Ph. D. Dissertation Presentation

변형률 완화 요소법: 향상된 저차 솔리드 및 쉘 유한요소 개발

The strain-smoothed element method: a method for developing improved low-order solid and shell finite elements

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2020.06.01

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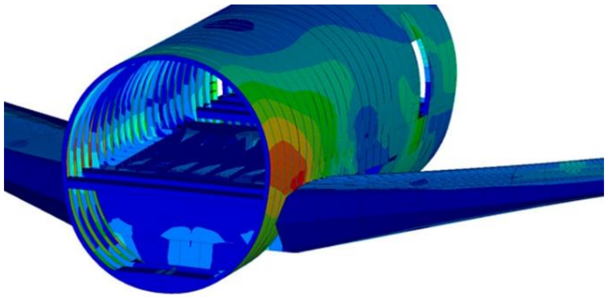
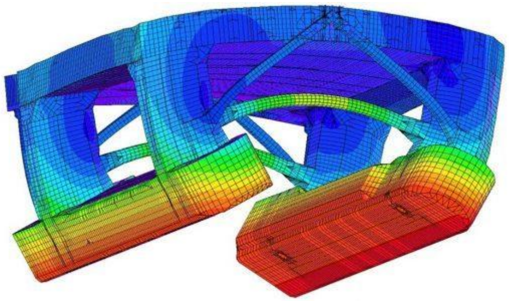
Contents

1. Introduction

FEM in Engineering Fields



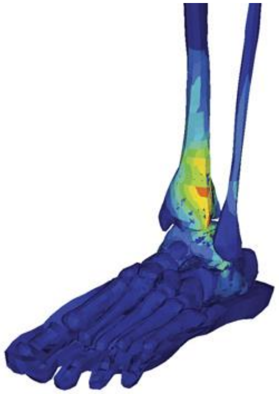
Aerospace Engineering



Offshore Engineering

Automotive Engineering

Architectural Engineering



Biomedical Engineering

- https://altairhyperworks.com/industry/
- http://www.lminnomaritime.com/application-of-fem-on-ships-structural-design/

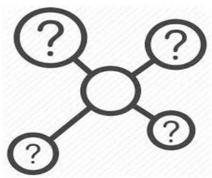
FEM in Engineering Fields

❖ **Commercial FEA software**

- ABAQUS
- ADINA
- ANSYS
- NASTRAN

Motivations

01



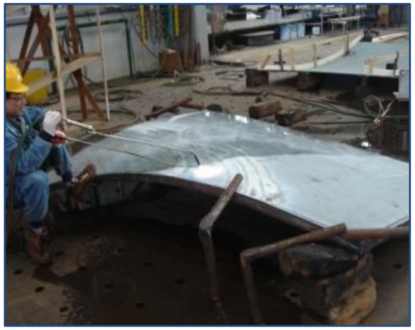
Problems

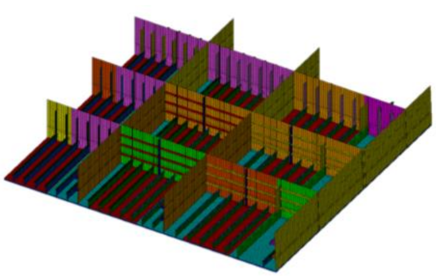
1. Accuracy and efficiency.
2. Instability.
3. New applications.

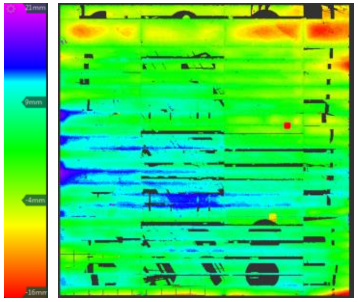
02

Needs

1. Larger model size.
2. More complex analysis.
3. Higher accuracy and efficiency.
4. Real-time analysis.







- 산학과제. 시뮬레이션 기반 고정도 선체 변형 및 정도 예측 시스템 개발. 한국과학기술원 이필승 외.

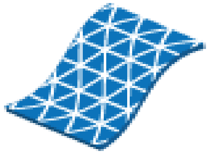
Motivations

03

Solutions

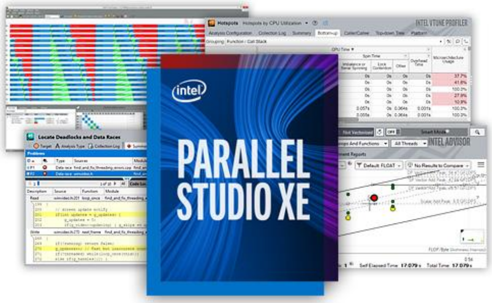
1. Powerful computer components.
2. Parallelization, optimization, faster algorithms.
3. Development of finite element technologies.

