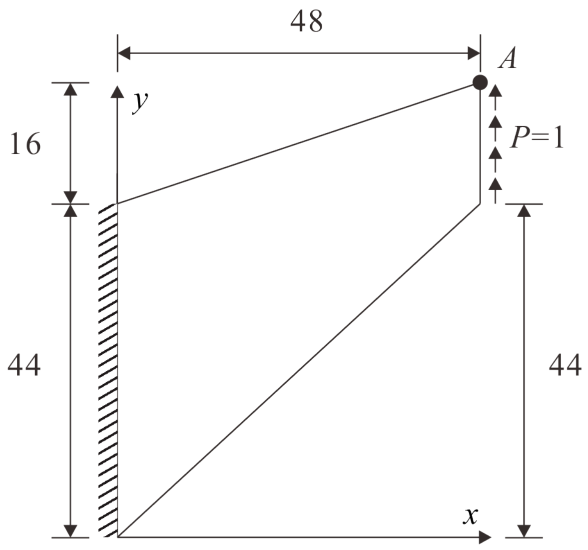
- **Convergence curves**





h=1/*N*

3) Cook's skew beam



- **Force**

  Distributed shearing force $P = 1$

- **Boundary condition**

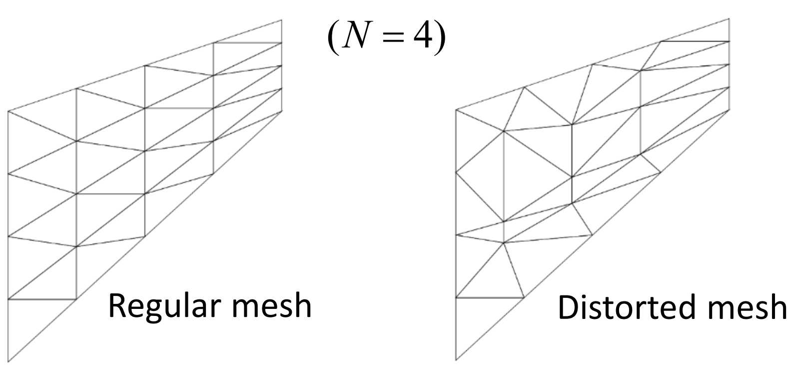
  Left edge is clamped

- **Material property (plane strain condition)**

  $E = 3 \times 10^7$, $\nu = 0.3, 0.49, 0.499$ and 0.4999

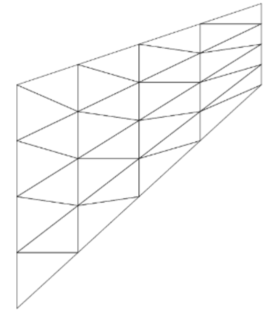
- **Regular and distorted Meshes**

  $N \times N$ elements; $N = 4$, 8, 16 and 32.

