Ph. D. degree dissertation presentation

확장된 Saint-Venant 비틀림 이론 및 이를 이용한 비틀림을 받는 빔의 위상 최적화

Extended Saint-Venant torsion theory and its application to topology optimization of beams under torsion

Dong-Hwa Lee

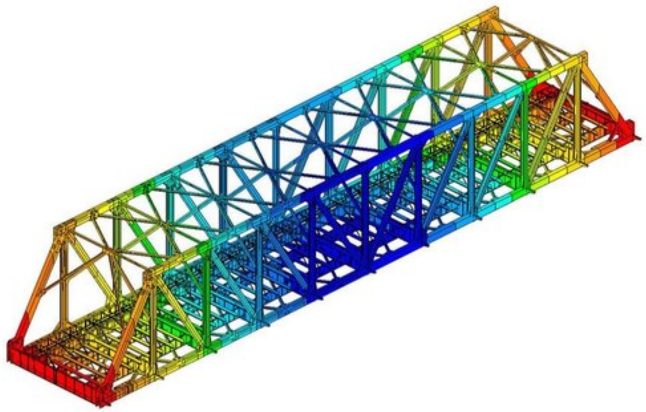
2021.12.06

KAIST Department of Mechanical Engineering

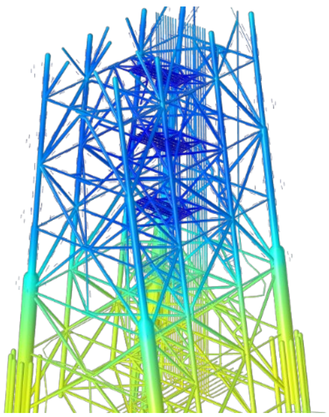
Contents

1. Introduction

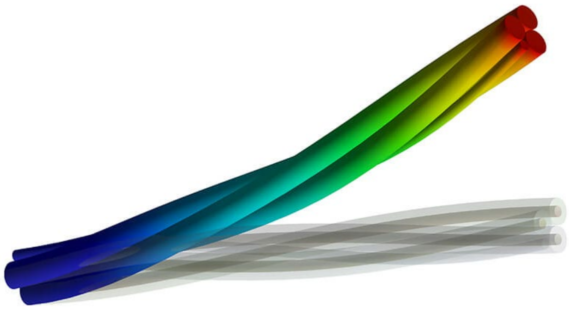
Applications of beam finite element analysis



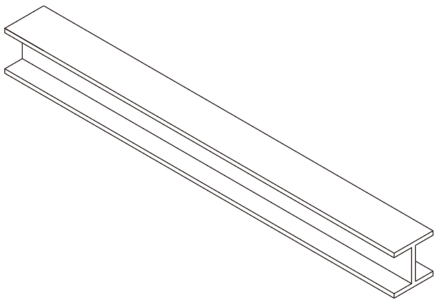
Bridge



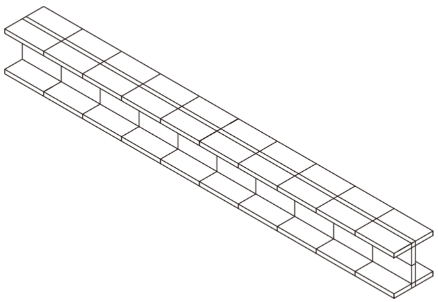
Offshore platform



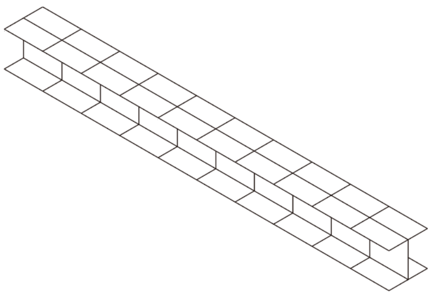
Wire cable



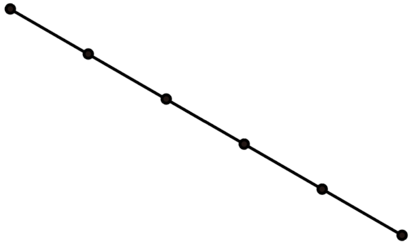
Physical model



Solid model
594 DOFs

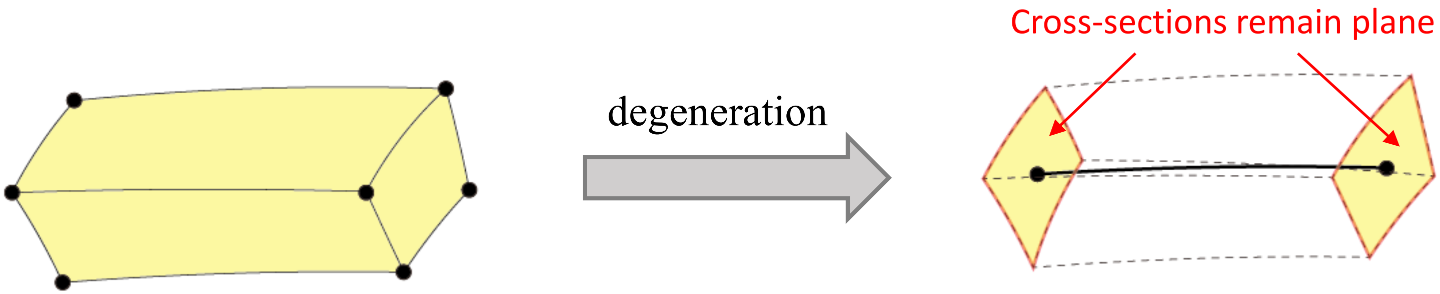


Shell model
352 DOFs

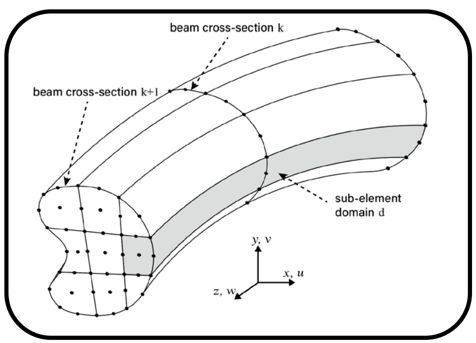


Beam model
36 DOFs

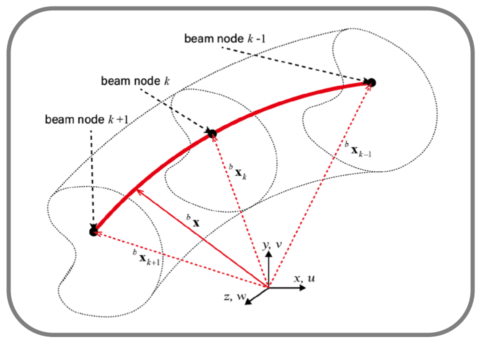
Concept of the continuum mechanics based beam model



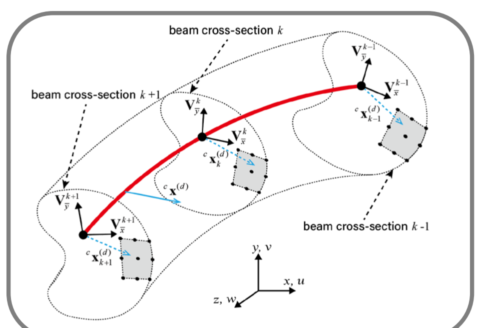
Solid **Beam**



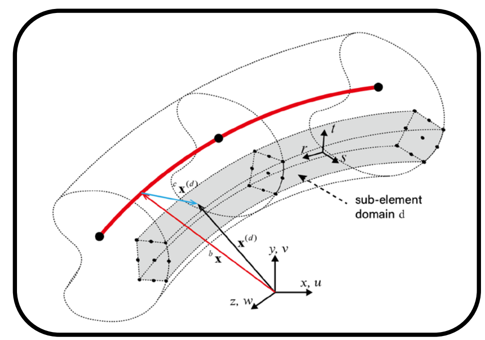
Solid



Nodal interpolation

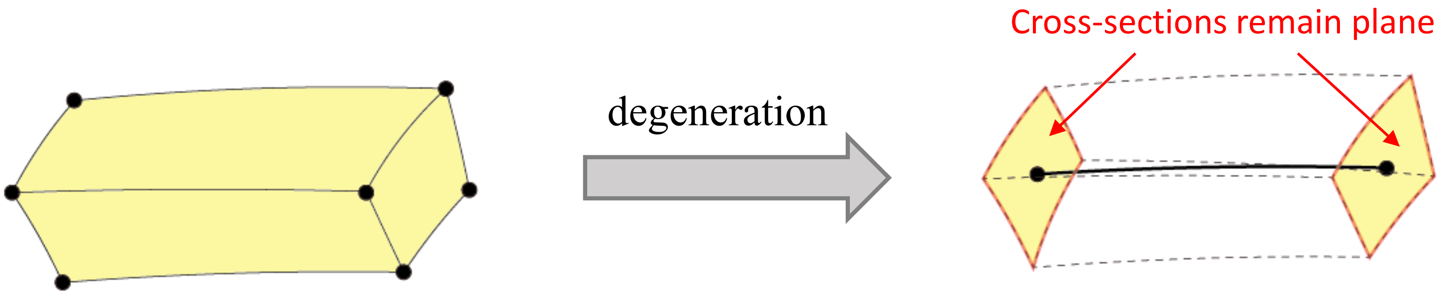


Cross-sectional interpolation

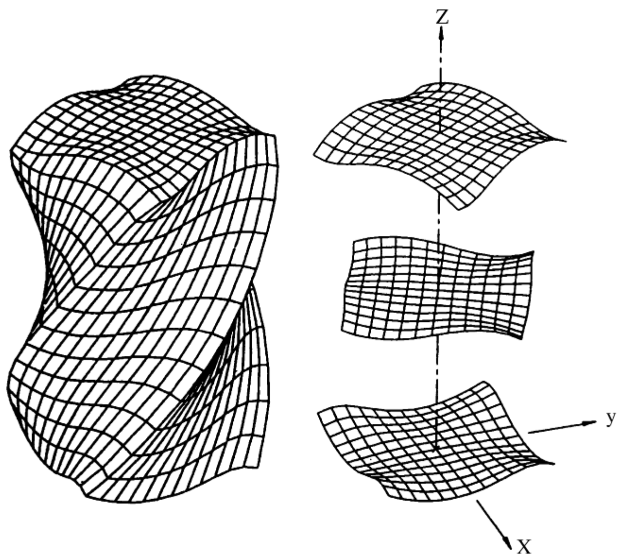


Continuum mechanics based beam

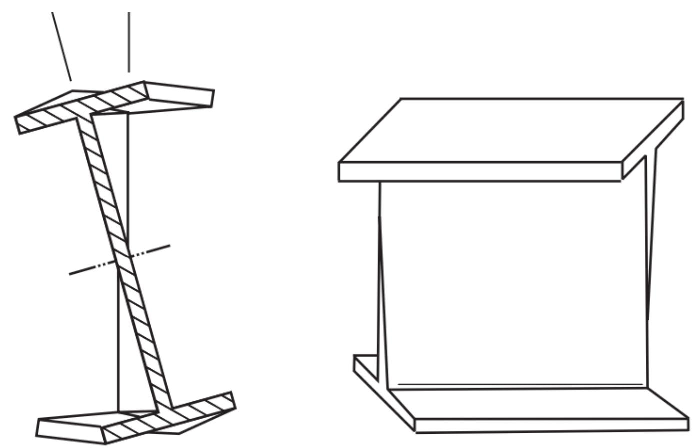
Concept of the continuum mechanics based beam model



Solid **Beam**

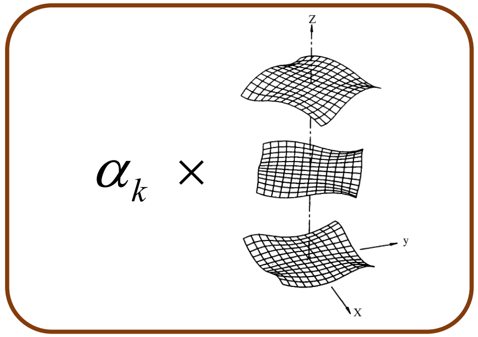


Warping of thick square beam

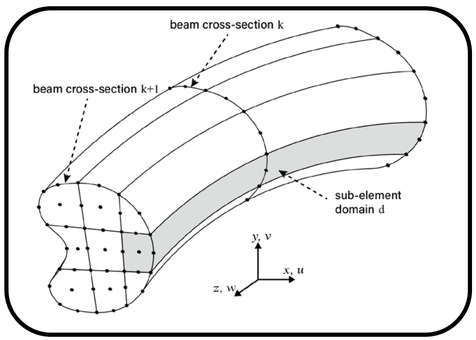


Warping of thin-walled beam

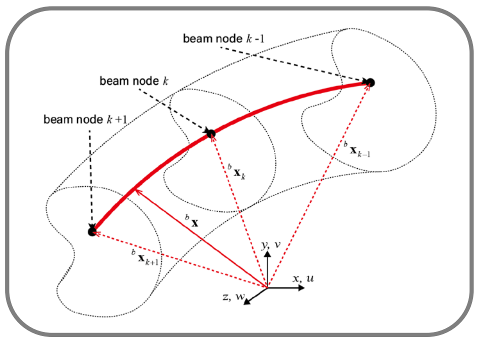
Imposing warping mode into beam element



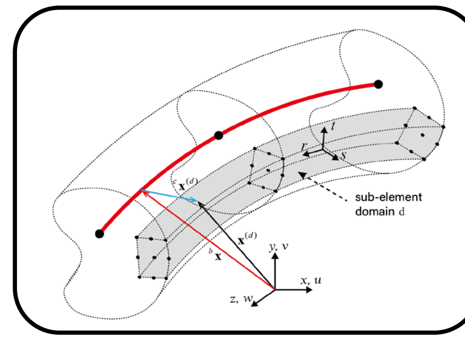
Warping interpolation



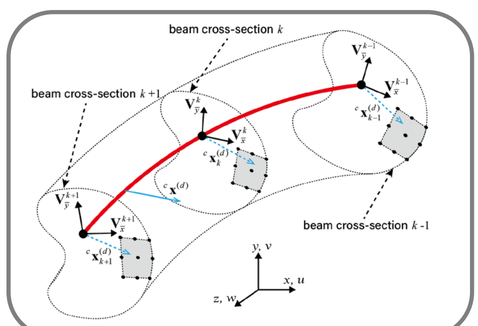
Solid



Nodal interpolation



Beam with warping mode



Cross-sectional interpolation

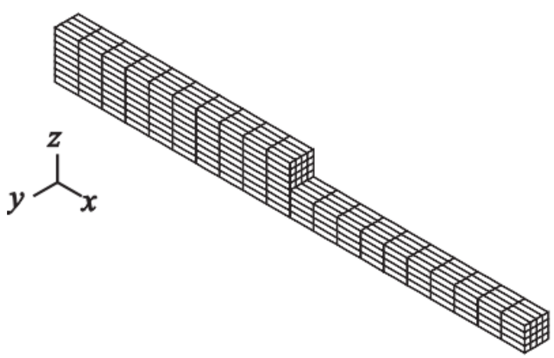
Imposing warping mode into beam element

❖ Benscoter type warping enrichment methods

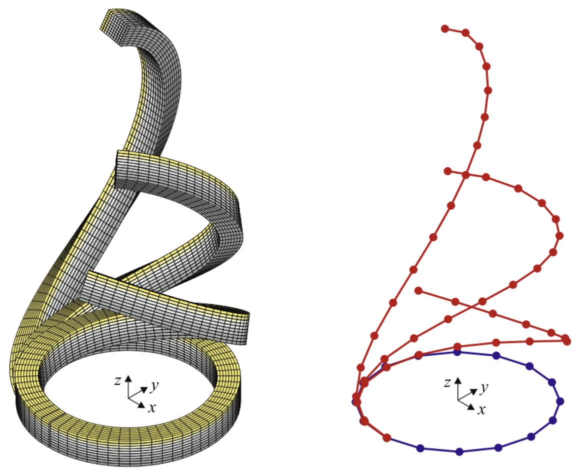
$$ {}^{e}\mathbf{x}^{(d)}(r,s,t)=\sum_{k=1}^{P}h_k(r)\left({}^{b}\mathbf{x}_k+\sum_{n=1}^{R}H^n(s,t)\left({}^{n}\bar{x}_k^{(d)}\,\mathbf{V}_{\bar{x}}^k+{}^{n}\,\bar{y}_k^{(d)}\,\mathbf{V}_{\bar{y}}^k\right)\right)+\underline{\sum_{k=1}^{P}h_k(r)\left(\sum_{n=1}^{R}H^n(s,t)\,{}^{n}f_k^{(d)}\,\mathbf{V}_{\bar{z}}^k\,\alpha_k\right)} $$

$$ =\sum_{k=1}^{P}h_k(r)\left({}^{b}\mathbf{x}_k+\sum_{n=1}^{R}H^n(s,t)\left({}^{n}\bar{x}_k^{(d)}\,\mathbf{V}_{\bar{x}}^k+{}^{n}\,\bar{y}_k^{(d)}\,\mathbf{V}_{\bar{y}}^k+\underline{{}^{n}\,f_k^{(d)}\,\mathbf{V}_{\bar{z}}^k\,\alpha_k}\right)\right) $$

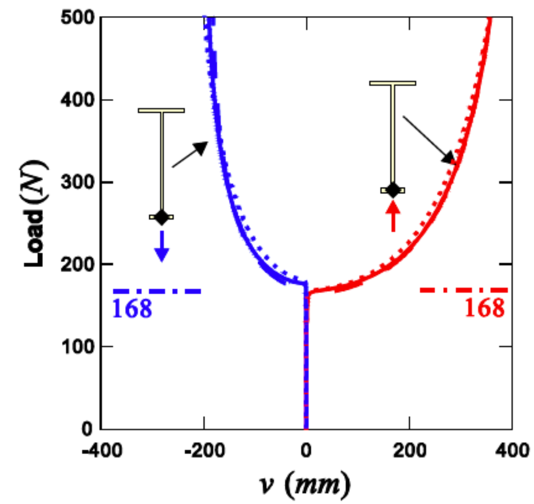
❖ Previous researches using Benscoter type warping enrichment methods



Discontinuous cross-section[1]



Functionally graded material[2]



Buckling analysis[3]

[1] Yoon, K., & Lee, P. S. (2014). Modeling the warping displacements for discontinuously varying arbitrary cross-section beams. Computers & Structures, 131, 56-69.
[2] Yoon, K., & Lee, P. S. (2014). Nonlinear performance of continuum mechanics based beam elements focusing on large twisting behaviors. Computer Methods in Applied Mechanics and Engineering, 281, 106-130.
[3] Yoon, K., Lee, P. S., & Kim, D. N. (2015). Geometrically nonlinear finite element analysis of functionally graded 3D beams considering warping effects. Composite Structures, 132, 1231-1247.

2. Classical Saint-Venant torsion theory

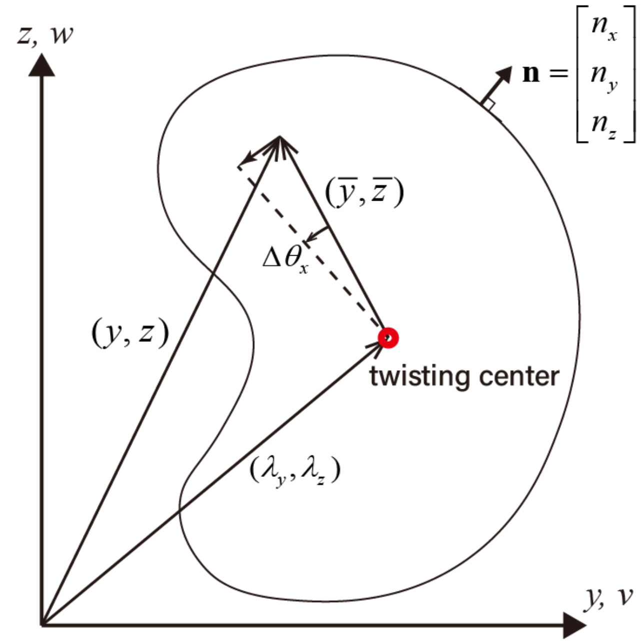
Classical Saint-Venant torsion theory

❖ Kinematics (Rigid body rotation)

$$u = -\left(y - \lambda_y\right)\alpha$$

$$v = \left(x - \lambda_x\right)\alpha$$

$$w = f(x, y)\frac{d\alpha}{dz}$$



❖ Governing equation

$$G\frac{\partial^2 w}{\partial x^2} + G\frac{\partial^2 w}{\partial y^2} + E\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{in } \Omega$$

❖ Boundary condition

$$G\frac{\partial w}{\partial x}n_x + G\frac{\partial w}{\partial y}n_y + E\frac{\partial w}{\partial z}n_z = -G\frac{\partial v}{\partial z}n_y - G\frac{\partial u}{\partial z}n_x \quad \text{on } \partial\Omega$$

- Rigid body rotation (from twisting center)
- Small rotation & strain
- Pure twisting condition
- Static condition with no body force