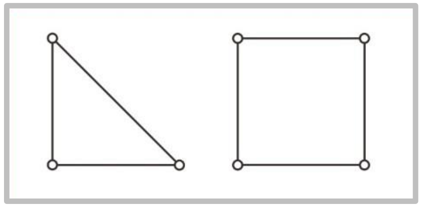
04



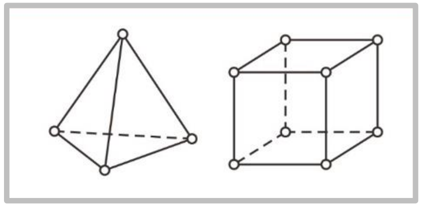
Finite Elements

- Fundamentally affecting the accuracy and efficiency of FEA.

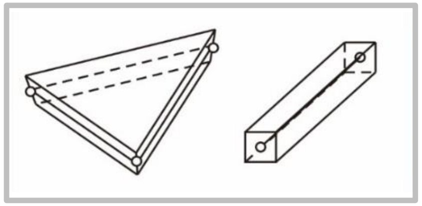




2D solid elements



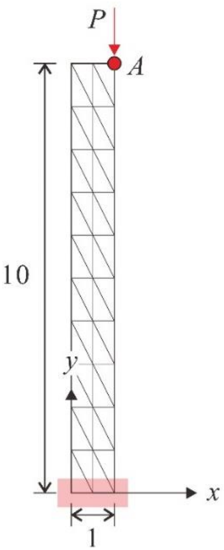
3D solid elements



Shell & beam elements

Motivations

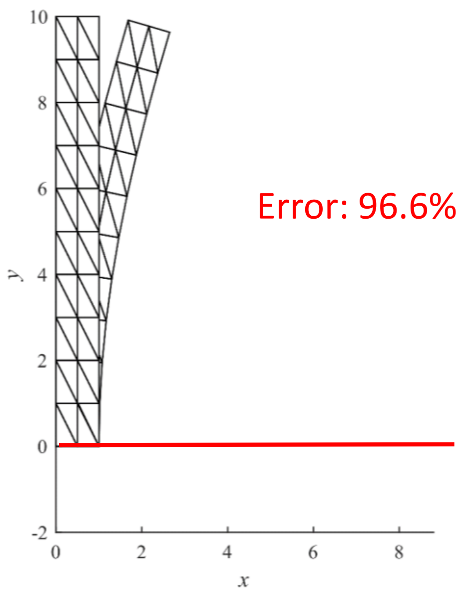
Example)



Relative errors (%) in vertical displacements at point *A*.

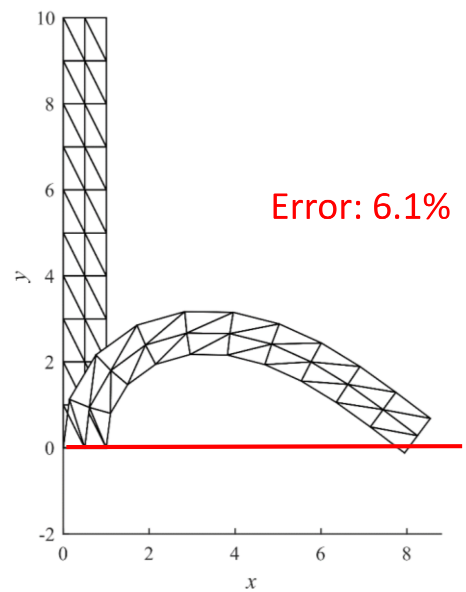
2×10 mesh

❖ **Finite element development ?**



Standard 3-node element

VS

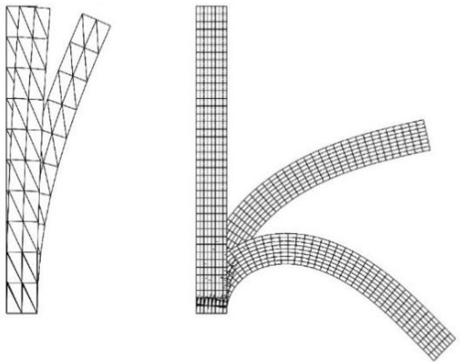


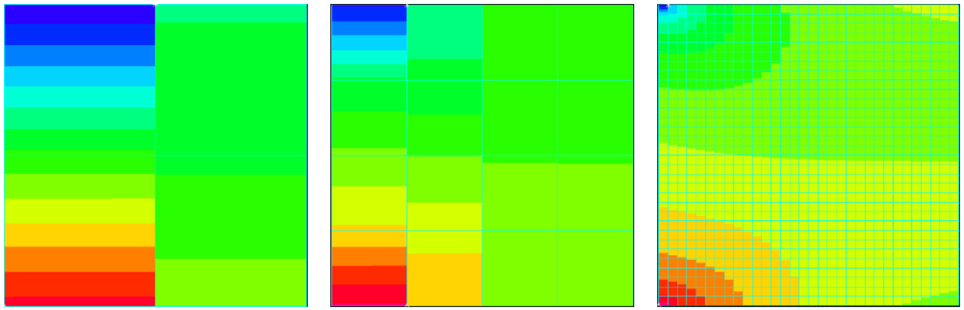
Proposed 3-node element

Finite Elements

❖ Stiff behaviors of finite elements

- Mesh refinement (*h* refinement, *p* refinement and *r* refinement) is tried first.