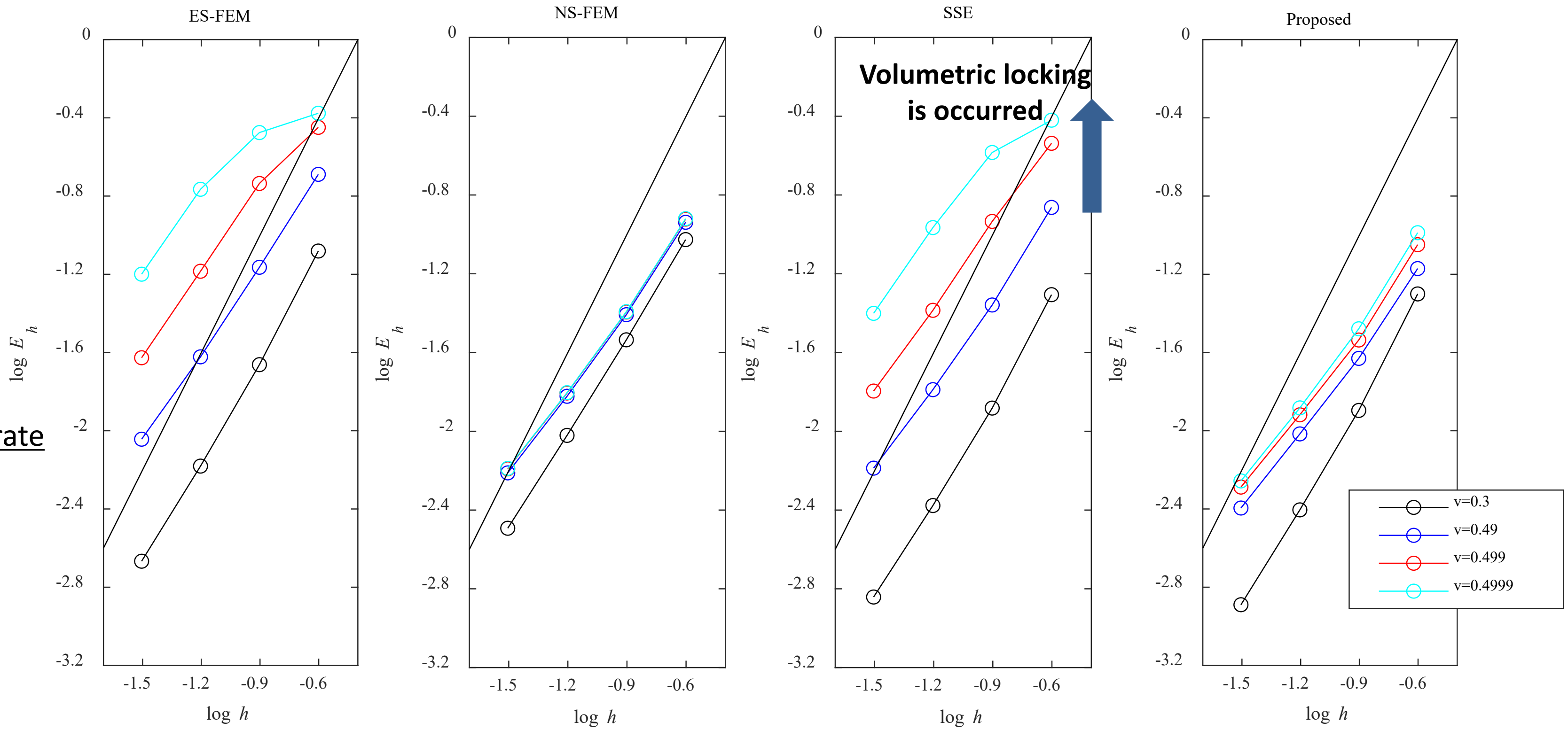


3) Cook's skew beam

- **Convergence curves (Regular meshes)**



Cook's skew beam problem (Regular meshes)