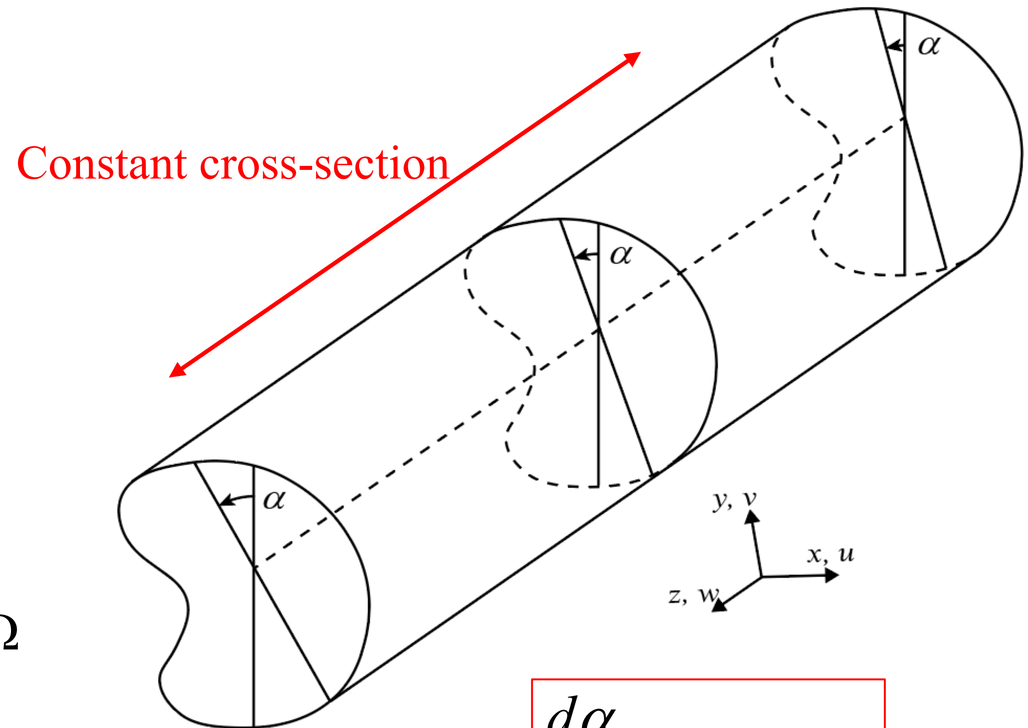
Classical Saint-Venant torsion theory

❖ Kinematics (Rigid body rotation)

$$u = -\left(y - \lambda_y\right)\alpha$$

$$v = \left(x - \lambda_x\right)\alpha$$

$$w = f(x, y)\cancelto{\text{constant}}{\frac{d\alpha}{dz}}$$



❖ Governing equation

$$G\frac{\partial^2 w}{\partial x^2} + G\frac{\partial^2 w}{\partial y^2} + E\cancelto{0}{\frac{\partial^2 w}{\partial z^2}} = 0 \qquad \text{in } \Omega$$

$$\boxed{\frac{d\alpha}{dz} = \text{constant}}$$

❖ Boundary condition

$$G\frac{\partial w}{\partial x}n_x + G\frac{\partial w}{\partial y}n_y + E\cancelto{0}{\frac{\partial w}{\partial z}}n_z = -G\frac{\partial v}{\partial z}n_y - G\frac{\partial u}{\partial z}n_x \qquad \text{on } \partial\Omega$$

- Rigid body rotation (from twisting center)
- Small rotation & strain
- Pure twisting condition
- Static condition with no body force

Discretized Classical Saint-Venant torsion theory

❖ Governing equation & Boundary condition

$$\vec{\nabla}^2(Gf)=0 \quad \text{in} \quad \Omega \qquad\qquad \vec{\nabla}(Gf)\mathbf{n}=G\left(y-\lambda_y\right)n_x-G\left(x-\lambda_x\right)n_y \quad \text{on} \ \partial\Omega$$

$$\mathbf{K}_w\mathbf{F}+\mathbf{N}_x\lambda_x-\mathbf{N}_y\lambda_y=\mathbf{B}$$

Weak form of Saint-Venant's torsion theory

$$\mathbf{H}_x\mathbf{F}=0, \ \mathbf{H}_y\mathbf{F}=0, \ \mathbf{H}_z\mathbf{F}=0$$

Orthogonality condition

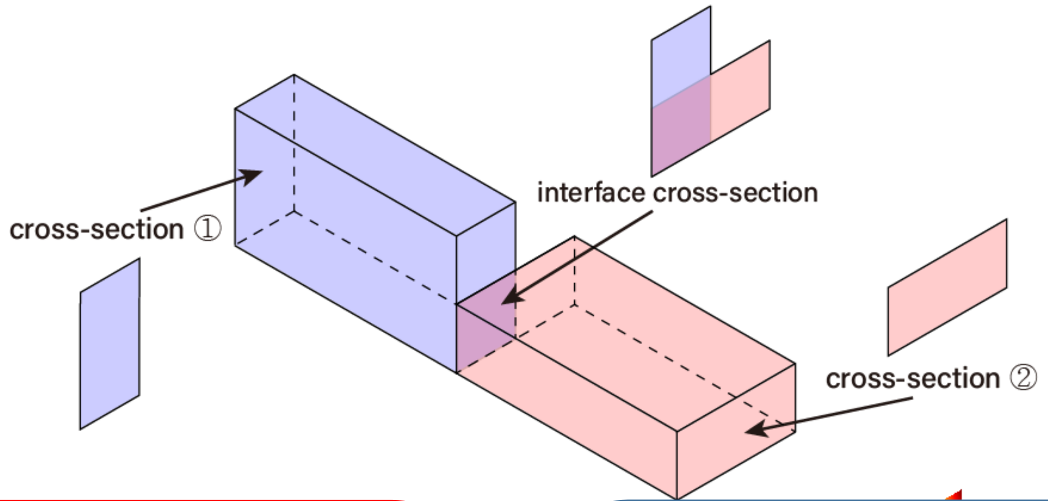
❖ Linearized matrix form of classical Saint-Venant's torsion theory

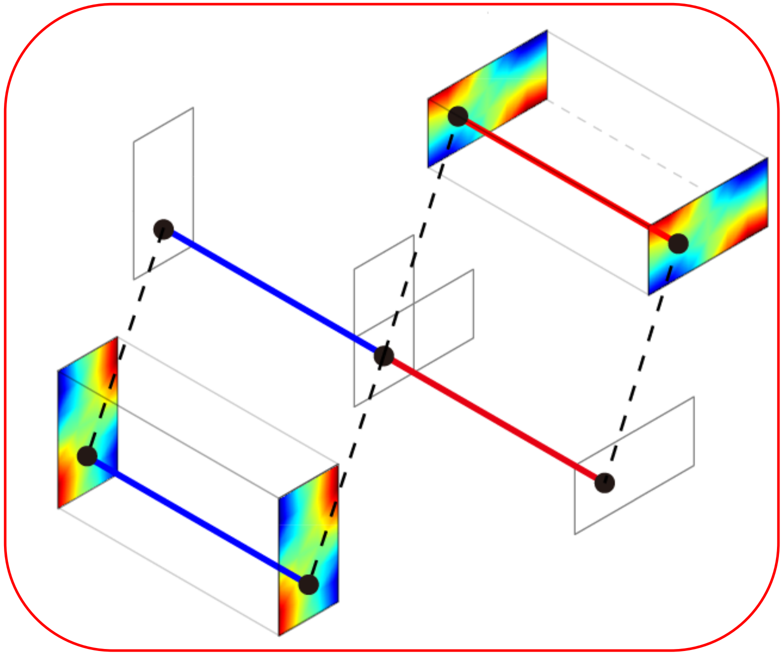
$$\begin{bmatrix} \mathbf{K}_w & \mathbf{N}_x & \mathbf{N}_y \\ \mathbf{H}_x & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_y & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_z & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{F} \\ \lambda_x \\ \lambda_y \end{bmatrix}=\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

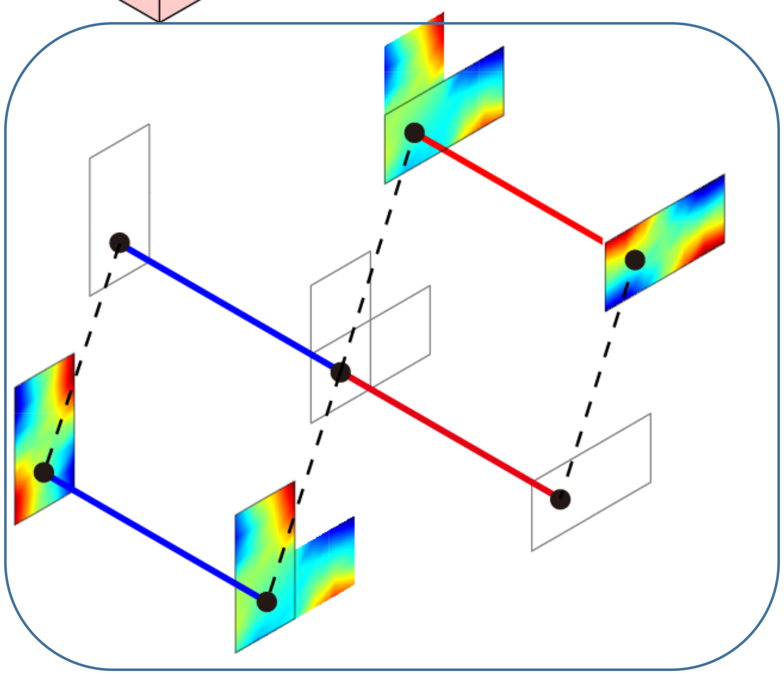
3. Extended Saint-Venant torsion theory

Research motivation

❖ Interface warping function for discontinuity

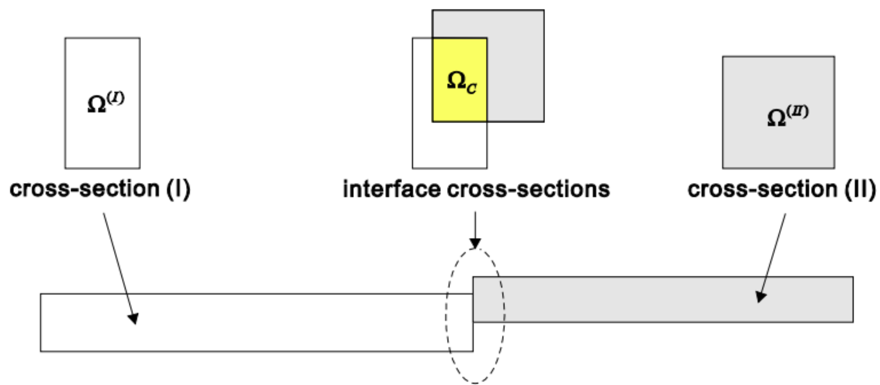


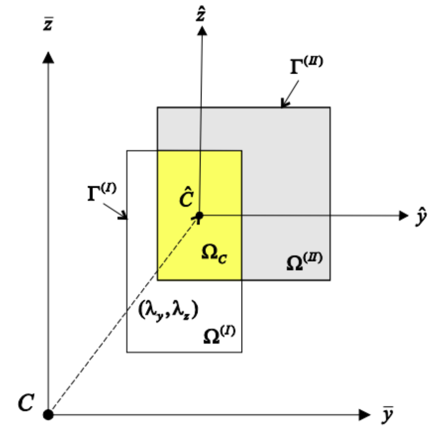


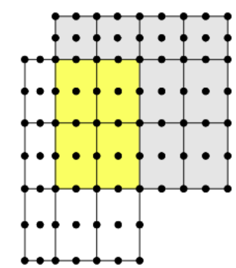


Research motivation

❖ Interface warping function using Lagrange Multiplier Method







$$\begin{bmatrix} G_1\mathbf{K}^{(I)} & G_1\mathbf{N}_{\bar{y}}^{(I)} & -G_1\mathbf{N}_{\bar{z}}^{(I)} & \mathbf{L}^{(I)} \\ G_1\mathbf{H}_{\bar{y}}^{(I)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_1\mathbf{H}_{\bar{z}}^{(I)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{(I)} \\ \lambda_{\bar{z}} \\ \lambda_{\bar{y}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{B}^{(I)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}$$

Cross-section (I)

$$\begin{bmatrix} G_2\mathbf{K}^{(II)} & G_2\mathbf{N}_{\bar{y}}^{(II)} & -G_2\mathbf{N}_{\bar{z}}^{(II)} & -\mathbf{L}^{(II)} \\ G_2\mathbf{H}_{\bar{y}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_2\mathbf{H}_{\bar{z}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{(II)} \\ \lambda_{\bar{z}} \\ \lambda_{\bar{y}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} G_2\mathbf{B}^{(II)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}$$

Cross-section (II)

$$\begin{bmatrix} G_1\mathbf{K}^{(I)} & \mathbf{0} & G_1\mathbf{N}_{\bar{y}}^{(I)} & -G_1\mathbf{N}_{\bar{z}}^{(I)} & \mathbf{L}^{(I)} \\ \mathbf{0} & G_2\mathbf{K}^{(II)} & G_2\mathbf{N}_{\bar{y}}^{(II)} & -G_2\mathbf{N}_{\bar{z}}^{(II)} & -\mathbf{L}^{(II)} \\ G_1\mathbf{H}_{\bar{y}}^{(I)} & G_2\mathbf{H}_{\bar{y}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_1\mathbf{H}_{\bar{z}}^{(I)} & G_2\mathbf{H}_{\bar{z}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{(I)T} & -\mathbf{L}^{(II)T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{(I)} \\ \mathbf{F}^{(II)} \\ \lambda_{\bar{z}} \\ \lambda_{\bar{y}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{B}^{(I)} \\ G_2\mathbf{B}^{(II)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}$$

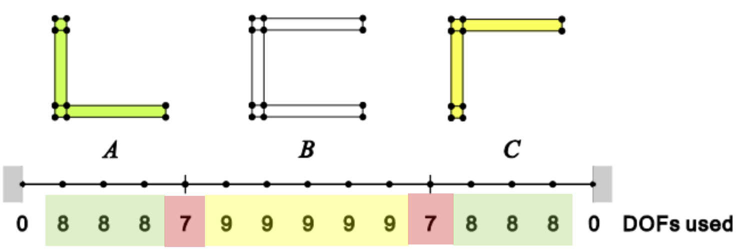
Combination of two equation using Lagrange multipliers

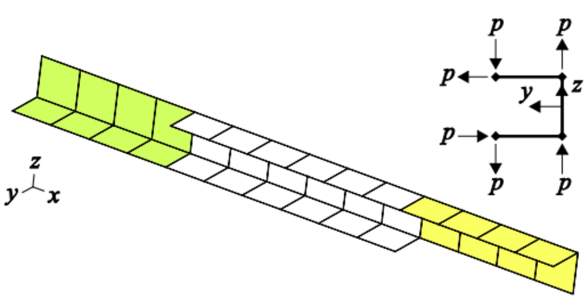
Yoon, K., & Lee, P. S. (2014). Modeling the warping displacements for discontinuously varying arbitrary cross-section beams. Computers & Structures, 131, 56-69.

Research motivation

❖ Interface warping function using Lagrange Multiplier Method

- **7~9 DOFs** per node, depending on boundary condition
- **Modified equations are required** to solve partially constrained interface
- Hard to model **discontinuously varying composite material**





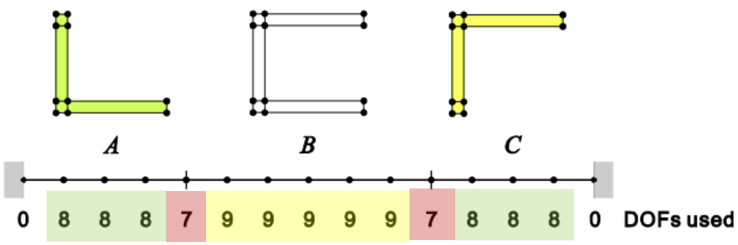
$$\begin{bmatrix} G_1\mathbf{K}^{(I)} & \mathbf{0} & G_1\mathbf{N}_{\bar{y}}^{(I)} & -G_1\mathbf{N}_{\bar{z}}^{(I)} & \mathbf{L}^{(I)} \\ \mathbf{0} & G_2\mathbf{K}^{(II)} & G_2\mathbf{N}_{\bar{y}}^{(II)} & -G_2\mathbf{N}_{\bar{z}}^{(II)} & -\mathbf{L}^{(II)} \\ G_1\mathbf{H}_{\bar{y}}^{(I)} & G_2\mathbf{H}_{\bar{y}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_1\mathbf{H}_{\bar{z}}^{(I)} & G_2\mathbf{H}_{\bar{z}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{(I)T} & -\mathbf{L}^{(II)T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{(I)} \\ \mathbf{F}^{(II)} \\ \lambda_{\bar{z}} \\ \lambda_{\bar{y}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{B}^{(I)} \\ G_2\mathbf{B}^{(II)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}^{(I)} & \mathbf{0} & \mathbf{N}_{\bar{y}}^{(I)} & -\mathbf{N}_{\bar{z}}^{(I)} & \mathbf{L}^{(I)} \\ \mathbf{0} & \mathbf{K}^{(II)} & \mathbf{N}_{\bar{y}}^{(II)} & -\mathbf{N}_{\bar{z}}^{(II)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\bar{y}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\bar{z}}^{(II)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^{(I)T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{(I)} \\ \mathbf{F}^{(II)} \\ \lambda_{\bar{z}} \\ \lambda_{\bar{y}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{(I)} \\ \mathbf{B}^{(II)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}$$

Yoon, K., & Lee, P. S. (2014). Modeling the warping displacements for discontinuously varying arbitrary cross-section beams. Computers & Structures, 131, 56-69.

Research purpose

- Objectives
  - ✓ Method of obtaining **exact warping function** with discontinuous cross-section
  - ✓ General method: **independent from boundary condition**
- Key idea
  - ✓ **Extending classical St.Venant torsion theory to the longitudinal direction**
- Which is expected?
  - ✓ **More accurate result** at discontinuous cross-section
  - ✓ **Cost efficient** with 7 DOF/node
  - ✓ **Generosity** with arbitrary boundary condition



Yoon, K., & Lee, P. S. (2014). Modeling the warping displacements for discontinuously varying arbitrary cross-section beams. Computers & Structures, 131, 56-69.

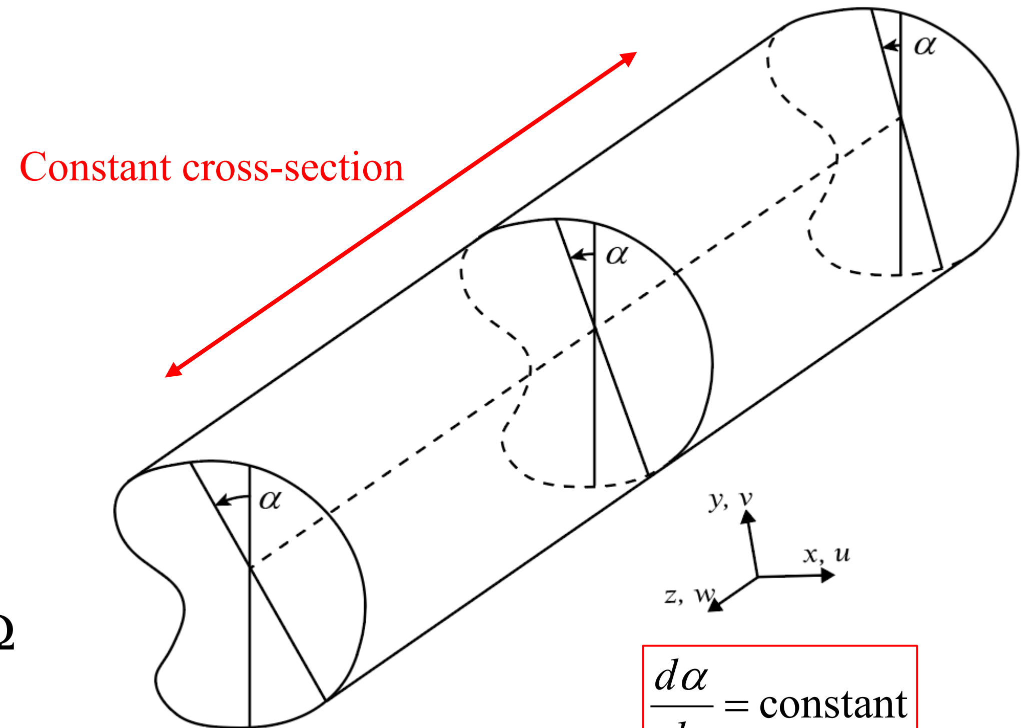
Extended Saint-Venant torsion theory

❖ Kinematics (Rigid body rotation)

$$u = -\left(y - \lambda_y\right)\alpha$$

$$v = \left(x - \lambda_x\right)\alpha$$

$$w = f(x, y)\cancelto{\text{constant}}{\frac{d\alpha}{dz}}$$



$$\boxed{\frac{d\alpha}{dz} = \text{constant}}$$

❖ Governing equation

$$G\frac{\partial^2 w}{\partial x^2} + G\frac{\partial^2 w}{\partial y^2} + \cancelto{0}{E\frac{\partial^2 w}{\partial z^2}} = 0 \qquad \text{in } \Omega$$

❖ Boundary condition

$$G\frac{\partial w}{\partial x}n_x + G\frac{\partial w}{\partial y}n_y + \cancelto{0}{E\frac{\partial w}{\partial z}n_z} = -G\frac{\partial v}{\partial z}n_y - G\frac{\partial u}{\partial z}n_x \qquad \text{on } \partial\Omega$$

- Rigid body rotation (from twisting center)
- Small rotation & strain
- Pure twisting condition
- Static condition with no body force

Extended Saint-Venant torsion theory

- Coordinate transformation

$$\tilde{x} = x,\ \tilde{y} = y,\ \text{and}\ \tilde{z} = \sqrt{\frac{G}{E}}\, z \qquad \frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}, \frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y},\ \text{and}\ \frac{\partial}{\partial \tilde{z}} = \sqrt{\frac{E}{G}} \frac{\partial}{\partial z}$$

- New governing equation and boundary condition

$$G\left(\frac{\partial^2 w}{\partial \tilde{x}^2} + \frac{\partial^2 w}{\partial \tilde{y}^2} + \frac{\partial^2 w}{\partial \tilde{z}^2}\right) = 0 \quad \text{in}\ \Omega$$

$$G\left(\frac{\partial w}{\partial \tilde{x}}\tilde{n}_x + \frac{\partial w}{\partial \tilde{y}}\tilde{n}_y + \frac{\partial w}{\partial \tilde{z}}\tilde{n}_z\right) = -G\sqrt{\frac{G}{E}}\left(\frac{\partial u}{\partial \tilde{z}}\tilde{n}_x + \frac{\partial v}{\partial \tilde{z}}\tilde{n}_y + \left(\frac{E}{G} - \sqrt{\frac{E}{G}}\right)\frac{\partial w}{\partial \tilde{z}}\tilde{n}_z\right) \quad \text{on}\ \partial\Omega$$

- Variational formulation & Principle of Virtual Work

$$0 = \int \vec{\vec{\nabla}}^2 (Gw)\delta f\, d\tilde{\Omega} = \int \vec{\vec{\nabla}}\left(\vec{\vec{\nabla}}(Gw)\right)\delta f\, d\tilde{\Omega}$$

$$= \int \vec{\vec{\nabla}}\left(\vec{\vec{\nabla}}(Gw)\,\delta f\right) d\tilde{\Omega} - \int \vec{\vec{\nabla}}(Gw)\vec{\vec{\nabla}}(\delta f)\, d\tilde{\Omega} \qquad \text{(Integration by parts)}$$

$$= \int \vec{\vec{\nabla}}(Gw)\,\delta f\,\mathbf{n}\, d\partial\tilde{\Omega} - \int \vec{\vec{\nabla}}(Gw)\vec{\vec{\nabla}}(\delta f)\, d\tilde{\Omega} \qquad \text{(Divergence theorem)}$$