❖ Various attempts for improving FE solutions

1. Reduced integrations & assumed strain methods
   - ✓ URI (Uniform Reduced Integration) and SRI (Selective Reduced Integration).
   - ✓ ANS (Assumed Natural Strain) and MITC (Mixed Integration of Tensorial Components).
2. Enrichment methods
   - ✓ Enriched FEM, XFEM (eXtended FEM) and GFEM (Generalized FEM).
3. Strain smoothing methods
   - ✓ Node, Edge, Face and Cell-based S-FEM (Smoothed FEM).

2. Research background

Linear elasticity problem

❖ Linear elastic boundary value problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded and polygonal domain. The boundary $\partial\Omega$ of Ω consists of two parts $\Gamma_D \neq \varnothing$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$.

$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$ in Ω, (equilibrium equations)

$\boldsymbol{\sigma}^T \mathbf{n} = \mathbf{t}$ on Γ_N, $\mathbf{u} = \mathbf{0}$ on Γ_D, (boundary conditions)

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ in Ω, (constitutive equations)

$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ in Ω, (strain-displacement equations)

where $\mathbf{D}$ is a 3×3 matrix of material constants that is symmetric and positive definite.

Linear elasticity problem

❖ Weak formulation

Find $\mathbf{u} \in V$ such that

$$a(\mathbf{u},\mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in V,$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{D}\boldsymbol{\varepsilon}[\mathbf{u}] : \boldsymbol{\varepsilon}[\mathbf{v}] \, d\Omega,$$

$$f(\mathbf{v}) = \int_\Omega \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v} \, d\Gamma.$$

space of displacement fields:

$$V = \left\{ \mathbf{u} \in (H^1(\Omega))^2 : \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

space of strain and stress fields:

$$W = (L^2(\Omega))^3.$$

❖ Finite element formulation

$$\mathbf{KU} = \mathbf{F} = \mathbf{F}_B + \mathbf{F}_S,$$

where

$$\mathbf{K} = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{B}^{(m)T} \mathbf{D} \mathbf{B}^{(m)} \, d\Omega, \quad \mathbf{F}_B = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{H}^{(m)T} \mathbf{b} \, d\Omega, \quad \mathbf{F}_S = \sum_{m=1}^{e} \int_{\Gamma^{(m)}} \mathbf{H}^{(m)T} \mathbf{t} \, d\Gamma.$$

$$\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)} \mathbf{u}^{(m)} \quad \text{where} \quad \mathbf{B}^{(m)T} = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}.$$

Strain smoothing

❖ Smoothing operation

$\boldsymbol{\varepsilon}$ → $\tilde{\boldsymbol{\varepsilon}}_k$

: strain defined for finite elements.

$$\int_{\Omega^{(k)}} \boldsymbol{\varepsilon}(\mathbf{x})\Phi_k(\mathbf{x})d\Omega.$$

✓ valid for strains with constant values.

: smoothed strain defined for smoothing domain.

❖ Smoothing domain

- The area where strain smoothing is performed.
- This domain crosses the finite elements.

$\Omega^{(k)}$: k^{th} smoothing domain.

❖ Smoothing function

$$\Phi_k(\mathbf{x}) = \begin{cases} 1/A^{(k)}, & \mathbf{x} \in \Omega^{(k)} \\ 0, & \mathbf{x} \notin \Omega^{(k)} \end{cases}.$$