More accurate

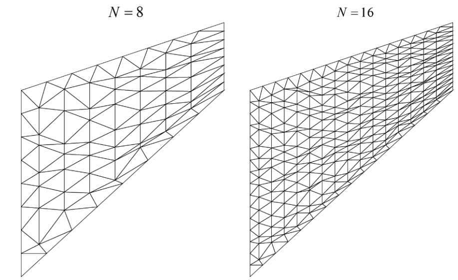
More elements

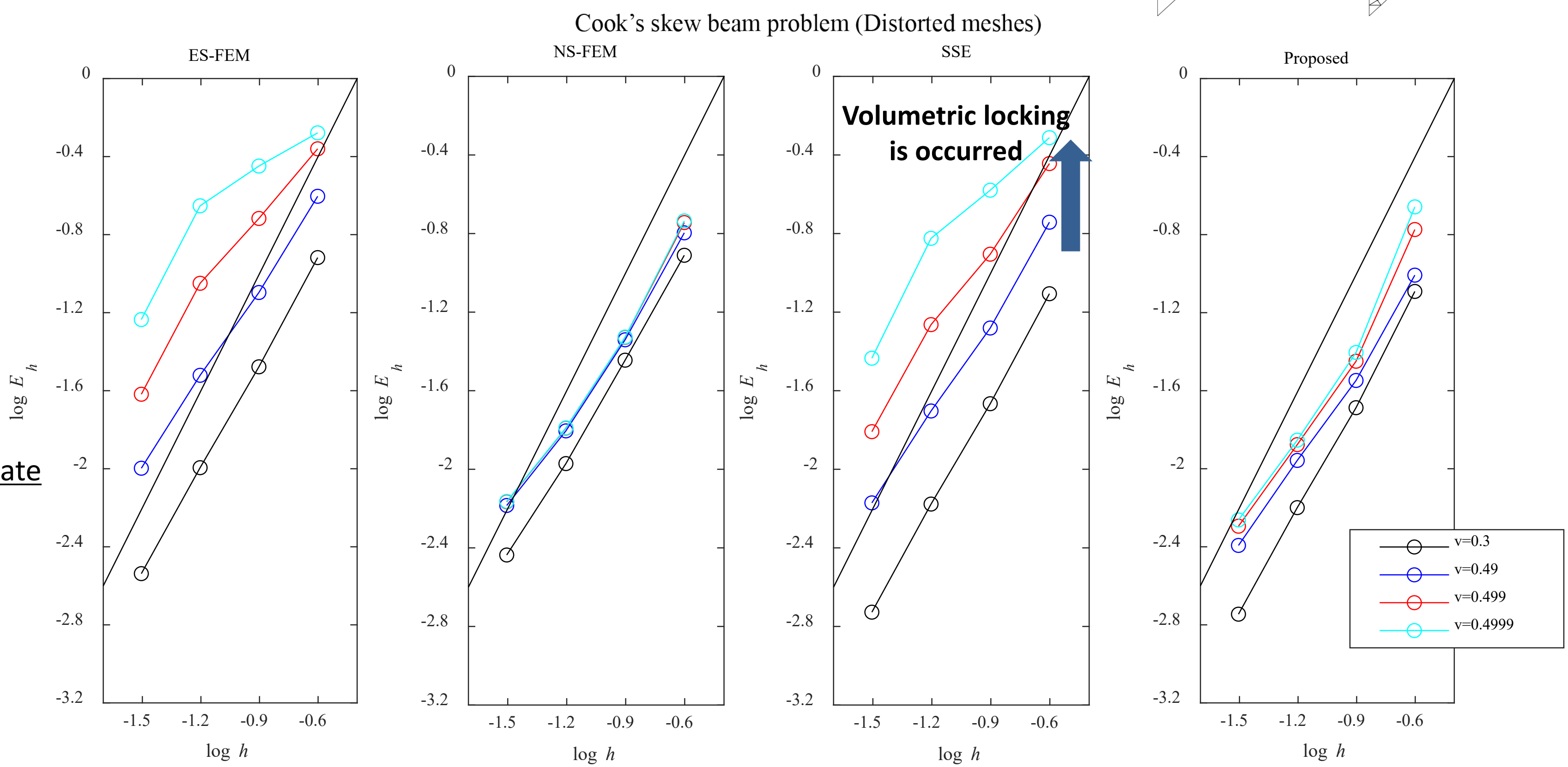
h=1/N

Numerical examples

3) Cook's skew beam

- **Convergence curves (Distorted meshes)**





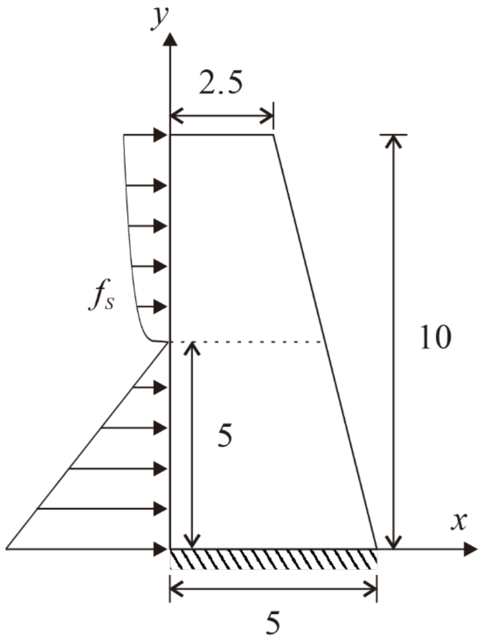
More accurate

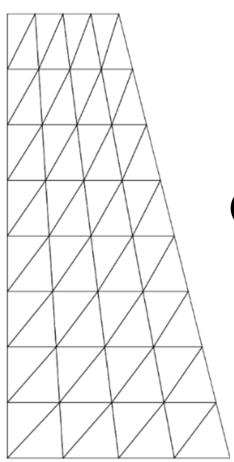
More elements

h=1/N

Numerical examples

4) Dam problem





$(N = 4)$

- **Force**

  Compressive surface force

$$f_S = \begin{cases} 5 - y & 0 \le y \le 5 \\ (y-5)^{1/5} & 5 \le y \le 10 \end{cases}$$

- **Boundary condition**

  The block is clamped along its bottom

- **Material property (plane strain condition)**

  $E = 3 \times 10^{10}$, $\nu = 0.3, 0.49, 0.499$ and 0.4999

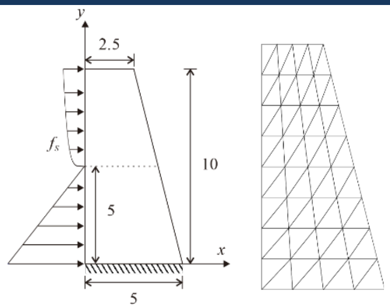
- **Meshes** ($N \times 2N$ elements)