Discretization of the extended Saint-Venant torsion theory

$$\mathbf{K}_w\mathbf{U} - \mathbf{N}_z\tilde{\boldsymbol{\Lambda}}_y + \mathbf{N}_y\tilde{\boldsymbol{\Lambda}}_z + \mathbf{B}_c\mathbf{A} = \mathbf{0}$$ Extended Saint-Venant's torsion theory

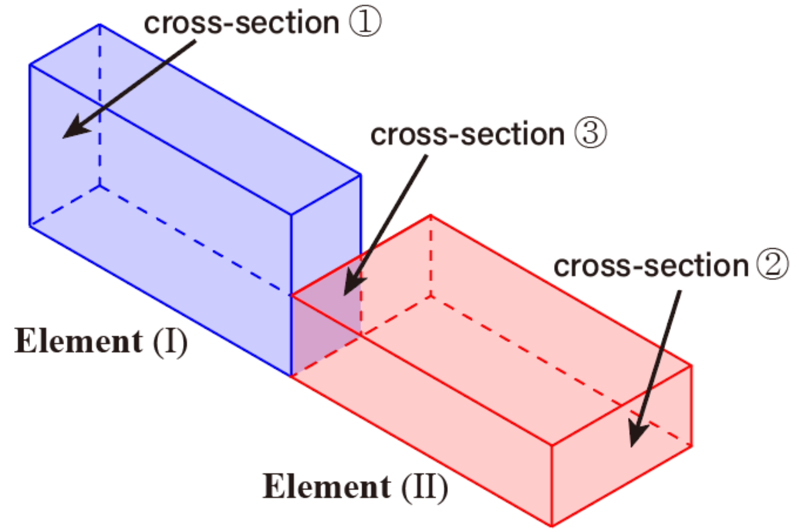
$$\mathbf{Q}_x\mathbf{U} = \mathbf{0}, \quad \mathbf{Q}_y\mathbf{U} = \mathbf{0}, \quad \mathbf{Q}_z\mathbf{U} = \mathbf{0}$$ Orthogonality condition

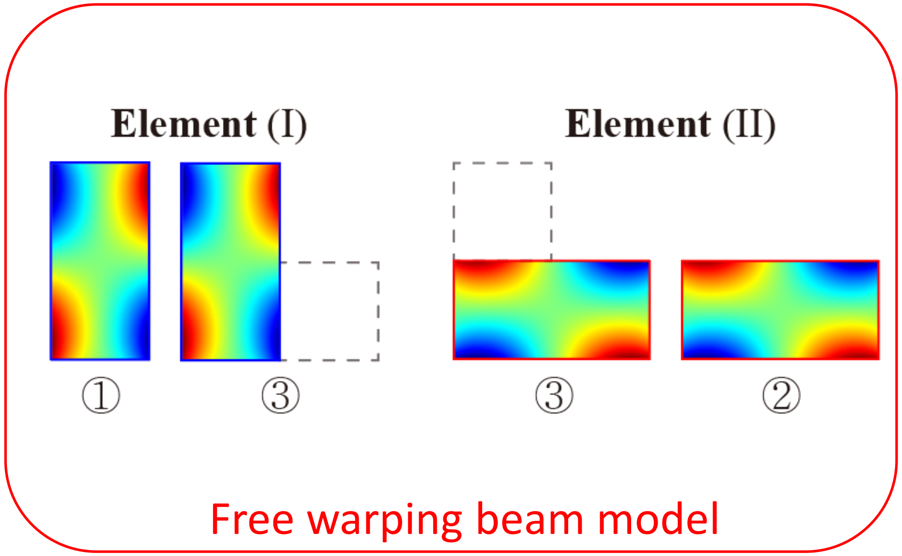
$$\mathbf{R}_w\mathbf{U} - \mathbf{S}_y\tilde{\boldsymbol{\Lambda}}_y - \mathbf{S}_z\tilde{\boldsymbol{\Lambda}}_z + \mathbf{J}_x\mathbf{A} = M_x\mathbf{1}$$ Stress equilibrium condition

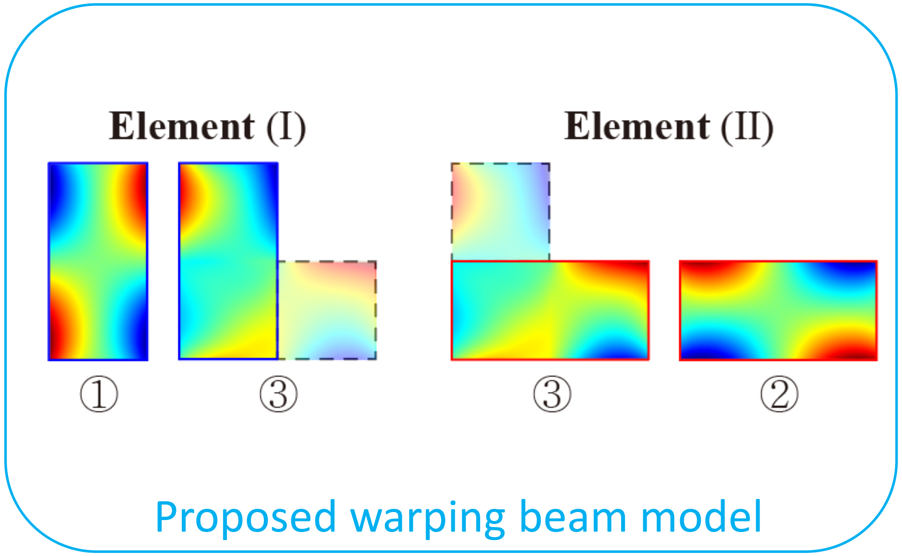
❖ Linearized matrix form of extended Saint-Venant's torsion theory

$$\begin{bmatrix} \mathbf{K}_w & -\mathbf{N}_z & \mathbf{N}_y & \mathbf{B}_c \\ \mathbf{Q}_x & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_y & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_z & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_w & -\mathbf{S}_y & -\mathbf{S}_z & \mathbf{J}_x \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \tilde{\boldsymbol{\Lambda}}_y \\ \tilde{\boldsymbol{\Lambda}}_z \\ \mathbf{A} \end{bmatrix} = M_x \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Numerical example



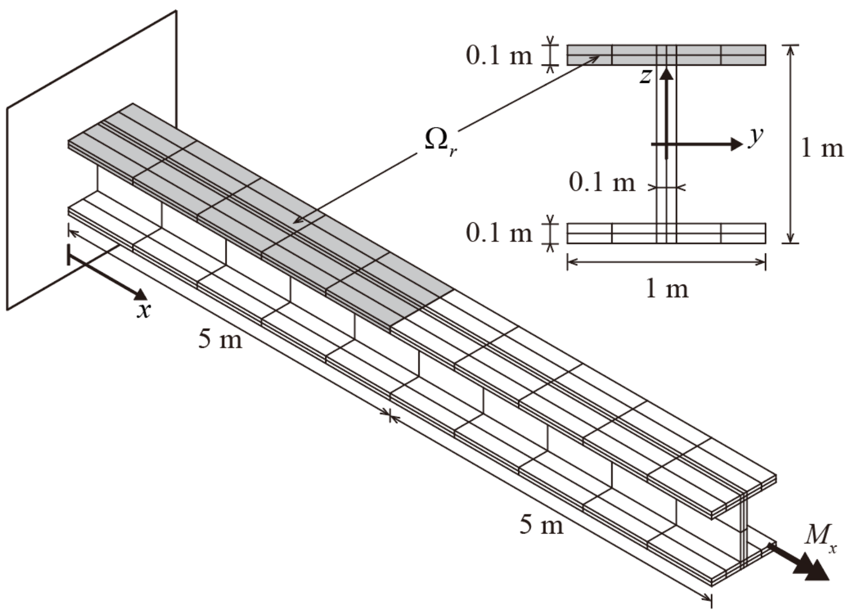




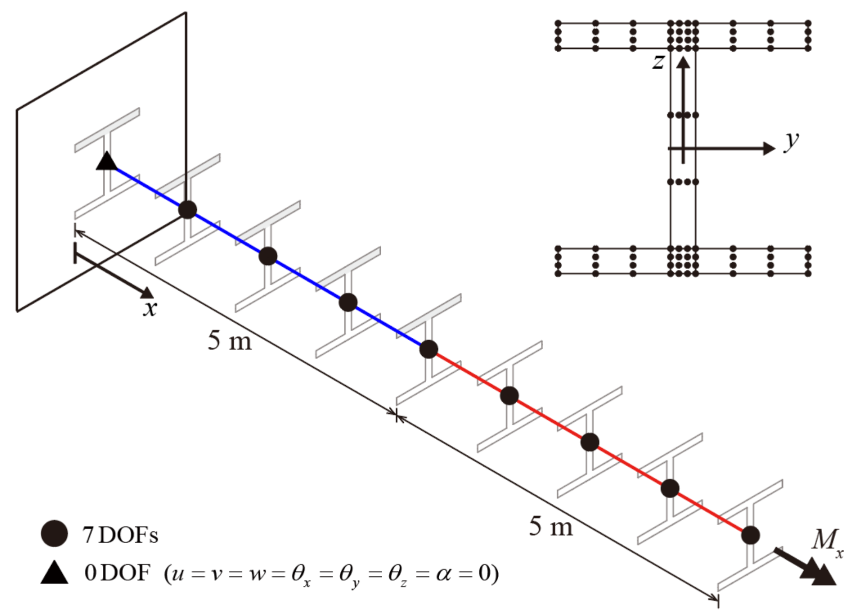
Ex1. Partially reinforced wide flange beam

❖ Problem description

$$\Omega_0 = 2\times 10^{11}[\text{Pa}]$$
$$\Omega_r = 6\times 10^{11}[\text{Pa}]$$
$$\nu = 0.0$$
$$M_x = 1[\text{Nm}]$$



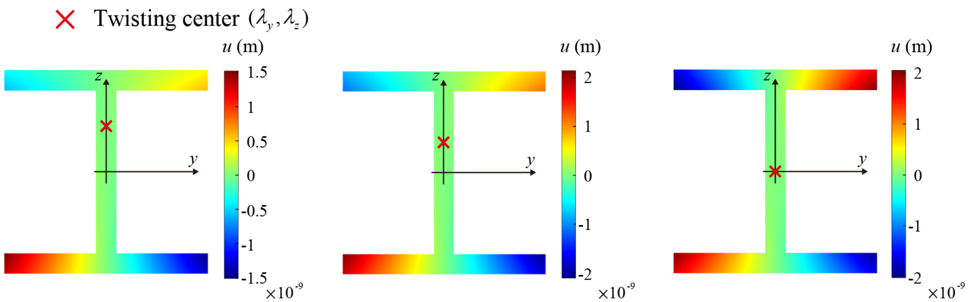
10,935 DOF
(27-node standard solid element)



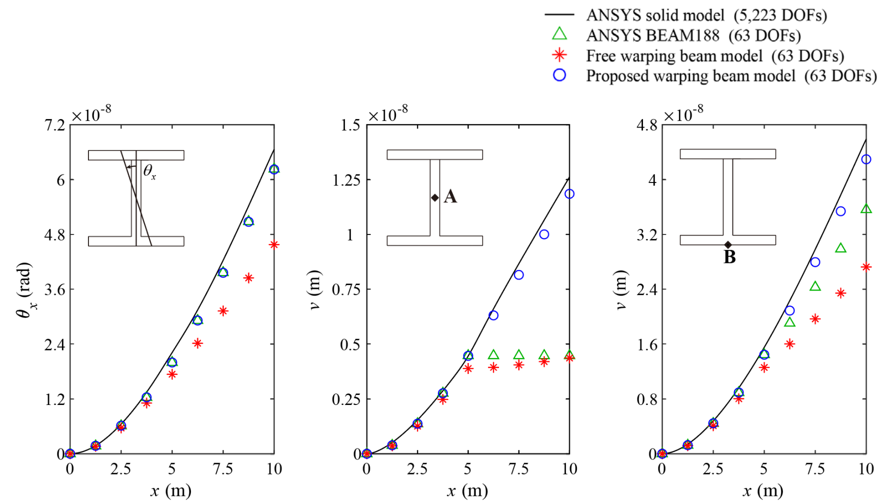
63 DOF
(2-node 7DOF beam element)

Ex1. Partially reinforced wide flange beam

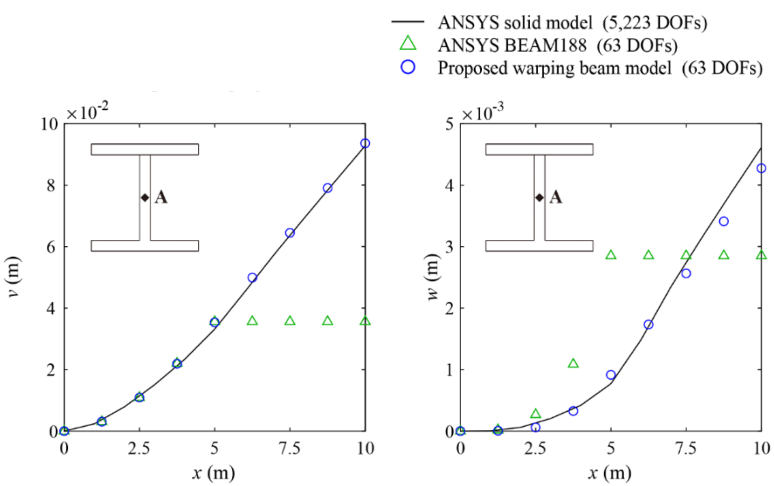
❖ Linear analysis result

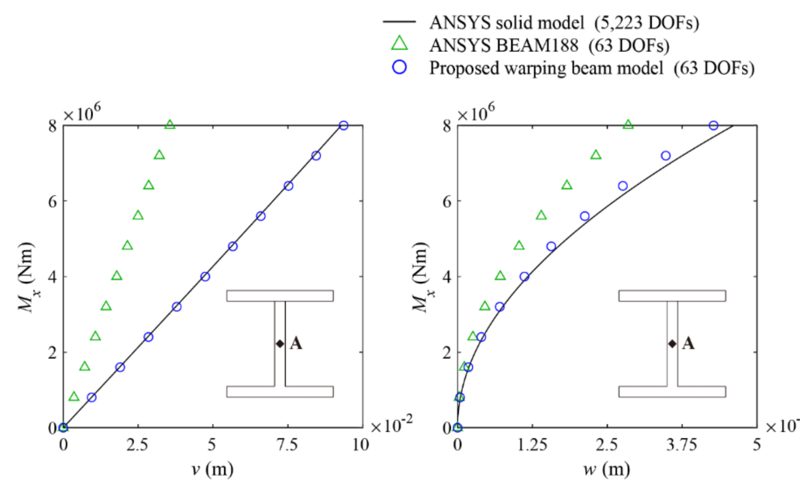


$(\lambda_y, \lambda_z) = (0, 0.2234)$ $(\lambda_y, \lambda_z) = (0, 0.1487)$ $(\lambda_y, \lambda_z) = (0, 0)$



❖ Nonlinear analysis result



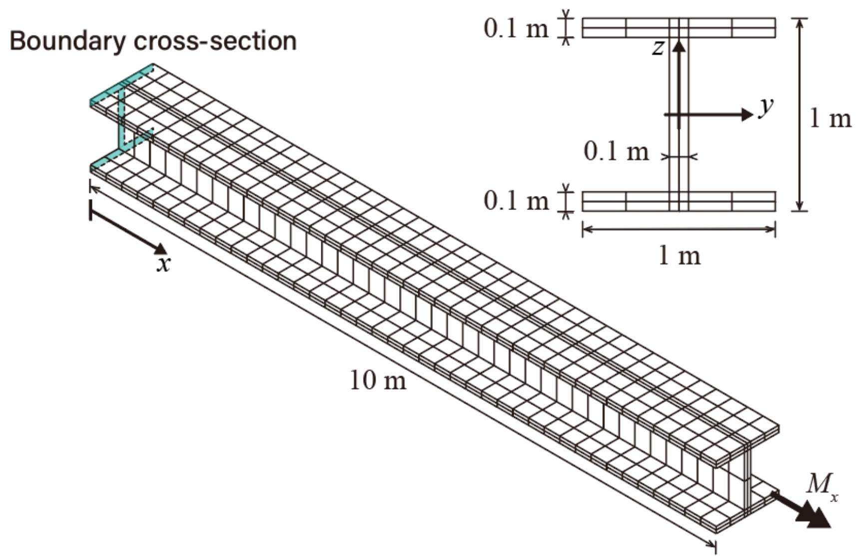


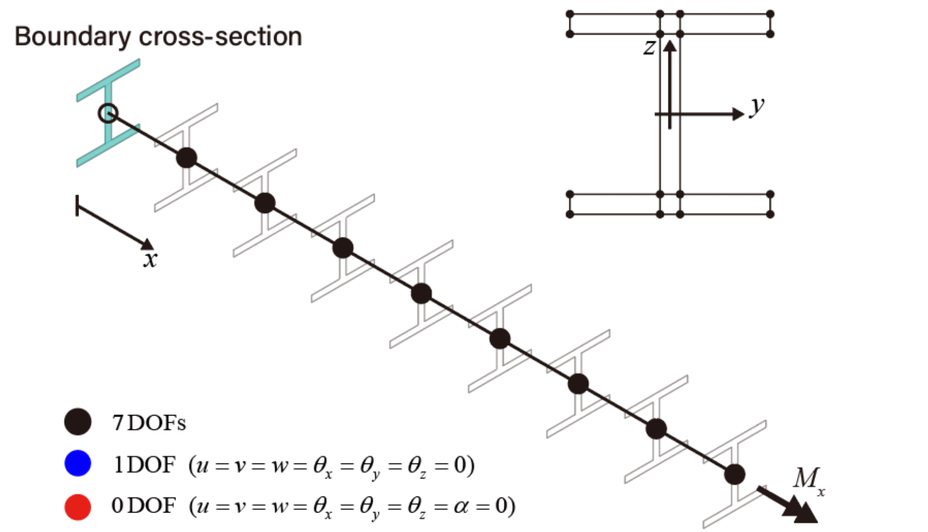
Ex2. Partially constrained warping problem

❖ Problem description

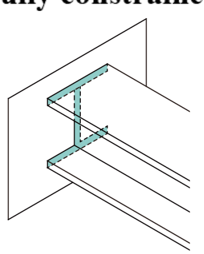
$E = 2\times10^{11}$[Pa]

$\nu = 0.0$

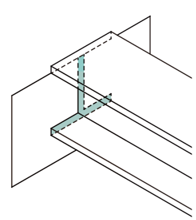




Fully constrained

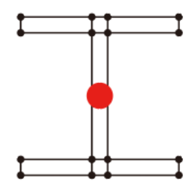


Partially constrained

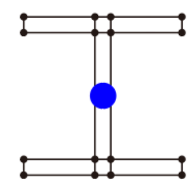


19,803 DOF
(27-node standard solid element)

Fully constrained



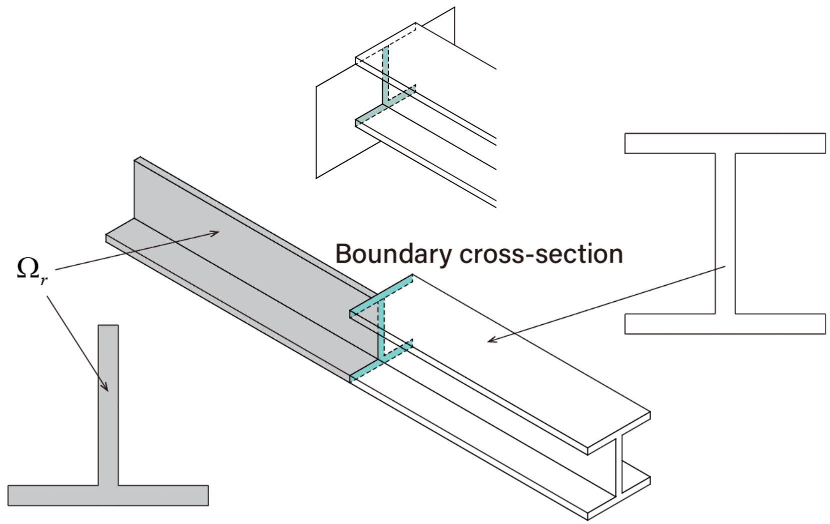
Partially constrained

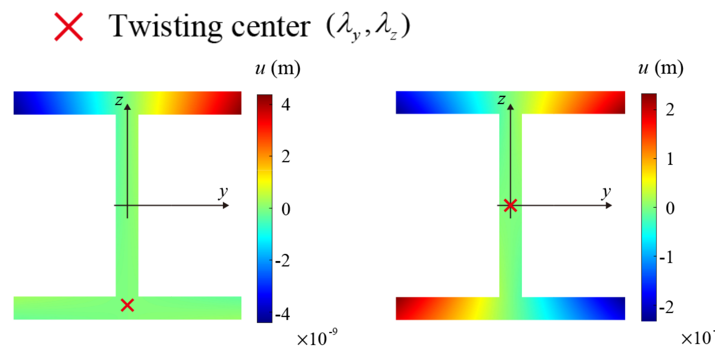


63 DOF
(2-node 7DOF beam element)

Ex2. Partially constrained warping problem

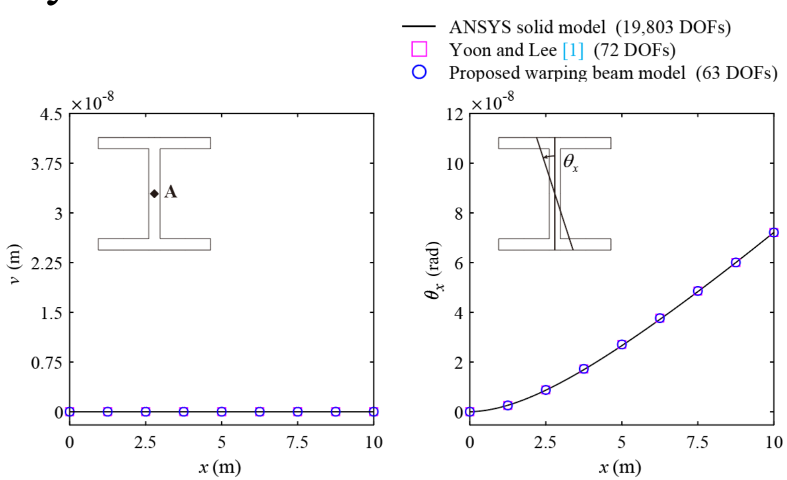
❖ Modeling of partially constrained condition and its warping functions



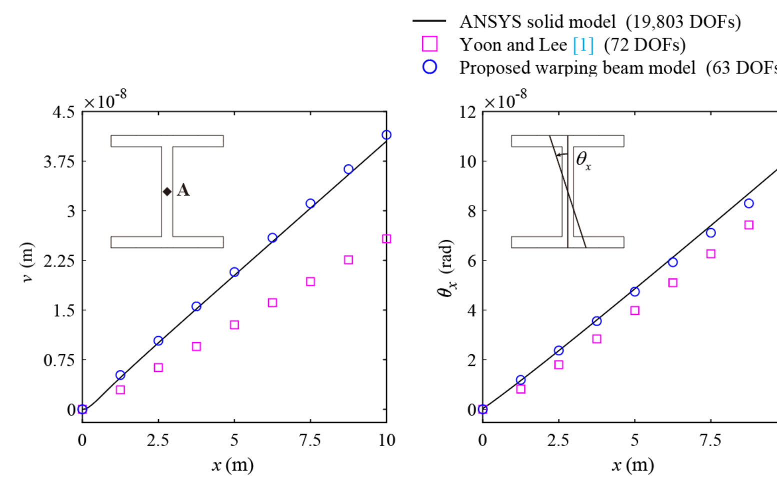


$(\lambda_y, \lambda_z) = (0, -0.4369)$ $(\lambda_y, \lambda_z) = (0, 0)$

❖ Fully constrained condition



❖ Partially constrained condition

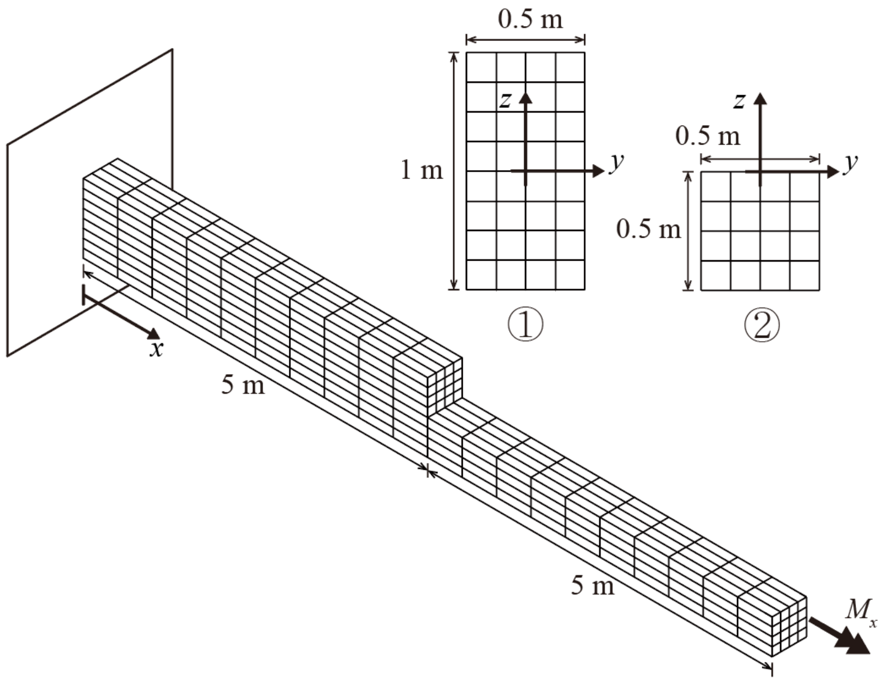


Ex3. Step varying rectangular cross-section beam

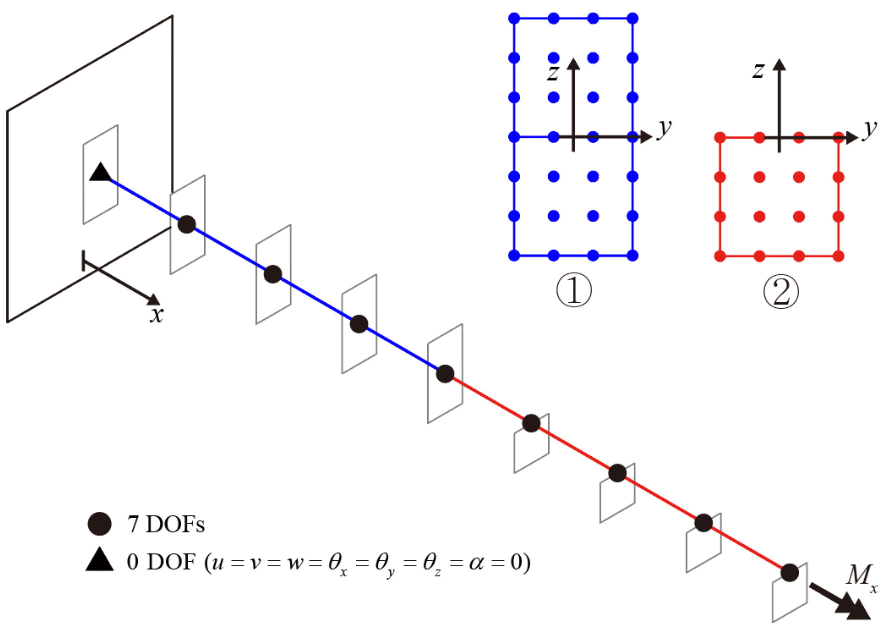
❖ Problem description

$E = 2\times10^{11}$[Pa]

$\nu = 0.0$



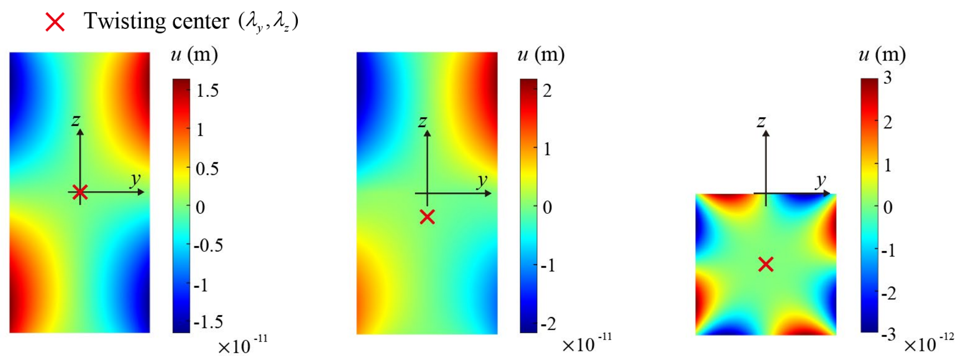
8,055 DOF
(27-node standard solid element)



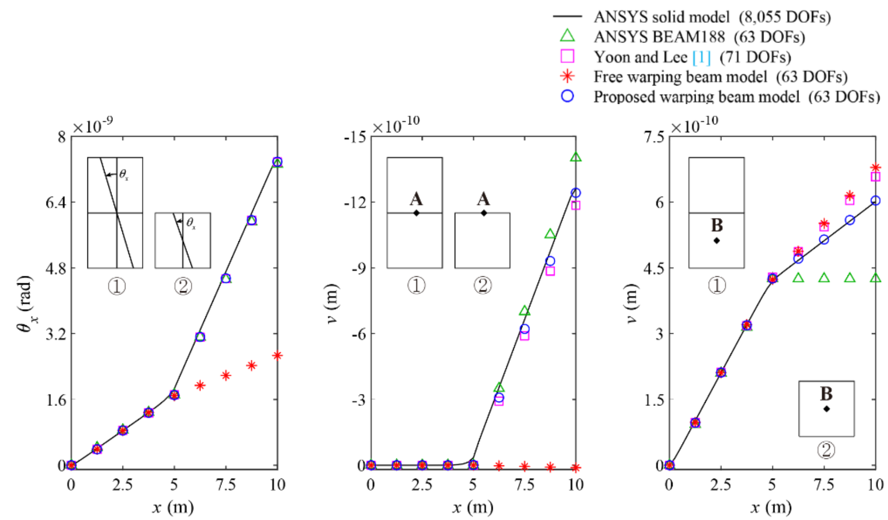
63 DOF
(2-node 7DOF beam element)

Ex3. Step varying rectangular cross-section beam

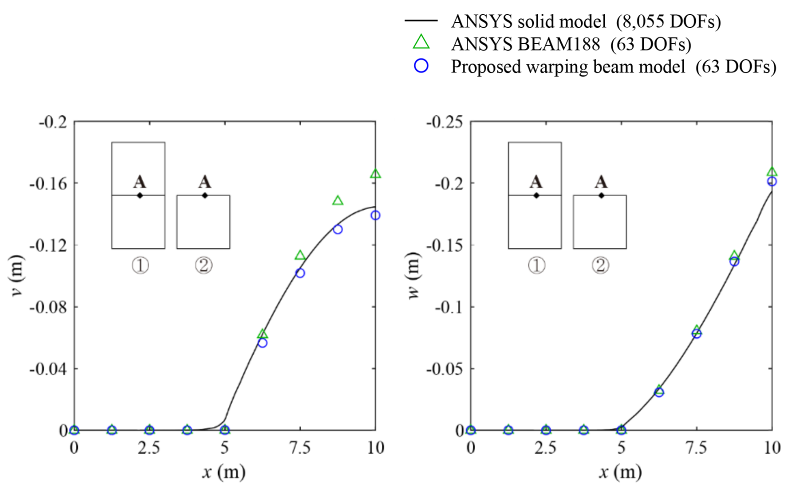
❖ Linear analysis result

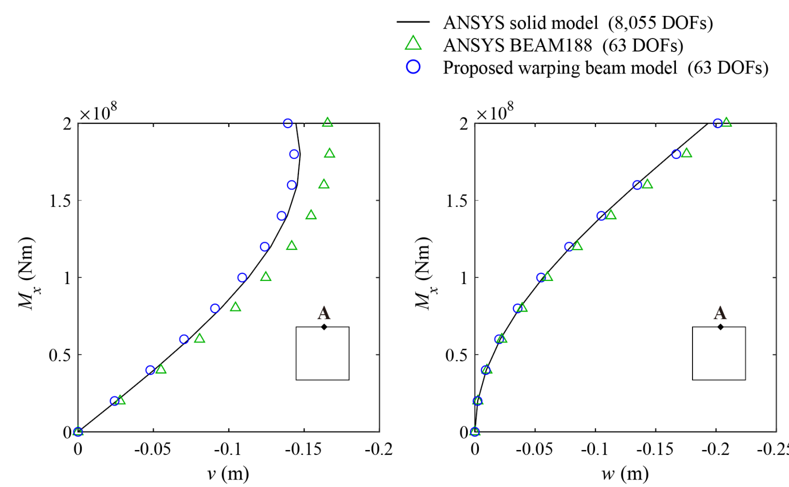


$(\lambda_y, \lambda_z) = (0, 0)$ $(\lambda_y, \lambda_z) = (0, -0.0833)$ $(\lambda_y, \lambda_z) = (0, -0.25)$



❖ Nonlinear analysis result



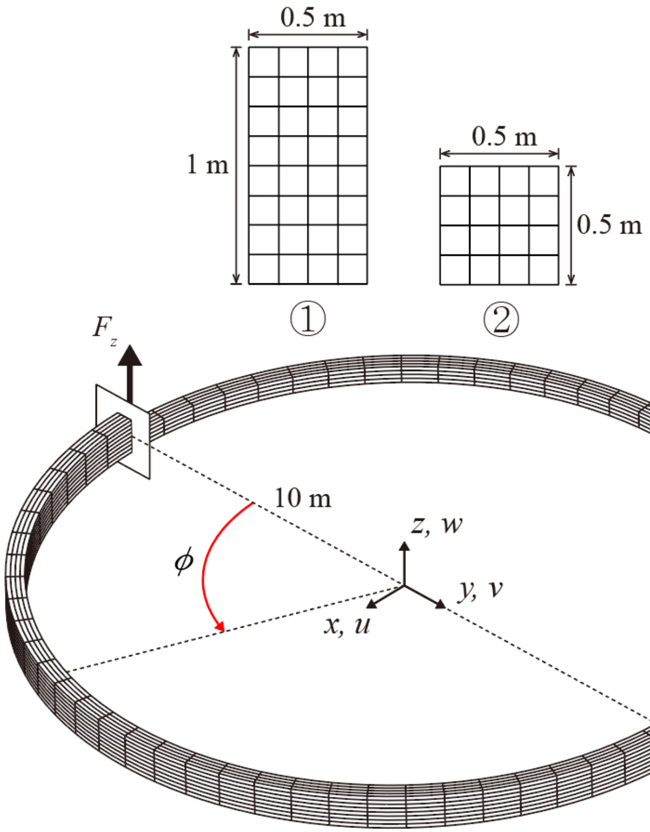


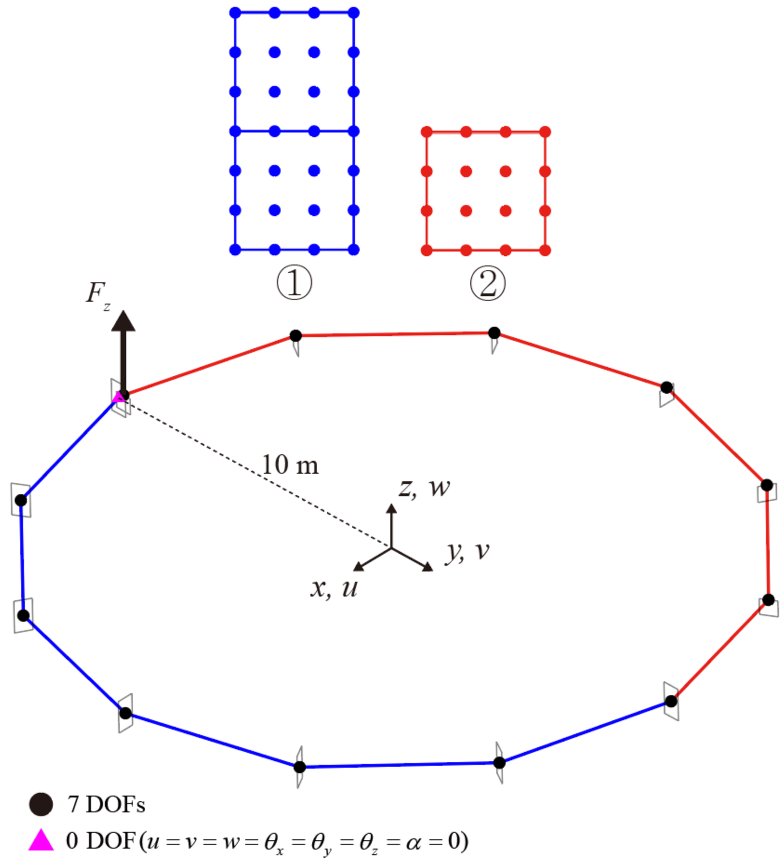
Ex4. Circular beam with step varying rectangular cross-section

❖ Problem description

$E = 2\times10^{11}$[Pa]

$\nu = 0.0$



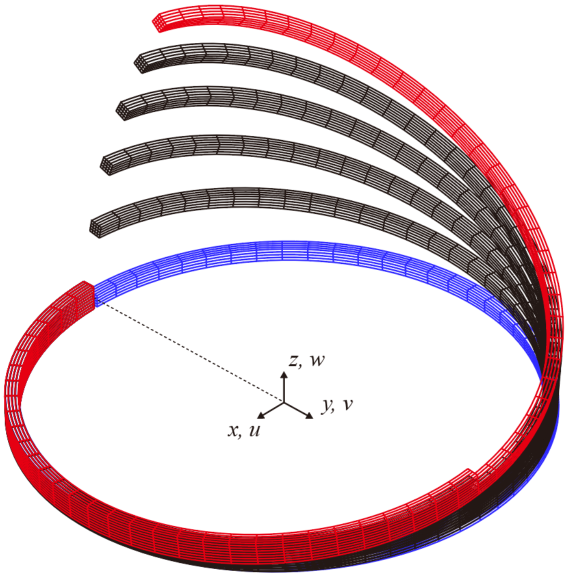


23,943 DOF
(27-node standard solid element)

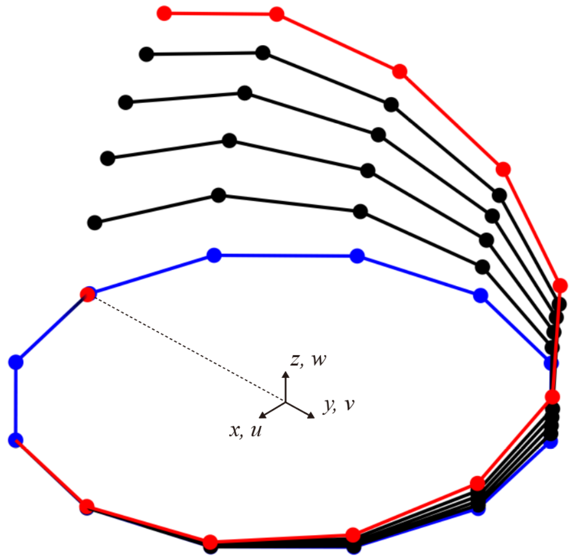
91 DOF
(2-node 7DOF beam element)

Ex4. Circular beam with step varying rectangular cross-section

❖ Deformed configurations

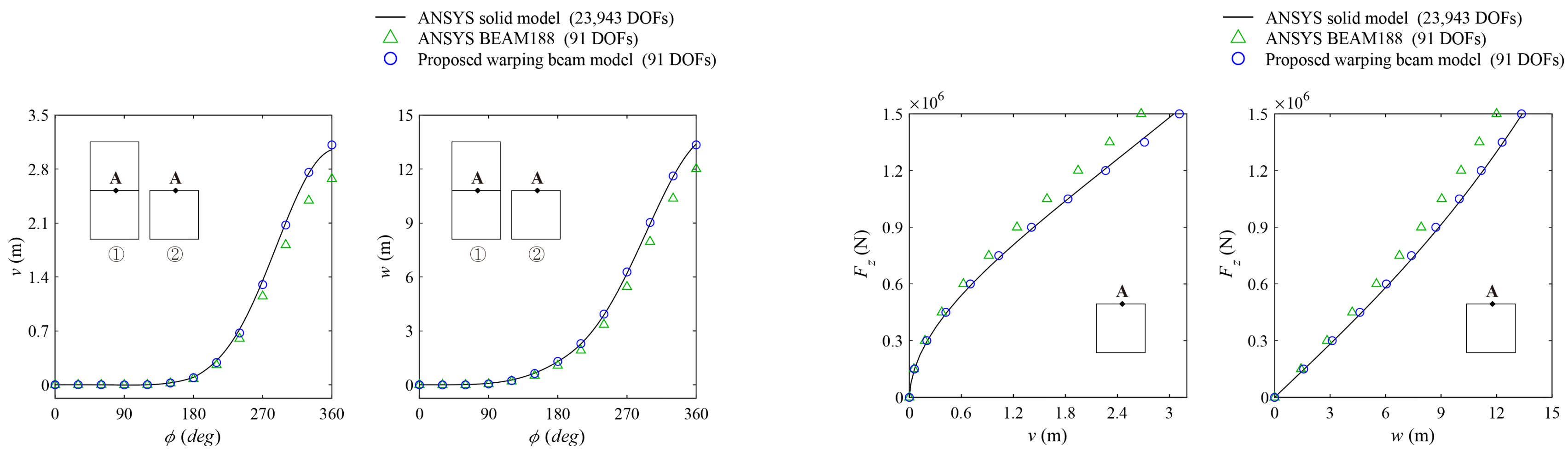


23,943 DOF
(27-node standard solid element)



91 DOF
(2-node 7DOF beam element)

❖ Nonlinear analysis result



4. Topology optimization of beam cross-sections

Structural topology optimization – Minimizing compliance

- Compliance

$$C = \mathbf{U}^T \mathbf{K} \mathbf{U} = \sum_{e=1}^{N} \mathbf{U}_e^T \mathbf{K}_e \mathbf{U}_e$$

- Solid Isotropic Material with Penalization method (SIMP)

$$\widetilde{\mathbf{C}}_e = \left(\rho_e\right)^p \widetilde{\mathbf{C}}_0 \qquad (\rho_e\text{: relative density, } \; p\text{ : penalization factor})$$