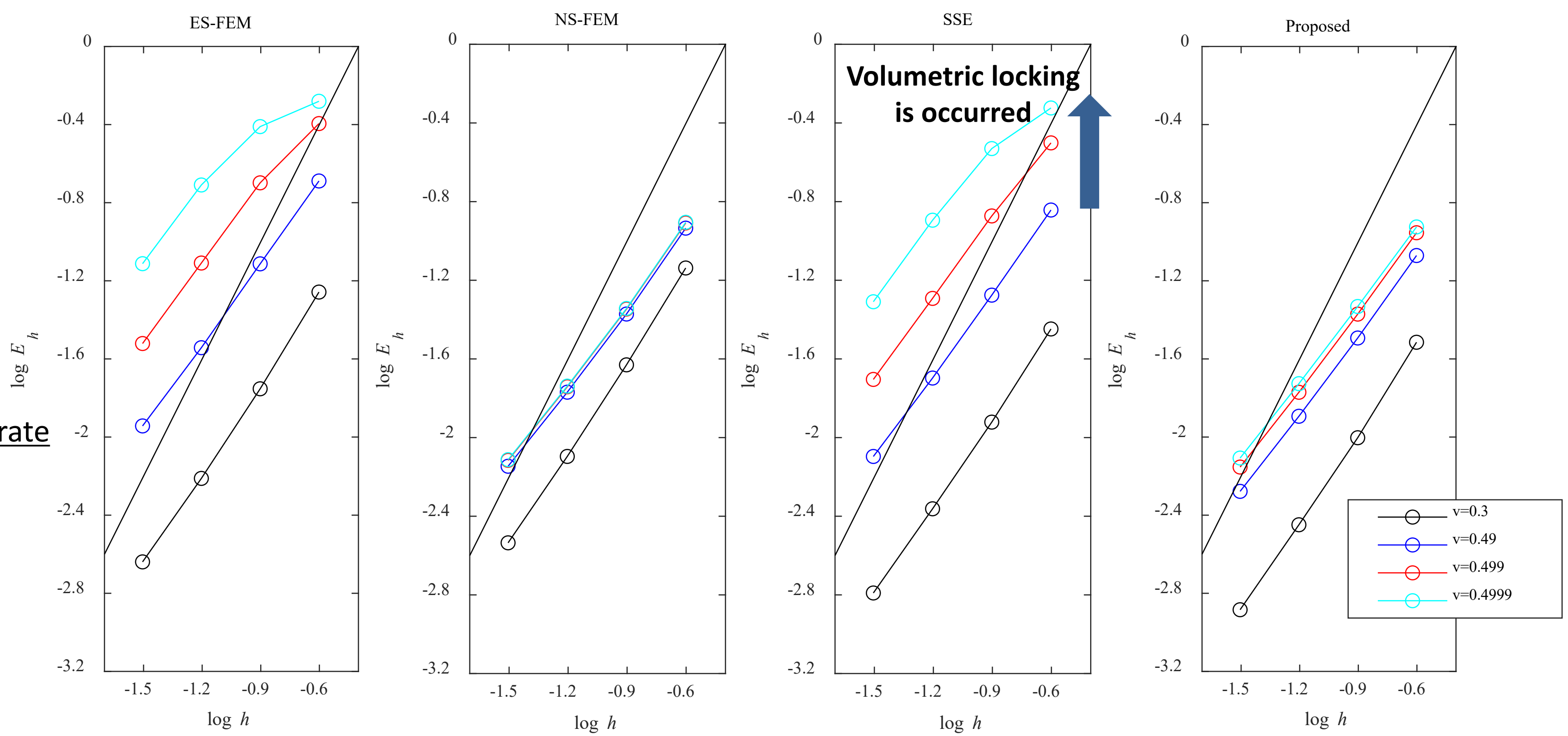
  $N =$ 4, 8, 16 and 32.

Numerical examples

4) Dam problem

- **Convergence curves**





More accurate

More elements

h=1/*N*

5. Conclusions & Future works

Conclusions

1. The SSE method has been applied to polygonal elements.

- To apply the SSE to the polygonal elements, <u>piecewise linear shape functions</u> are employed to triangulate the elements.
- smoothed strains are assigned to the center point of the sub-quadrilaterals of the elements and a piecewise linear strain field is conducted in elements.
- The elements showed improved convergence behaviors compared with previously developed elements in various numerical examples.
- The elements still show improved performance even when considering various meshing techniques of polygonal elements.

2. The treatment of volumetric locking in 3-node triangular SSE has been developed.

- To address volumetric locking, the <u>smoothed volumetric strains</u> are defined on a node-wise basis.
- Based on the volumetric strains at the nodes, the strain field within the element is constructed.
- This treatment preserves the convergence performance of the SSE even when no locking occurs.
- When volumetric locking occurs, it shows improved convergence performance compared to general SSE.

Future works

The polygonal strain-smoothed elements

- Application to contact / crack problems
- Geometry / material nonlinear analyses

The treatment of volumetric locking in 3-node triangular SSE

- Modeling of rubberlike material / nearly incompressible flow
- Material nonlinear analyses
- Geometry nonlinear analyses (Large deformation)
- Extension to 3D tetrahedral element

Thank you for listening