- Structural optimization to minimizing compliance

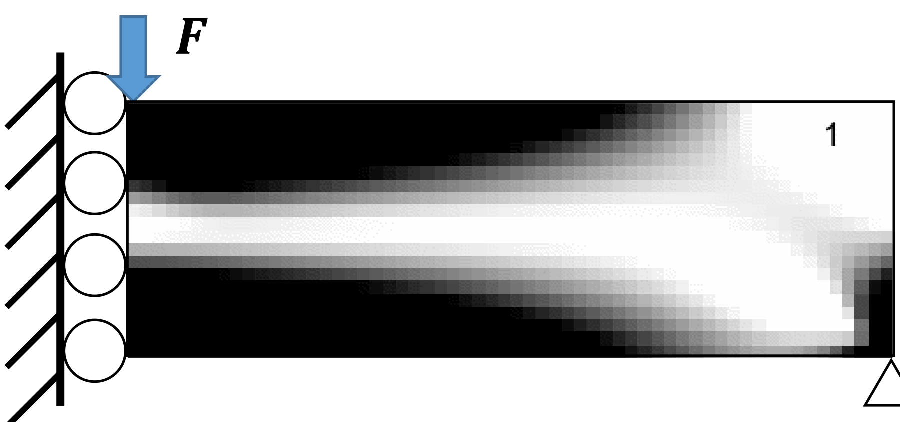
Find: $\boldsymbol{\rho} = [\rho_1, \, \rho_2, \, \cdots, \, \rho_e, \, \cdots, \, \rho_n]^T$

to minimize: $C = \mathbf{U}^T \mathbf{K} \mathbf{U}$

subject to : $g = \sum_{e=1}^{N} \rho_e V_e - V_0 \le 0$

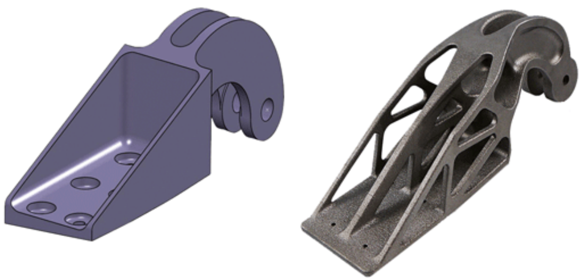
$\mathbf{K}\mathbf{U} = \mathbf{F}$

$0 \le \rho_e \le 0, \quad e = 1, \cdots, N$



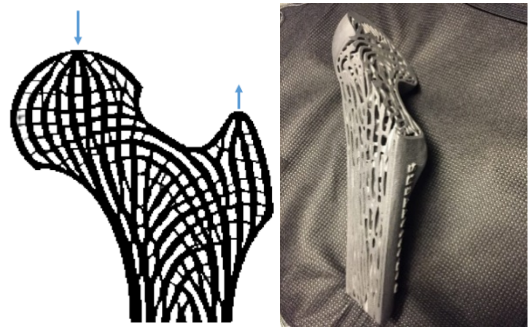
Motivation

➢ Structural optimization using solid elements



Aircraft assembly part

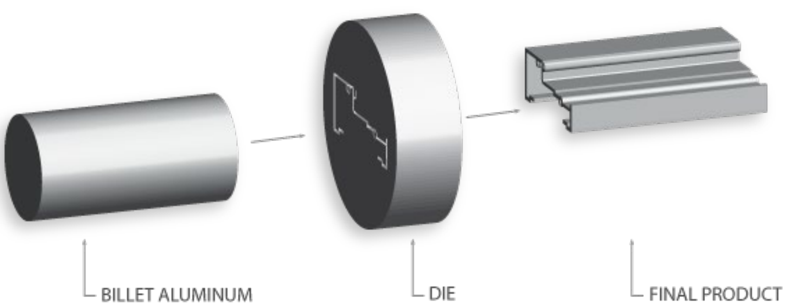
Seat belt bracket



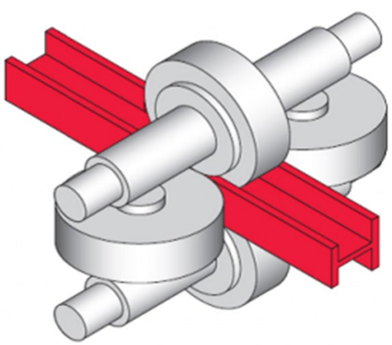
Medical implant

✓ Limited to small parts

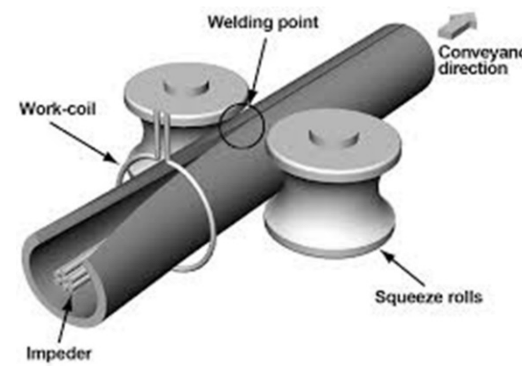
➢ Beam manufacturing methods



Extrusion



Rolling

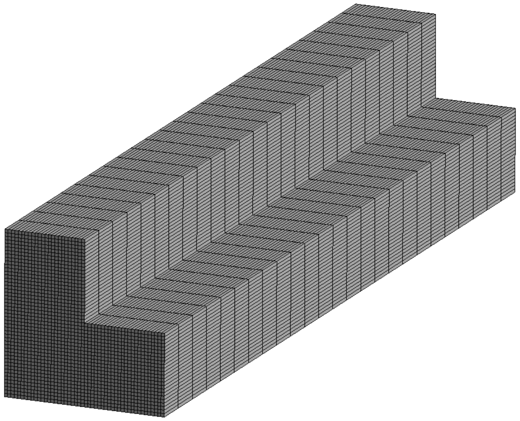


Welding

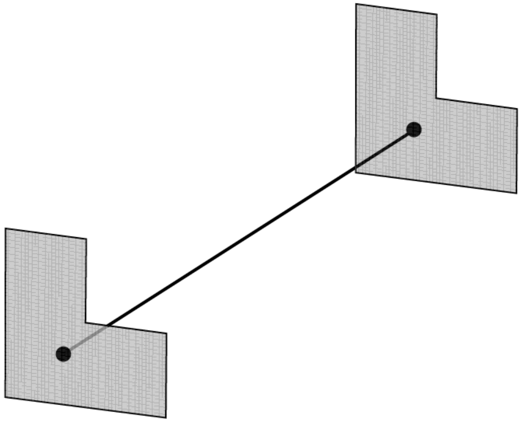
✓ Cross-sections are constant

Motivation

- Finite element modeling of beam structure

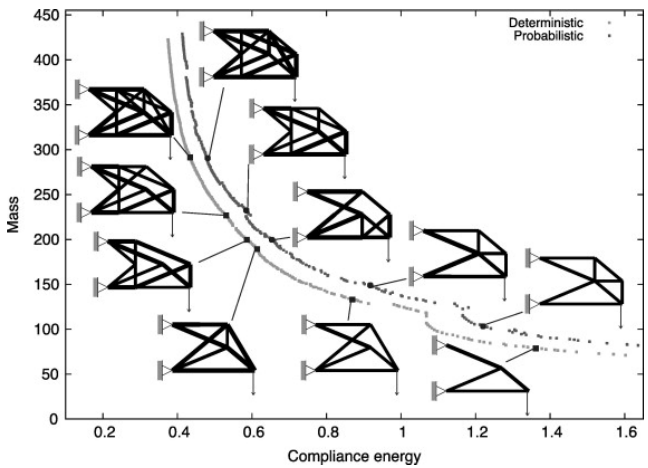


Solid element
(44,928 element)

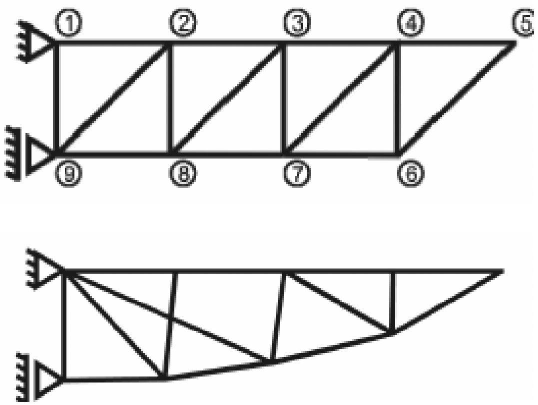


Beam element
(1,728 sub-beam)

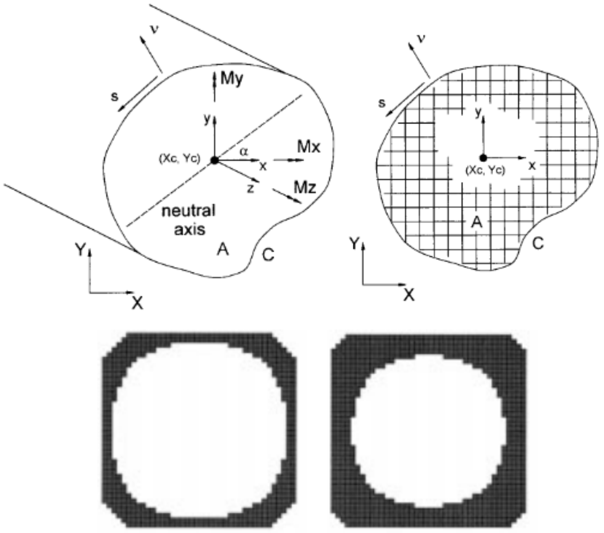
- Beam optimization



Size optimization

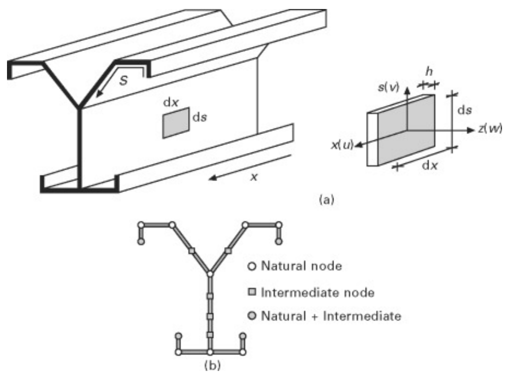


Shape optimization

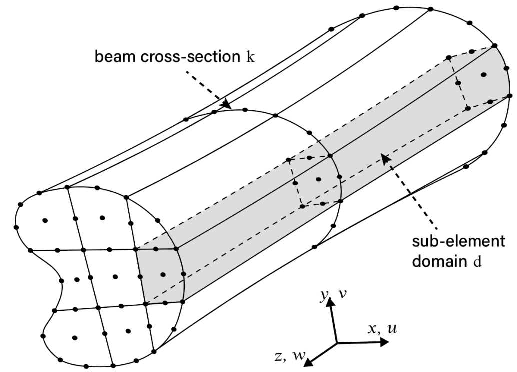


Topology optimization

Related works

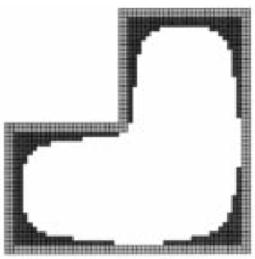


Thin-walled beam composed of nodes

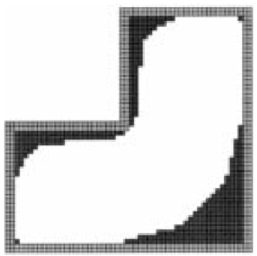


Thick beam composed of sub-beams

- Kim and Kim (2000)
  - Linear combination of torsional stiffness(w_i) & principle bending stiffness (w_j)
  - Combination weights are artificially selected
  - Warping is considered
  - No torsion-bending coupling effect
  - Based on straight constant beam

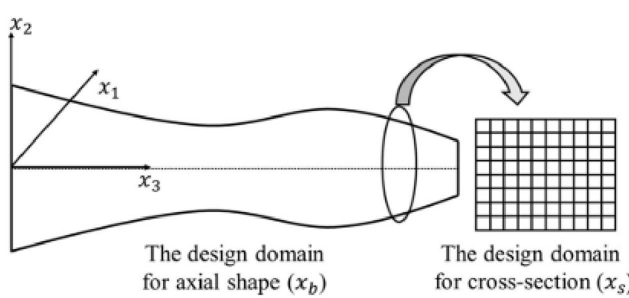


w_i=1.0



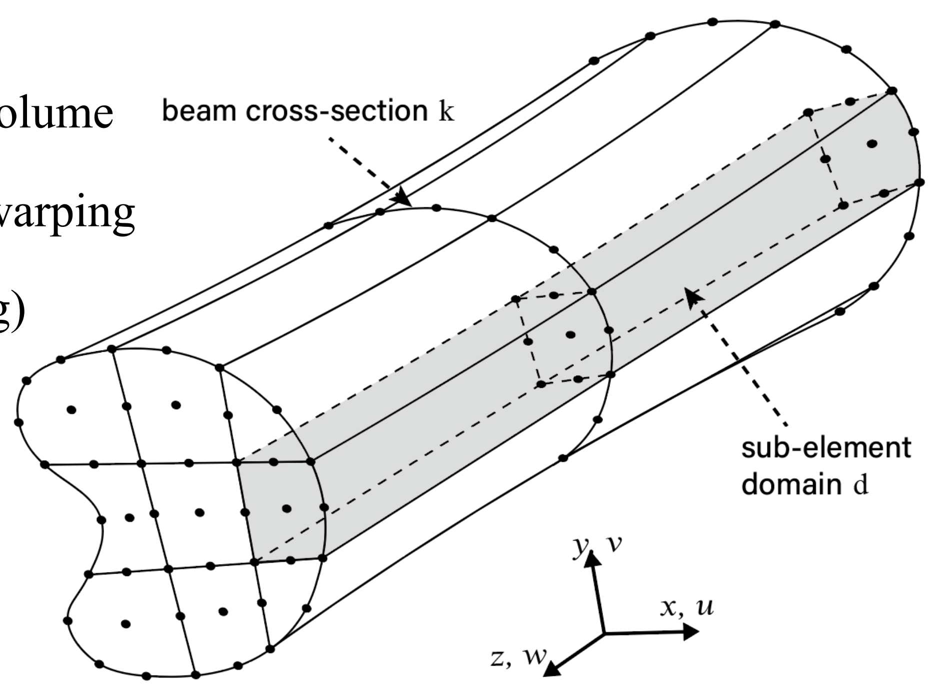
w_i=1.0, w_j=0.02

- Liu *et al*. (2008)
  - Based on solid-like beam model
  - 5 Local deformation modes are considered
  - High computational cost
  - Based on straight beam



Research purpose

- Objectives
  - ✓ Design of **optimized beam cross-sectional shapes** at a given load
  - ✓ Beam topology optimization considering **warping effect**
- Key idea
  - ✓ **Interface warping function** with Benscoter type enrichment
- Which is expected?
  - ✓ **Highest stiffness** at a given mass and volume
  - ✓ **Accurate result** considering interface warping
  - ✓ **Fully coupled effects** (bending-twisting)
  - ✓ **Cost efficient** with 7 DOF/node



Sensitivity derivation

❖ Sensitivity derivation with warping displacement enrichment

➢ Sensitivity with warping function

$$\frac{dC}{d\rho_e^b} = -\mathbf{U}^T \frac{d\mathbf{K}}{d\rho_e^b}\mathbf{U} = -\mathbf{U}^T\left(\frac{\partial\mathbf{K}}{\partial\rho_e^b} + \frac{d\,\mathbf{f}}{d\,\rho_e^b}\frac{\partial\mathbf{K}}{\partial\mathbf{f}}\right)\mathbf{U}$$

➢ Calculation from the continuum mechanics based beam

$$\frac{d\mathbf{K}}{df_j} = \iiint \frac{d\widetilde{\mathbf{B}}^T}{df_j}\widetilde{\mathbf{C}}\widetilde{\mathbf{B}}\left\|\mathbf{J}\right\| dr\,ds\,dt + \iiint \widetilde{\mathbf{B}}^T\widetilde{\mathbf{C}}\frac{d\widetilde{\mathbf{B}}}{df_j}\left\|\mathbf{J}\right\| dr\,ds\,dt$$

$$\frac{d\widetilde{\mathbf{B}}_{nm}}{df_j} = \frac{1}{2}\left[\frac{d}{df_j}\left(\frac{d\mathbf{L}}{dr_k}\right)\mathbf{g}_l + \frac{d}{df_j}\left(\frac{d\mathbf{L}}{dr_l}\right)\mathbf{g}_k\right]\left(\mathbf{t}_n\cdot\mathbf{g}^k\right)\left(\mathbf{t}_m\cdot\mathbf{g}^l\right)$$

➢ Calculation from the interface warping function

$$\begin{bmatrix} \mathbf{K}_w & -\mathbf{N}_z & \mathbf{N}_y & \mathbf{B}_c \\ \mathbf{Q}_x & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_y & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_z & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_w & -\mathbf{S}_y & -\mathbf{S}_z & \mathbf{J}_x \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \tilde{\Lambda}_y \\ \tilde{\Lambda}_z \\ \mathbf{A} \end{bmatrix} = M_x\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \Rightarrow \mathbf{WF} = \mathbf{M} \Rightarrow \frac{d\,\mathbf{F}}{d\,\rho_e^b} = -\mathbf{W}^{-1}\frac{d\mathbf{W}}{d\,\rho_e^b}\mathbf{F}$$

Calculation technique

❖ Eliminating chain-rule summation

➢ Summation into a single matrix form

$$\frac{d\mathbf{f}}{d\rho_e^b}\frac{\partial \mathbf{K}}{\partial \mathbf{f}} = \mathbf{f}'\frac{\partial \mathbf{K}}{\partial \mathbf{f}} = \sum_j f_j' \frac{\partial \mathbf{K}}{\partial f_j}$$

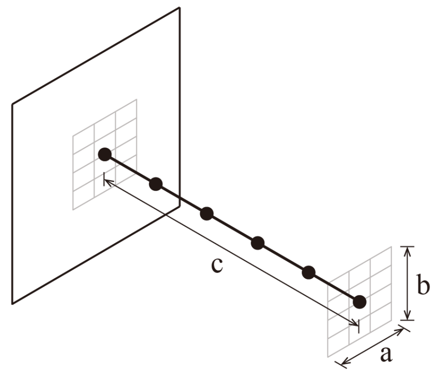
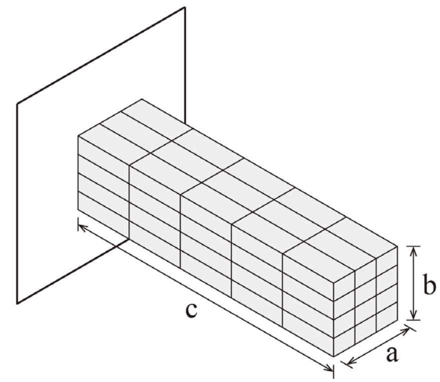
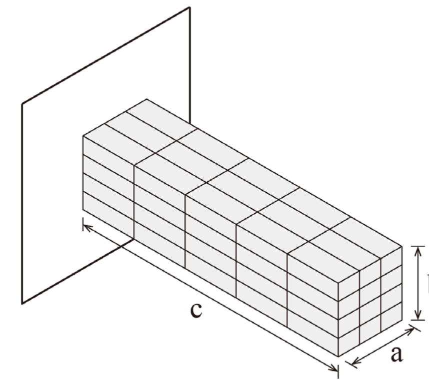
$$= \sum_j \left[\iiint \left(f_j' \frac{\partial \widetilde{\mathbf{B}}^T}{\partial f_j}\right) \widetilde{\mathbf{C}}\widetilde{\mathbf{B}} \left\|\mathbf{J}\right\| dr\, ds\, dt + \iiint \widetilde{\mathbf{B}}^T \widetilde{\mathbf{C}} \left(f_j' \frac{\partial \widetilde{\mathbf{B}}}{\partial f_j}\right)\left\|\mathbf{J}\right\| dr\, ds\, dt \right]$$

$$\widetilde{\mathbf{B}}' = \sum_j \frac{\partial\left(f_j' \; \widetilde{\mathbf{B}}_{nm}\right)}{\partial f_j} = \frac{1}{2}\left[\frac{d\mathbf{L}'}{dr_k}\mathbf{g}_l + \frac{d\mathbf{L}'}{dr_l}\mathbf{g}_k \right]\left(\mathbf{t}_n \cdot \mathbf{g}^k\right)\left(\mathbf{t}_m \cdot \mathbf{g}^l\right)$$

$$\mathbf{L}' = h_i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \bar{f}' V_{rx}^i \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{f}' V_{ry}^i \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{f}' V_{rz}^i \end{bmatrix} \qquad \text{with} \quad \bar{f}' = H_j f_j'$$

$$= \iiint \widetilde{\mathbf{B}}'^T \widetilde{\mathbf{C}}\widetilde{\mathbf{B}} \left\|\mathbf{J}\right\| dr\, ds\, dt + \iiint \widetilde{\mathbf{B}}^T \widetilde{\mathbf{C}}\widetilde{\mathbf{B}}' \left\|\mathbf{J}\right\| dr\, ds\, dt$$

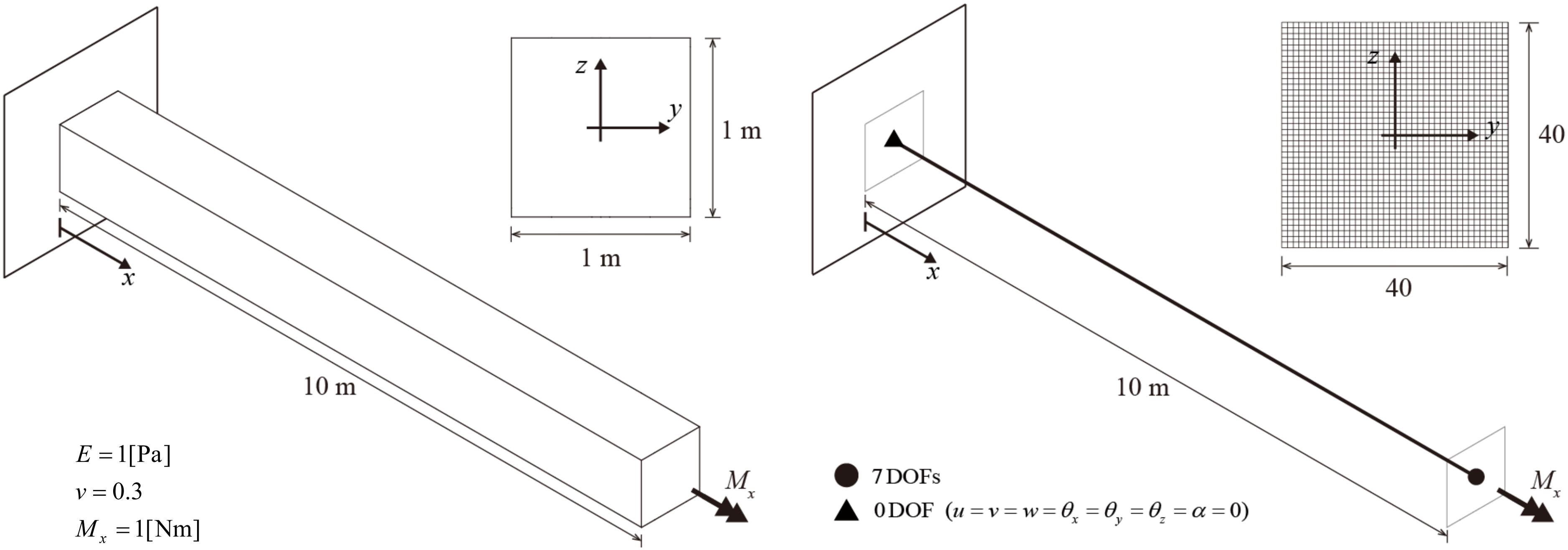
Numerical example

		Proposed beam model	**Reference solid model**	**ANSYS**
		In-House Code (MATLAB)	Revised Code [1] (MATLAB)	Commercial Software (Fortran, C/C++)
				
FEM	Matrix size to solve	[3ab] × c + [7c]	[3abc]	[3abc]
	Parallel	Partially vectorized	Fully vectorized	Parallel code
	Sparsity	Non-Sparse matrix	Sparse matrix	Sparse matrix
	Inverse solver	QR Decomposition	Symmetric Cholesky	Symmetric PCG
Optimization	Methodology	SIMP	SIMP	SIMP
	Optimizer	OC/MMA	OC/MMA	OC
	Tolerance	0.001	0.001	0.001

[1] Liu, K., & Tovar, A. (2014). An efficient 3D topology optimization code written in Matlab. Structural and Multidisciplinary Optimization, 50(6), 1175-1196.

Ex1. Square-section beam problem

❖ Problem description



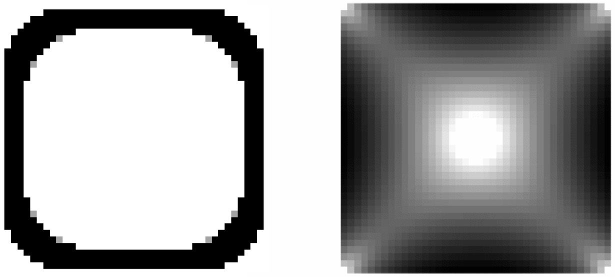
	Proposed beam	Reference solid	ANSYS
Volume fraction	0.3 / 0.5 / 0.6	0.3 / 0.5 / 0.6	0.3 / 0.5 / 0.6
Penalization factor	p=3	p=3	p=3
Optimizer	OC	OC	OC
Tolerance	0.001	0.001	0.001

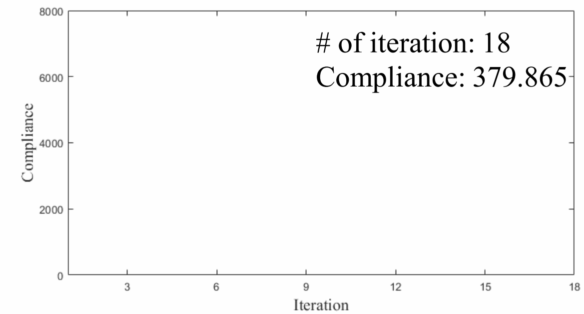
Ex1. Square-section beam problem

Proposed Beam

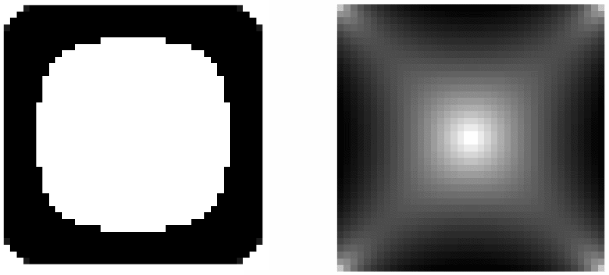
Objective function

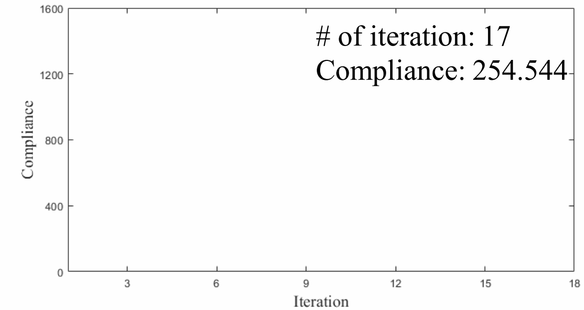
30% mass constraint



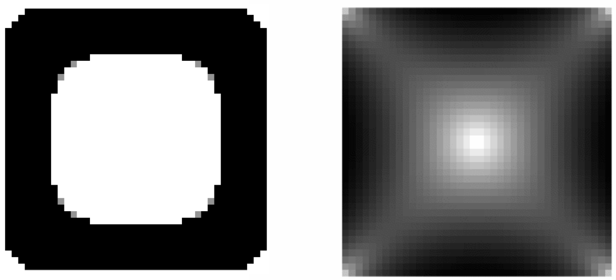


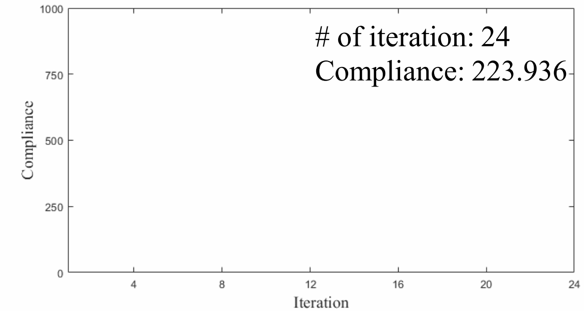
50% mass constraint



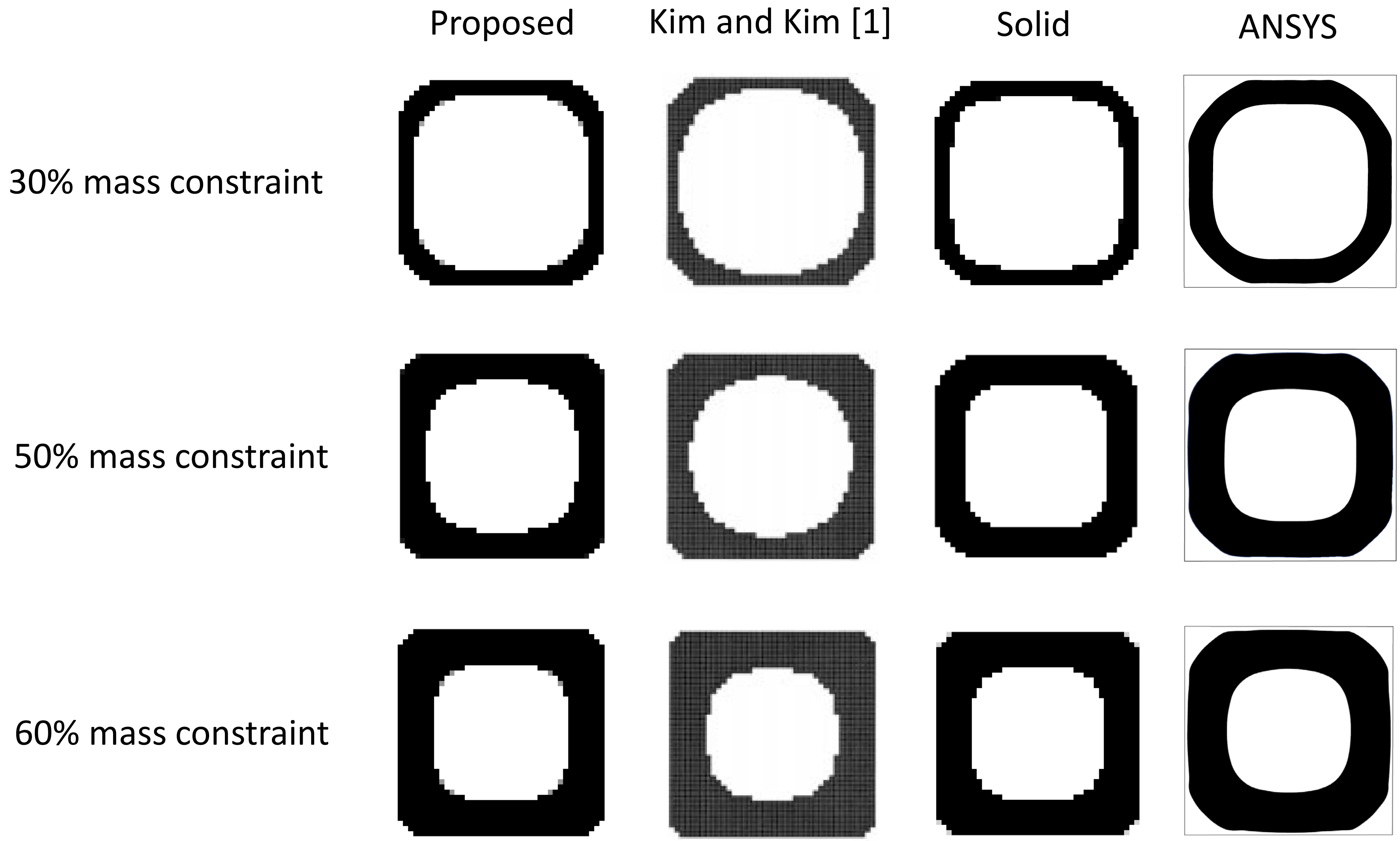


60% mass constraint





Ex1. Square-section beam problem

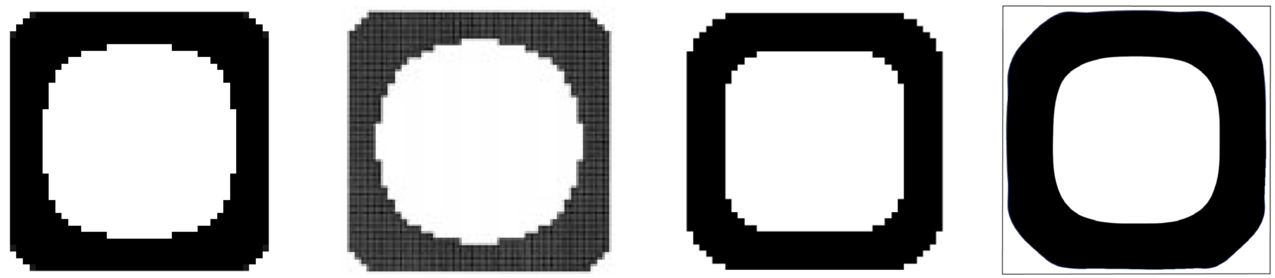


[1] Kim, Y. Y., & Kim, T. S. (2000). Topology optimization of beam cross sections. International journal of solids and structures, 37(3), 477-493.

Ex1. Square-section beam problem

Proposed Kim and Kim [1] Solid ANSYS

50% mass constraint



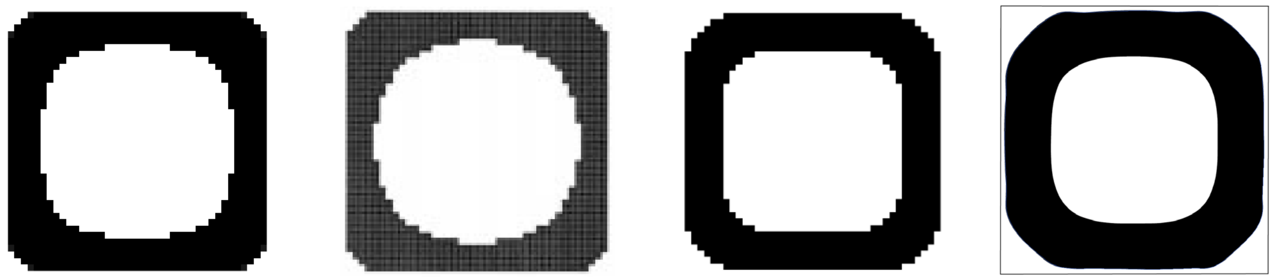
		Proposed beam	Reference solid	ANSYS
Compliance		254.5439	255.9978	-
# of design variables		40×40×1	40×40×15	40×40×25
DOF of inverse problem		5,052	45,387	126,075
Overhead		3.5 [s]	593 [s]	59 [s]
Iteration	Average iteration time	5.5 [s/iter]	76.3 [s/iter]	40.6 [s/iter]
	# of iterations	12 [iter]	31 [iter]	18 [iter]
Total time		69.5 [s]	2958.2 [s]	789.9 [s]

[1] Kim, Y. Y., & Kim, T. S. (2000). Topology optimization of beam cross sections. International journal of solids and structures, 37(3), 477-493.

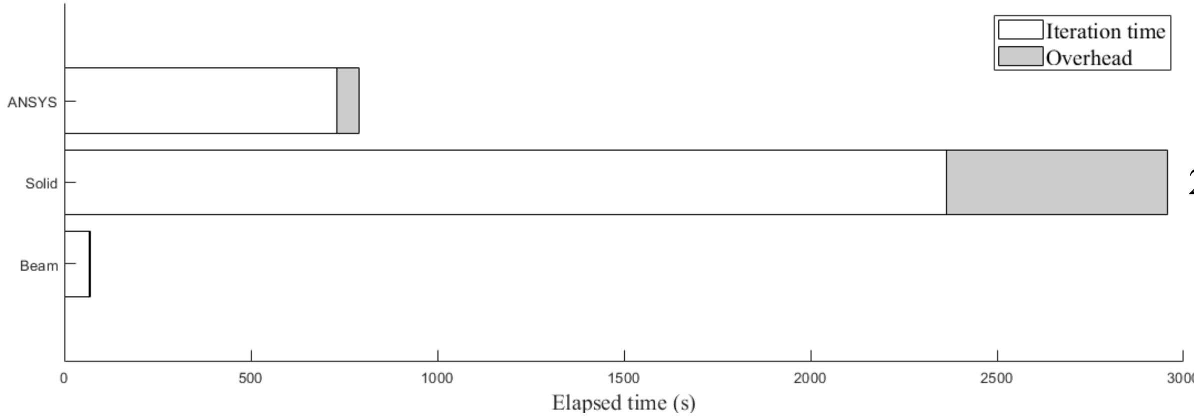
Ex1. Square-section beam problem

Proposed | Kim and Kim [1] | Solid | ANSYS

50% mass constraint



	Proposed beam	Reference solid	ANSYS
Compliance	254.5439	255.9978	-
# of design variables	40×40×1	40×40×15	40×40×25
DOF of inverse problem	5,052	45,387	126,075



789.9 [s] ×11.4 slower

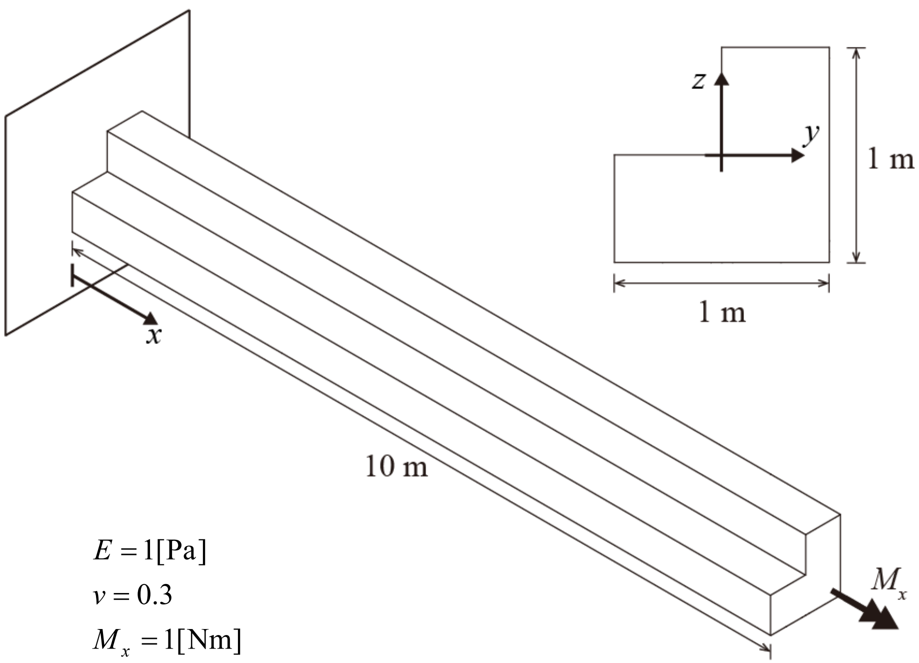
2958.2 [s] ×42.6 slower

69.5 [s]

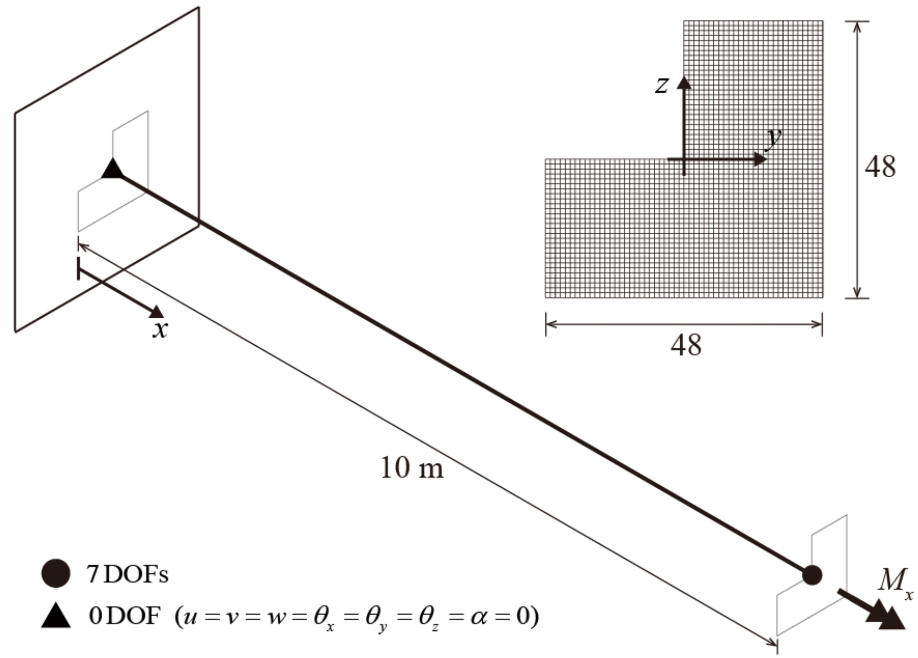
[1] Kim, Y. Y., & Kim, T. S. (2000). Topology optimization of beam cross sections. International journal of solids and structures, 37(3), 477-493.

Ex2. L-section beam problem

❖ Problem description



$E = 1$[Pa]
$\nu = 0.3$
$M_x = 1$[Nm]

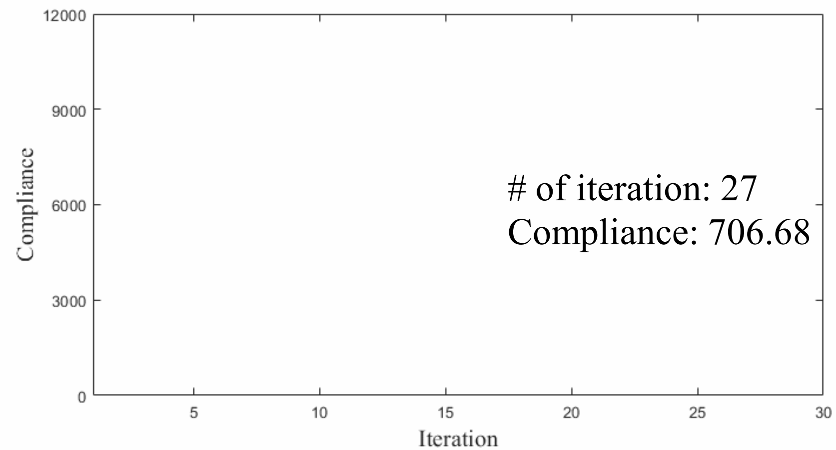


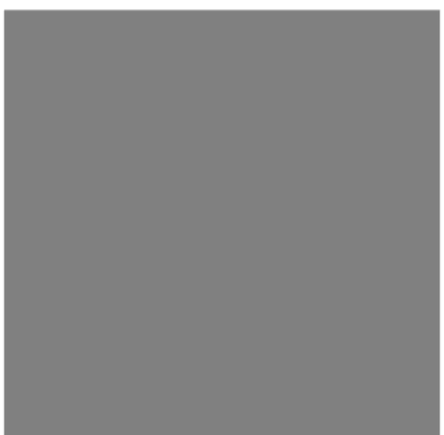
	Proposed beam	Reference solid	ANSYS
Volume fraction	0.5	0.5	0.5
Penalization factor	p=3	p=3	p=3
Optimizer	OC / MMA	OC / MMA	OC
Tolerance	0.001	0.001	0.001

Ex2. L-section beam problem

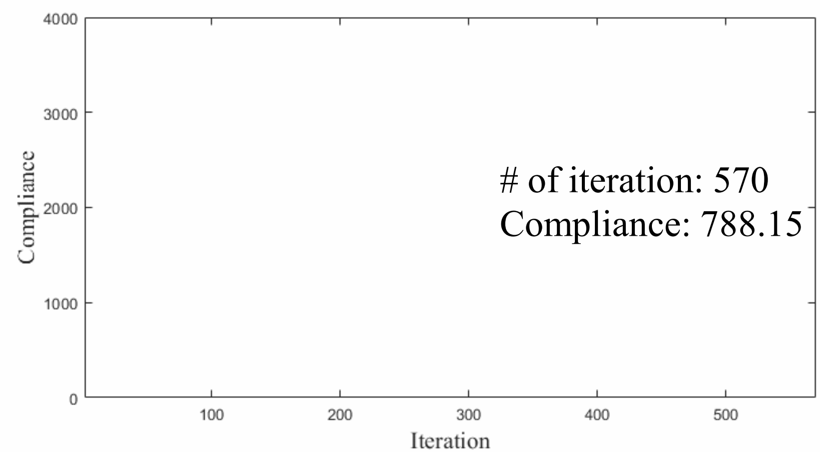
❖ Optimized using MMA

Proposed beam



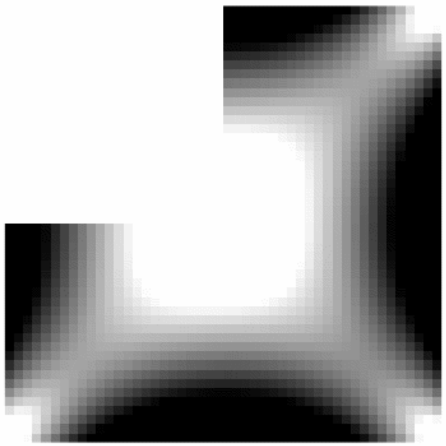


Solid reference

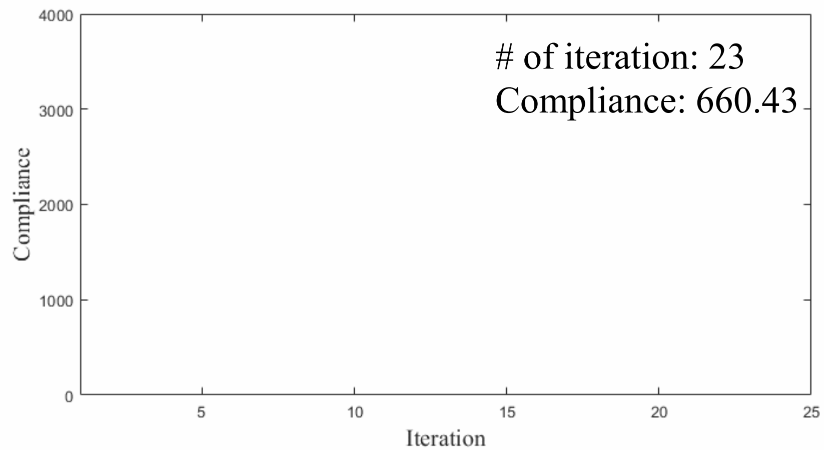


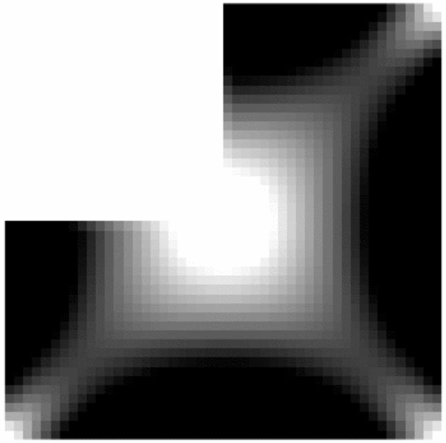
Ex2. L-section beam problem

❖ Optimized using OC

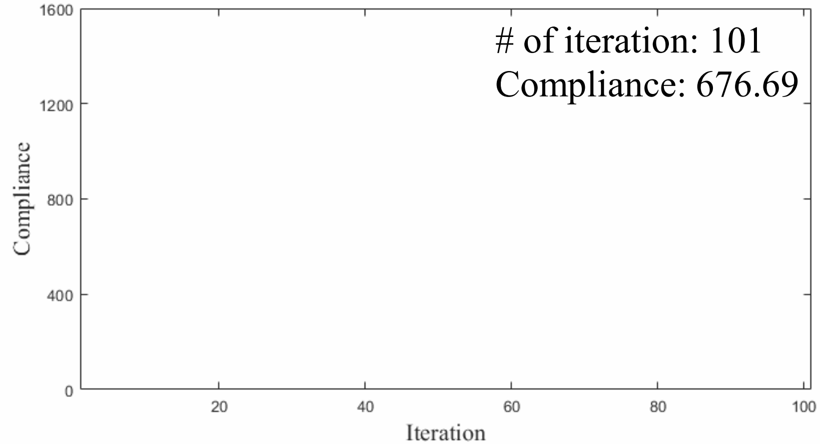


Proposed beam





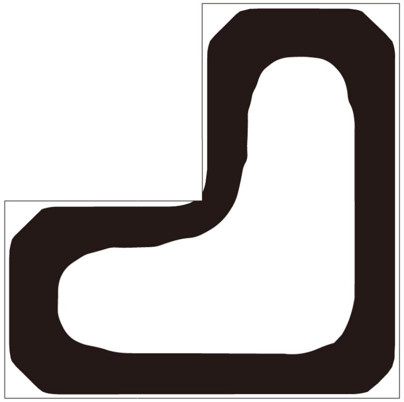
Solid reference



Ex2. L-section beam problem

Proposed beam

Solid reference



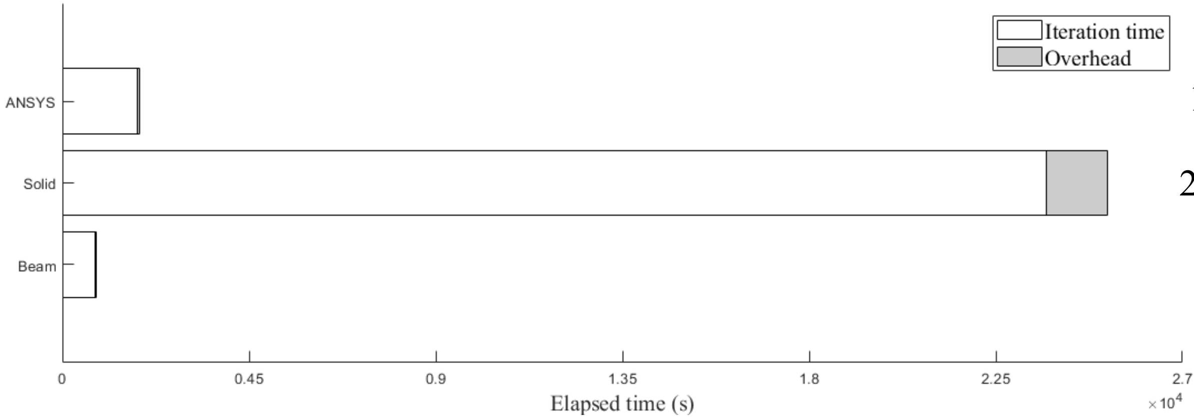
ANSYS

		Proposed beam	Reference solid	ANSYS
Compliance		660.4315	676.6892	-
# of design variables		48×48×1	48×48×9	48×48×26
DOF of inverse problem		5,484	38,325	147,825
Overhead		5.9 [s]	1469 [s]	51.6 [s]
Iteration	Average iteration time	34.7 [s/iter]	234.9 [s/iter]	106.7 [s/iter]
	# of iterations	23 [iter]	101 [iter]	17 [iter]
Total time		804.0 [s]	25,193.9 [s]	1,865.5 [s]

Ex2. L-section beam problem

Proposed beam | Solid reference | ANSYS

	Proposed beam	Reference solid	ANSYS
Compliance	660.4315	676.6892	-
# of design variables	48×48×1	48×48×9	48×48×26
DOF of inverse problem	5,484	38,325	147,825



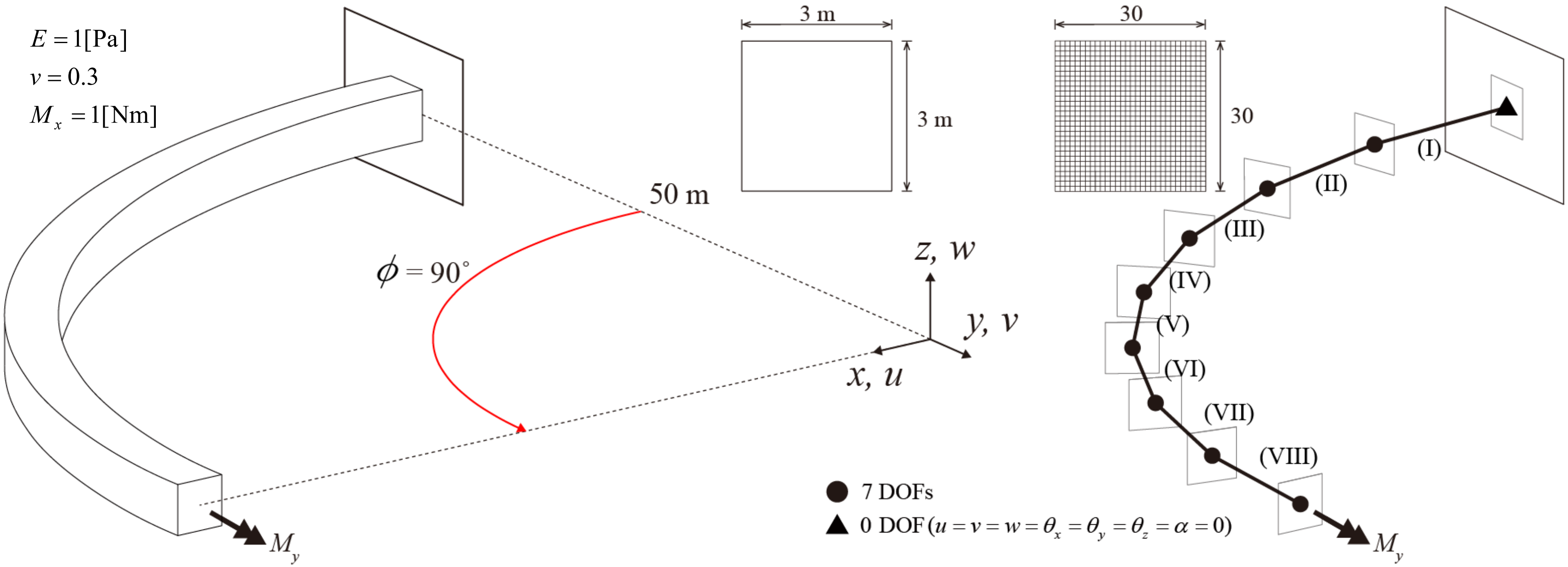
1865.5 [s] ×2.32 slower

25193.9 [s] ×31.3 slower

804.0 [s]

Ex3. 90-degree curved beam problem

❖ Problem description

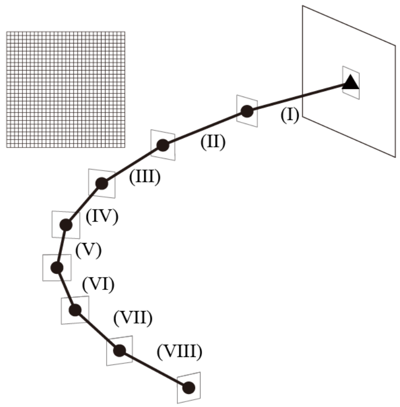


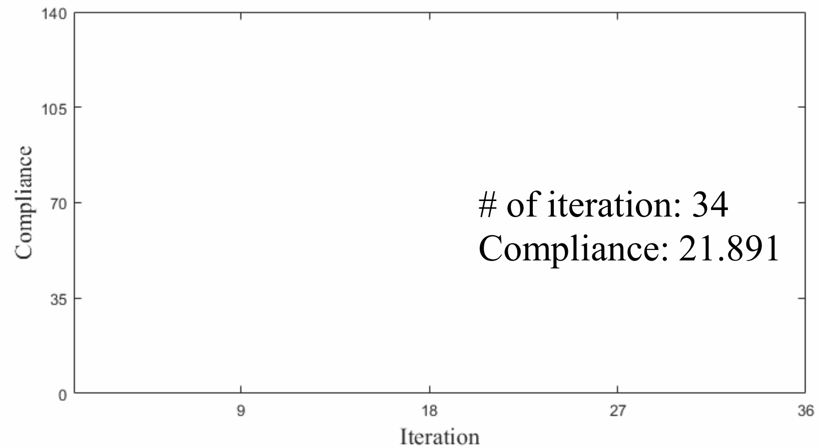
	Proposed beam	Reference solid	ANSYS
Volume fraction	0.5	0.5	0.5
Penalization factor	*p*=3	*p*=3	*p*=3
Optimizer	OC	OC	OC
Tolerance	0.001	0.001	0.001

Ex3. 90-degree curved beam problem

❖ Proposed beam

Optimizer: OC
Mesh: 30×30






(I)


(II)


(III)


(IV)


(V)


(VI)

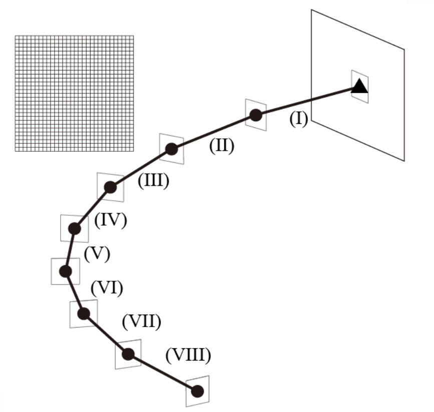

(VII)

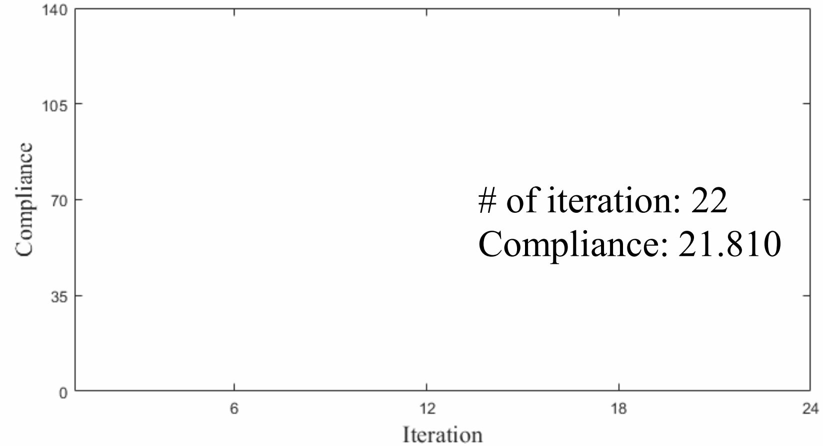

(VIII)

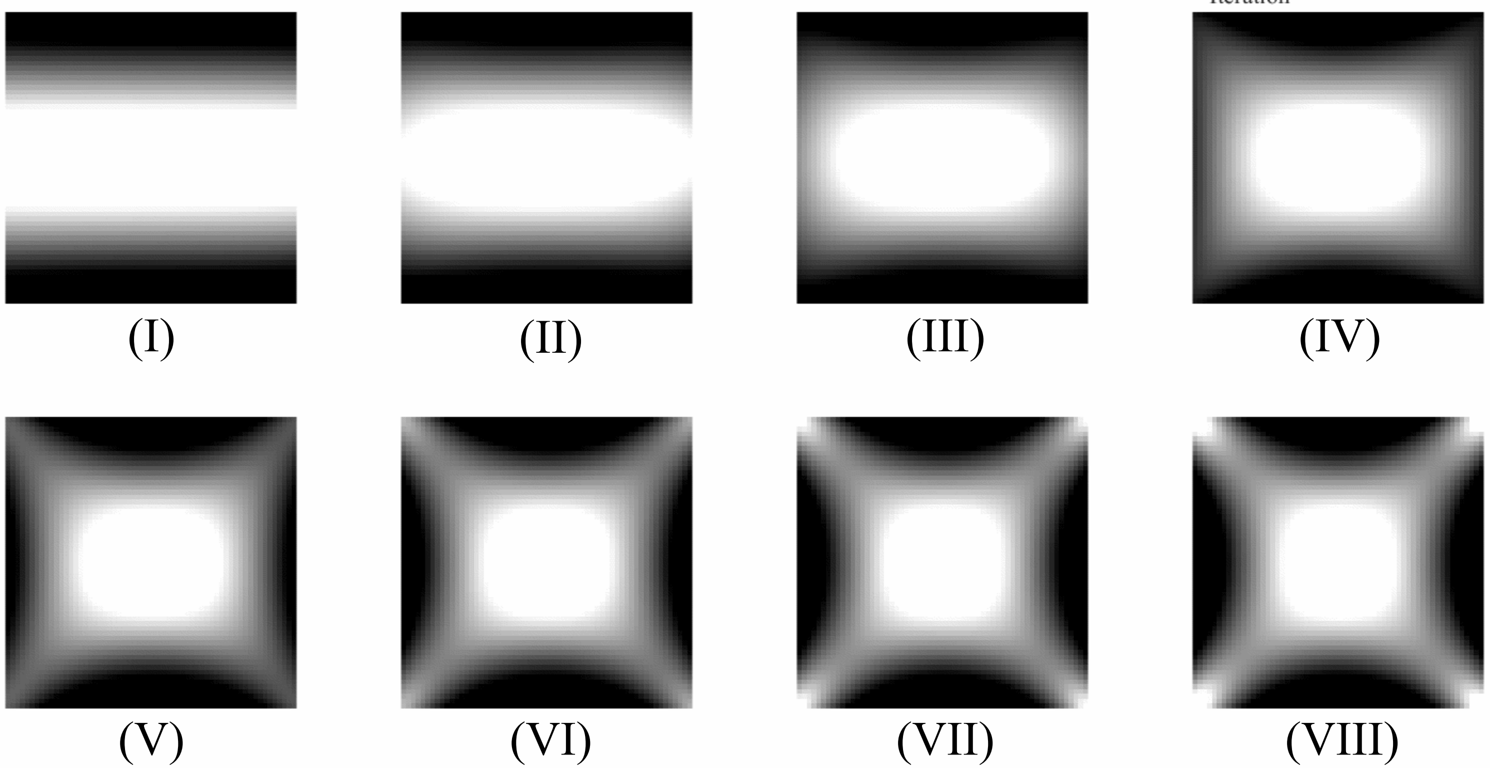
Ex3. 90-degree curved beam problem

❖ Proposed beam

Optimizer: OC
Mesh: 60×60



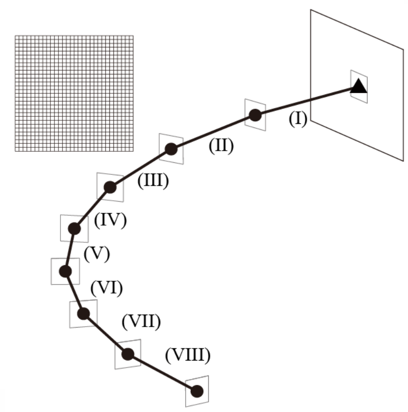


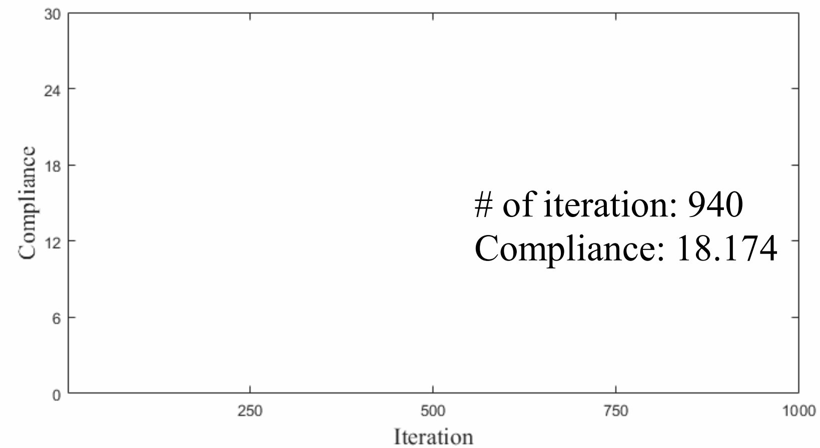


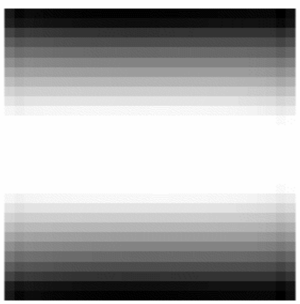
Ex3. 90-degree curved beam problem

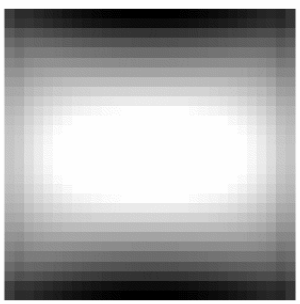
❖ Solid reference

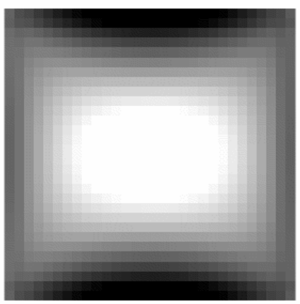
Optimizer: OC
Mesh: 30×30

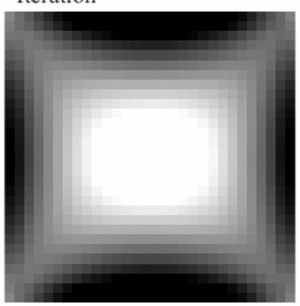


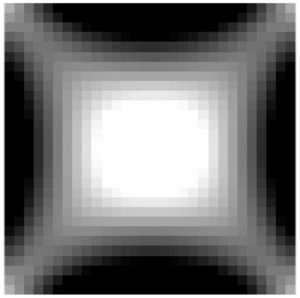


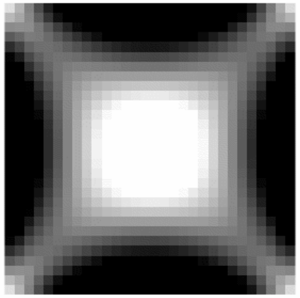

(I)


(II)


(III)


(IV)


(V)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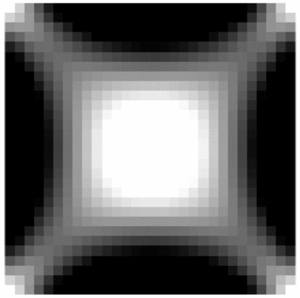
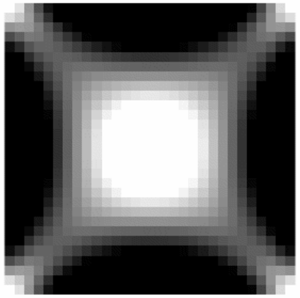

(VII)


(VIII)

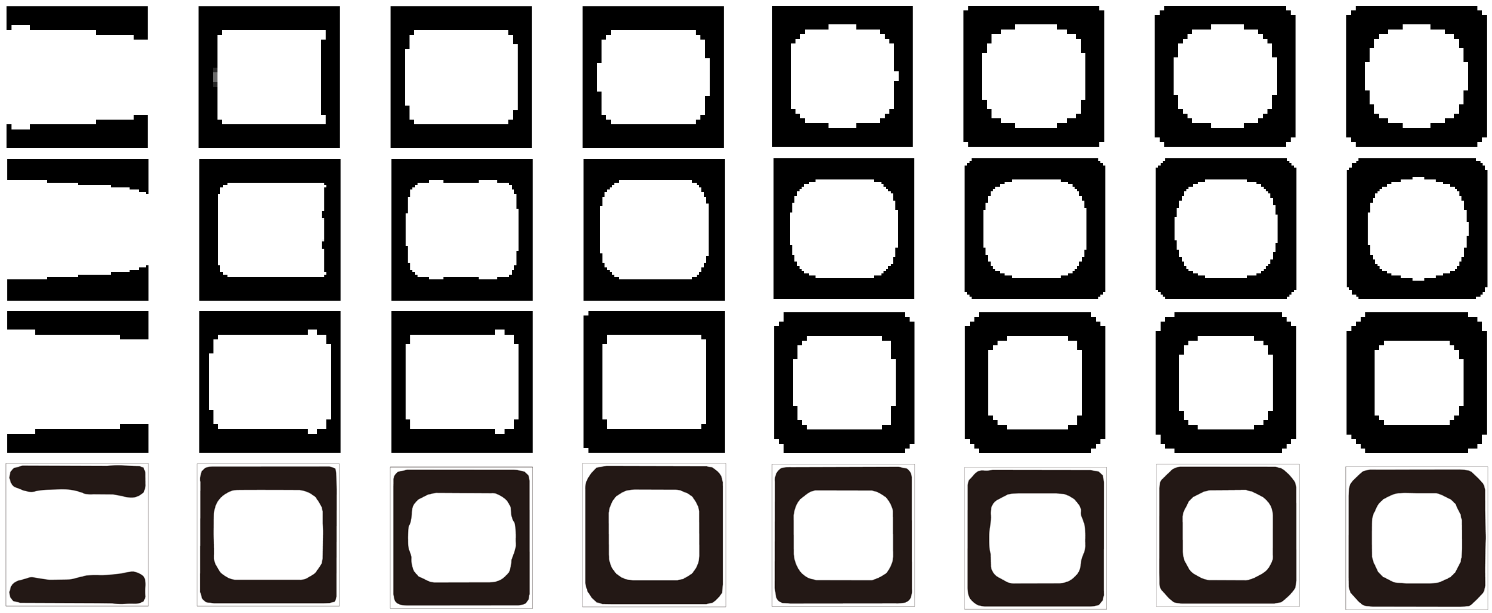
Ex3. 90-degree curved beam problem

Proposed Beam (30×30)

Proposed Beam (60×60)

Solid reference

ANSYS



		Proposed beam	Reference solid	ANSYS
Compliance		21.8909	18.1737	-
# of design variables		30×30×8	30×30×24	30×30×112
DOF of inverse problem		2,895	72,075	325,779
Stiffness construction		14 [s]	1556 [s]	116 [s]
Iteration	Average iteration time	27.3 [s/iter]	92.7 [s/iter]	100.1 [s/iter]
	# of iterations	34 [iter]	940 [iter]	14 [iter]
Total time		942.2 [s]	88,756 [s]	1,518 [s]

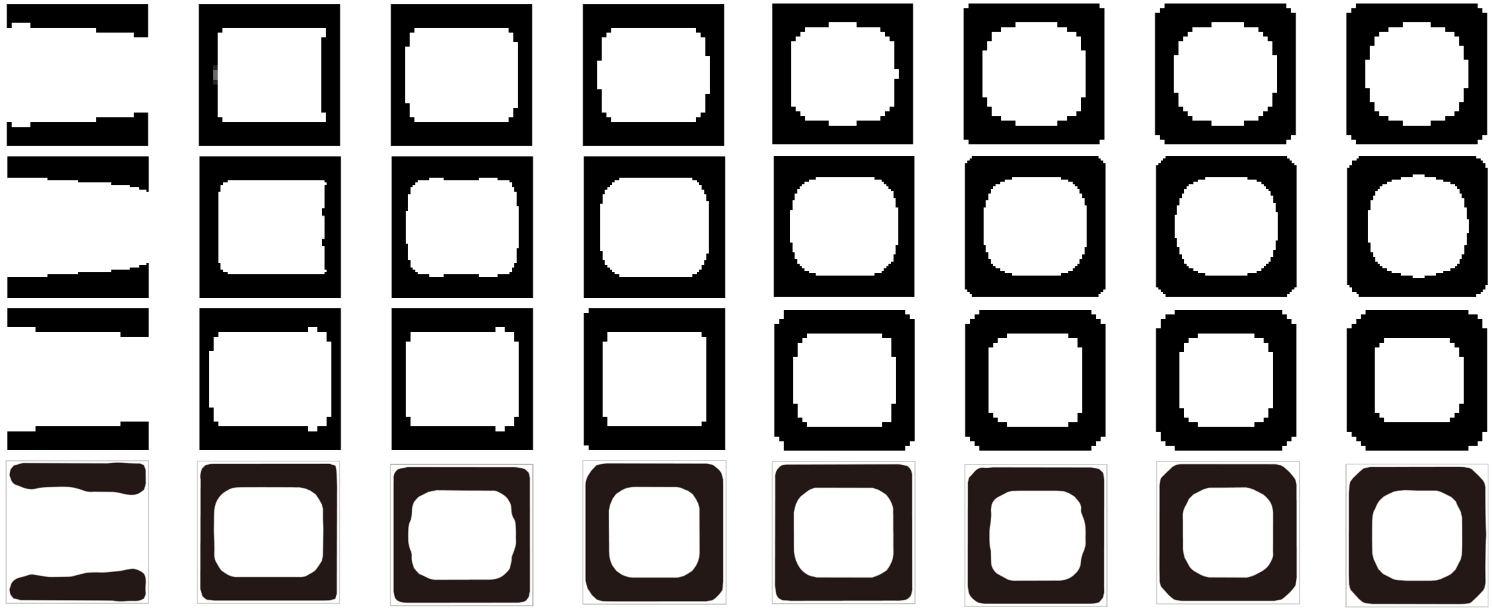
Ex3. 90-degree curved beam problem

Proposed Beam (30×30)

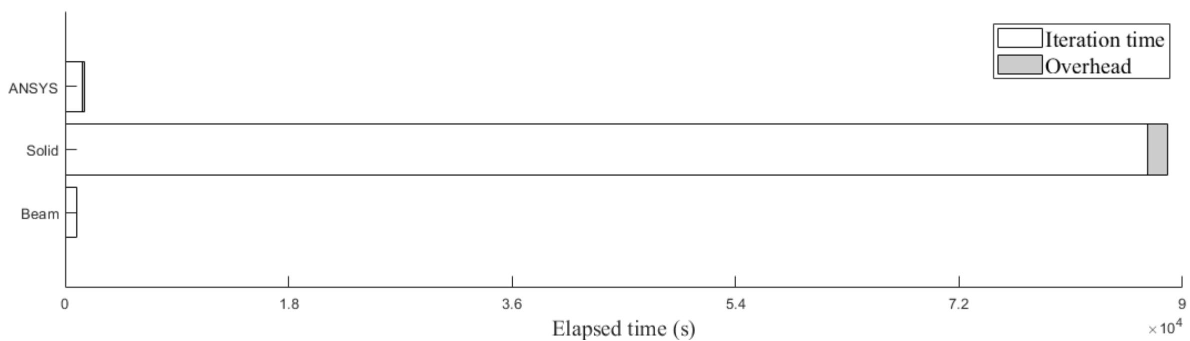
Proposed Beam (60×60)

Solid reference

ANSYS



	Proposed beam	Reference solid	ANSYS
Compliance	21.8909	18.1737	-
# of design variables	30×30×8	30×30×24	30×30×112
DOF of inverse problem	2,895	72,075	325,779



1,518 [s] ×1.61 slower

88,756 [s] ×94.2 slower

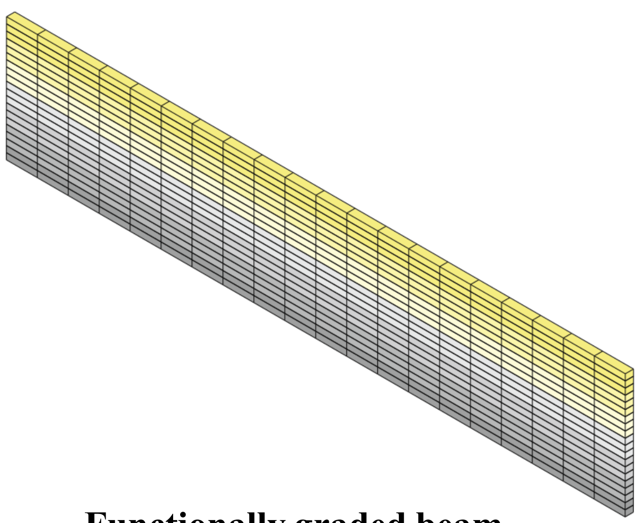
942.2 [s]

5. Conclusion & Future works

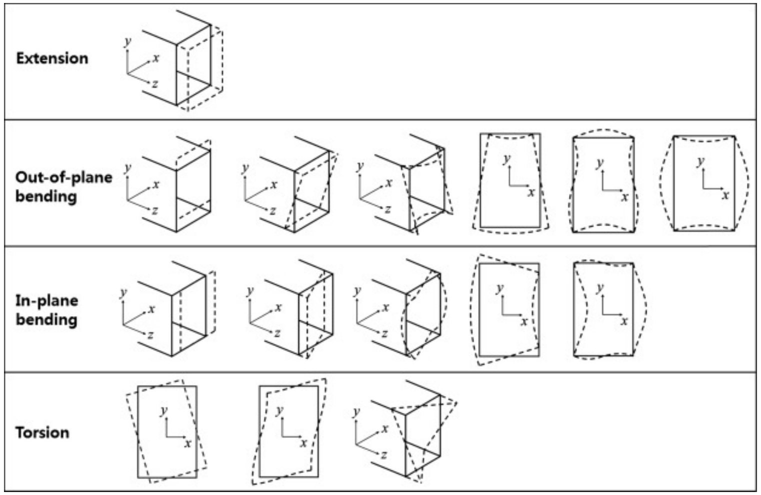
Conclusion & Future work

1. Method of calculating an **interface warping function** is proposed

- ✓ The proposed method can be used to solve the beam with material/geometric discontinuity which cannot be solved using ANSYS.
- ✓ It provides high modeling capabilities, including arbitrary-shaped curved composite beams.
- ✓ The method can be extended to solve non-linear problems using 7 DOF/node.
- ✓ It has better accuracy and a lower computational cost than the previous work.



Functionally graded beam

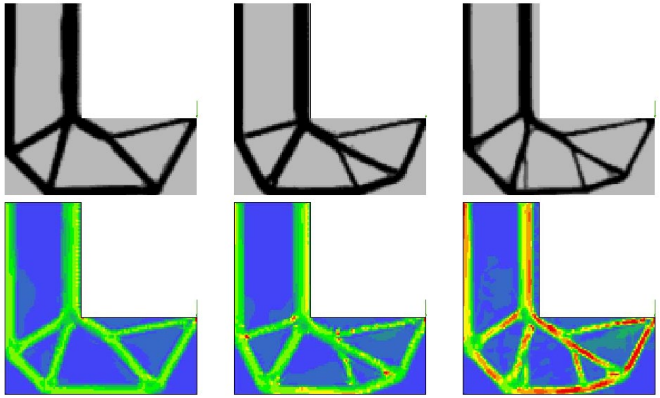


Generalized beam theory

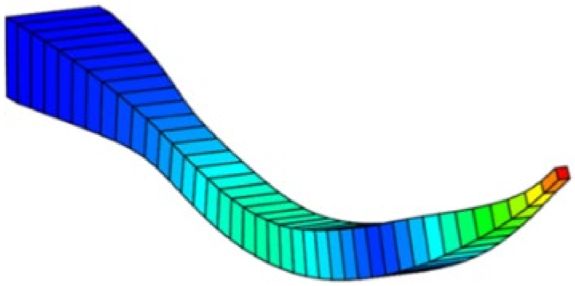
Conclusion & Future work

2. **Topology optimization of beam** considering interface warping is proposed

- ✓ The proposed method can be applied to design thin/thick beams with arbitrarily shaped design space.
- ✓ The method inherits the high modeling capability and fully coupled bending-twisting behavior from the continuum mechanics based beam.
- ✓ It provides faster optimization results with fewer iterations than optimizations using solid elements.
- ✓ It can be easily extended to design beams with different objective functions.



Stress based optimization problem



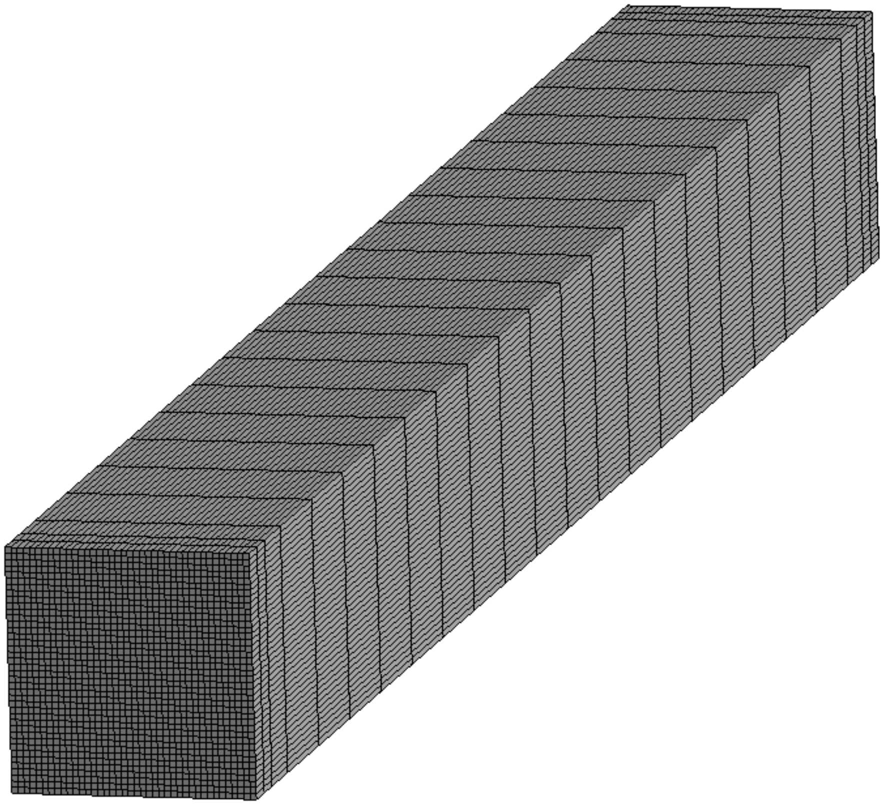
Vibration optimization problem

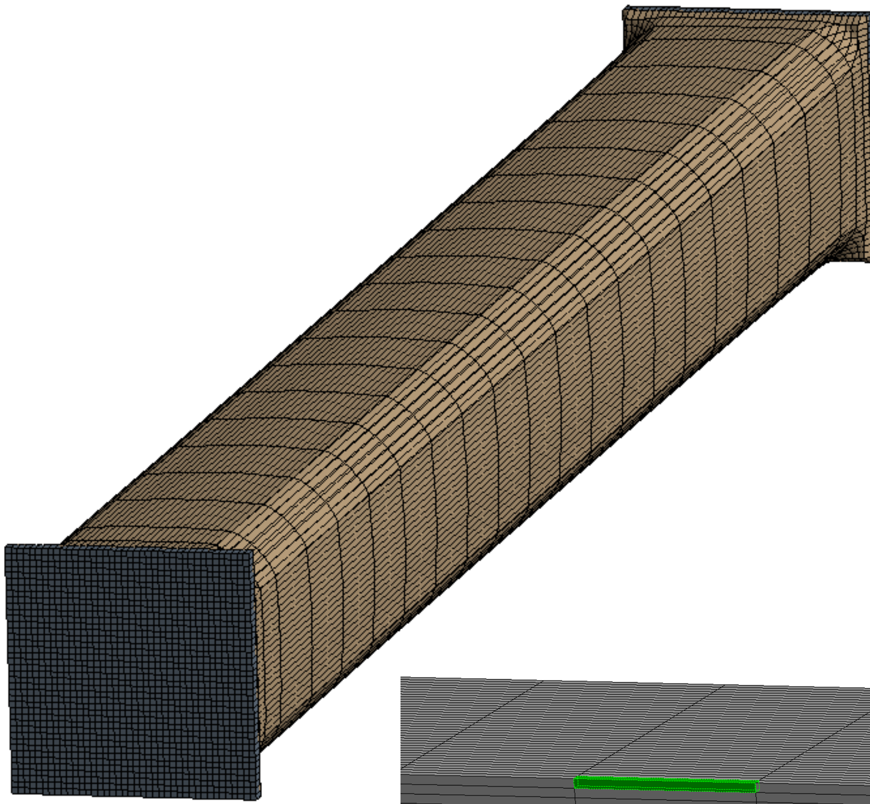
Thank you

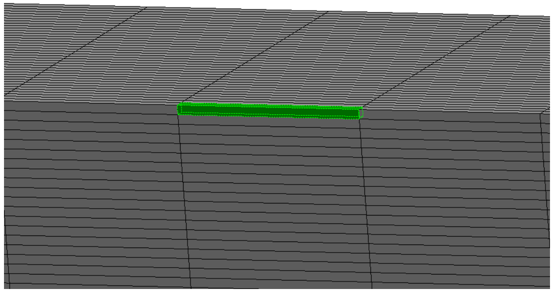
Appendix

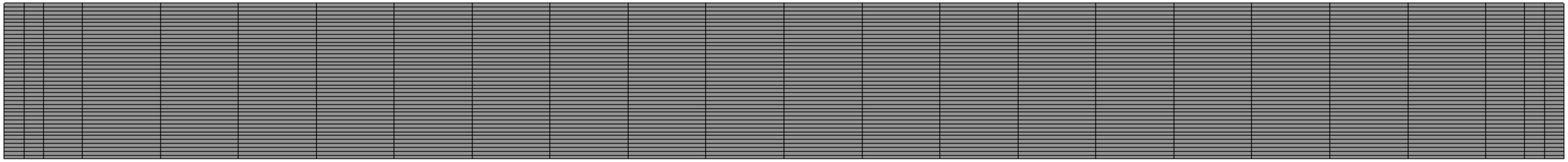
Ex1. Square-section beam problem

❖ ANSYS Topology Optimization Result



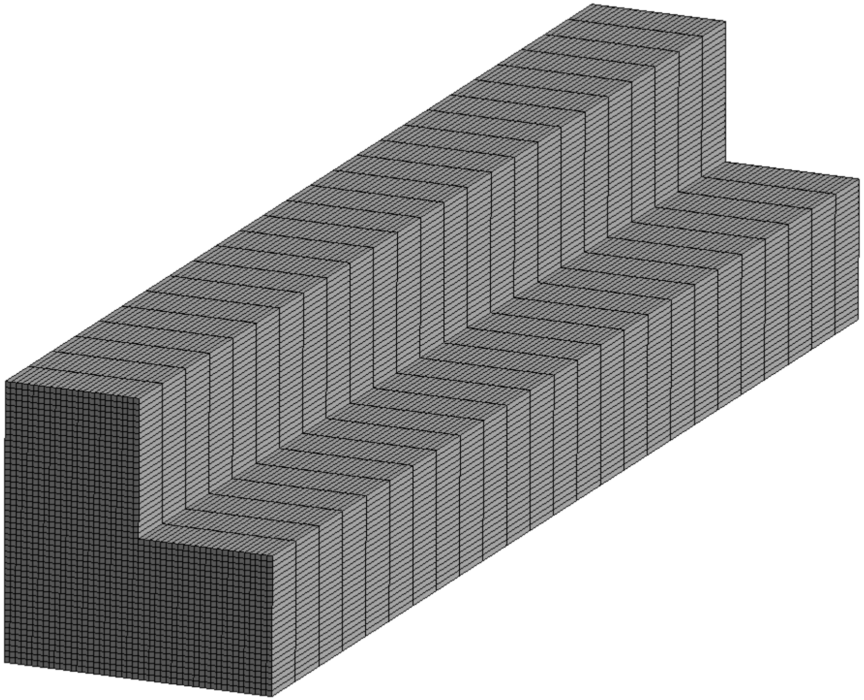


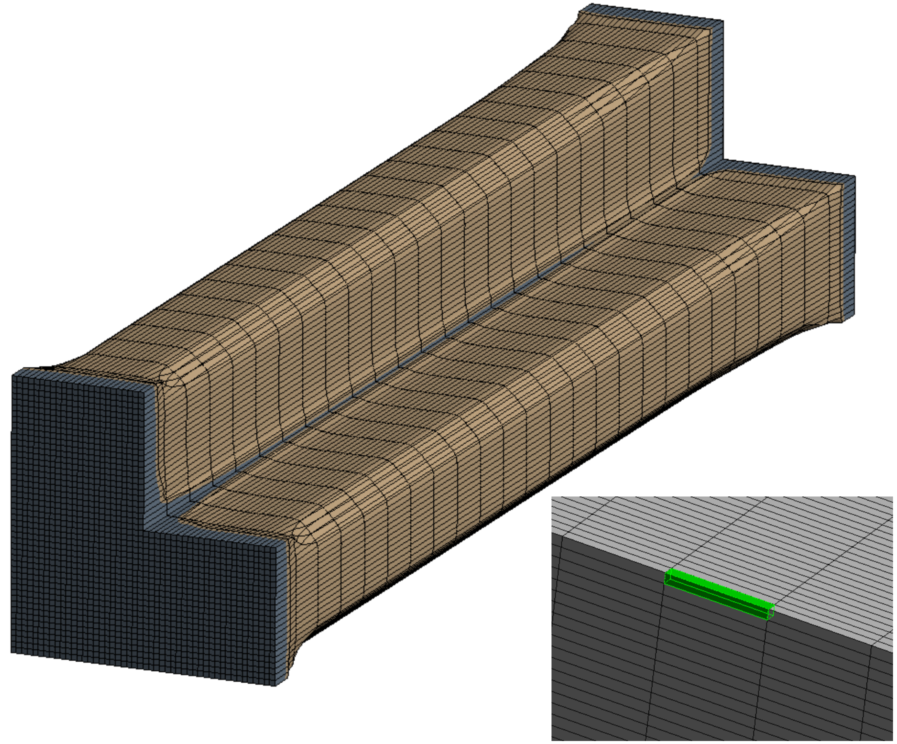


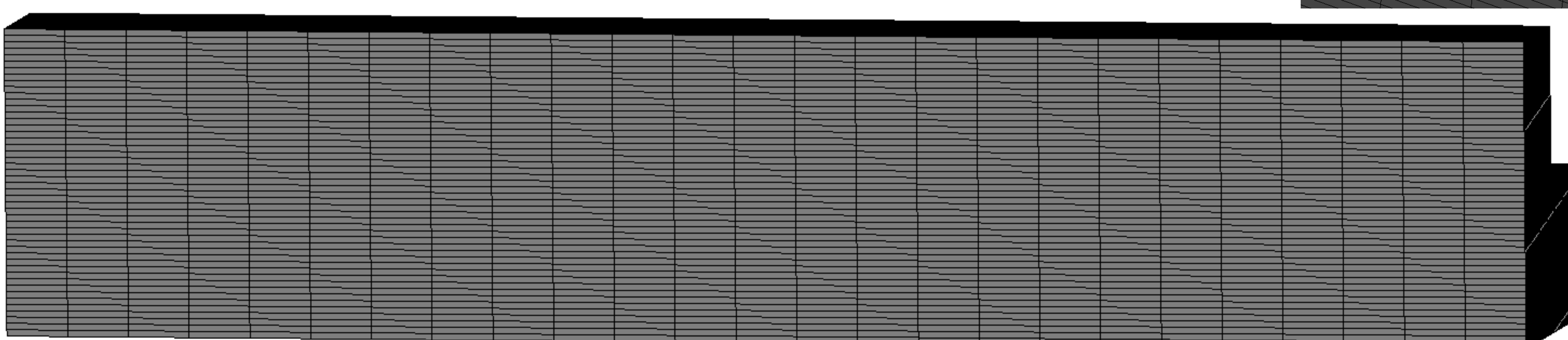


Ex2. L-section beam problem

❖ ANSYS Topology Optimization Result







Ex3. 90-degree curved beam problem

❖ ANSYS Topology Optimization Result