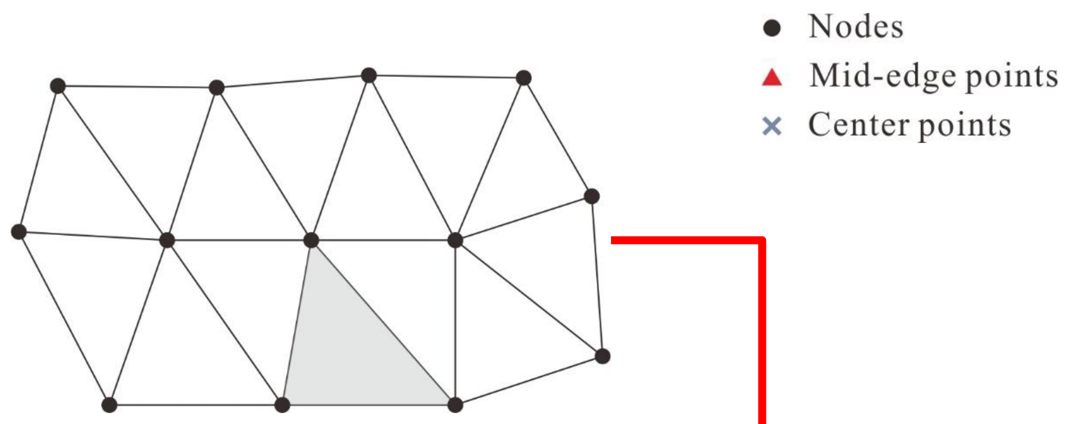
$A^{(k)}$: area of the smoothing domain $\Omega^{(k)}$.

$\Phi_k(\mathbf{x})$: smoothing function for domain $\Omega^{(k)}$.

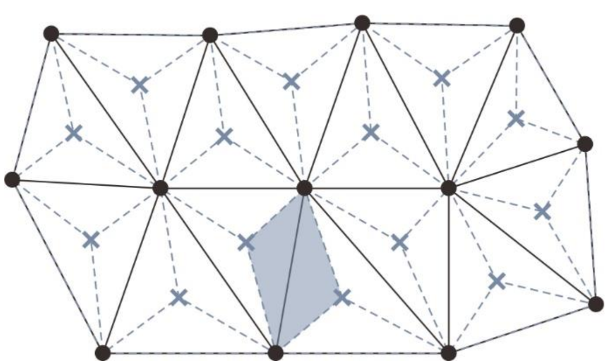
❖ Smoothing domain (continued)

$$\tilde{\boldsymbol{\varepsilon}}_k = \frac{1}{A_c^{(1)} + A_c^{(2)}}\left(A_c^{(1)}\boldsymbol{\varepsilon}^{(1)} + A_c^{(2)}\boldsymbol{\varepsilon}^{(2)}\right).$$

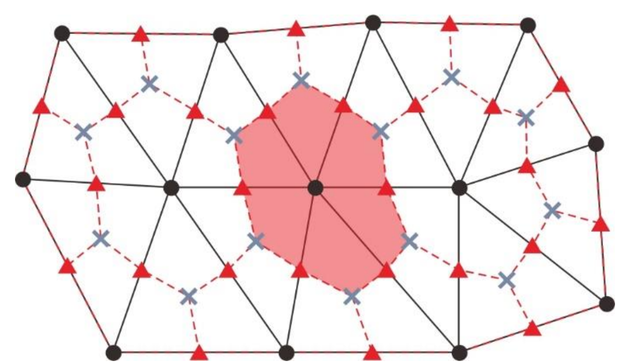
: smoothed strain defined for smoothing domain.



FEM domain



Edge-based smoothing domain

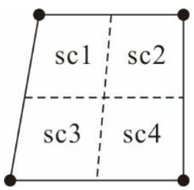


Node-based smoothing domain

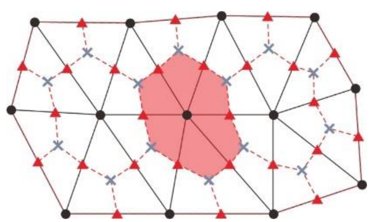
1. Development of smoothing methods for 2D & 3D linear solid elements.

- **Chen et al. (2001)**
  - The strain smoothing method was first proposed for the Galerkin mesh-free method.

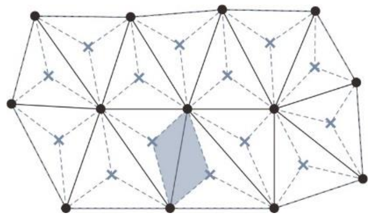
- **Liu et al. (2007)**
  - A cell-based S-FEM was proposed for 2D solid mechanics problems.
  - An element is subdivided into finite number of smoothing cells (SCs).
  - Depending on the number of SCs, it possesses spurious zero energy mode.



- **Liu et al. (2009)**
  - A node-based S-FEM was proposed for 2D solid mechanics problems.
  - It is effective for solving volumetric locking.
  - It gives overly soft solutions.



- **Liu et al. (2009)**
  - An edge-based S-FEM was proposed for 3-node triangular 2D solid element.
  - It shows the best performance among the previous strain smoothing methods.



- **Nguyen-Thoi et al. (2009)**
  - A face-based S-FEM was proposed for 4-node tetrahedral 3D solid element.
  - The improvement of performance is not significant.

History

2. Extension to Polygonal & Polyhedral Solid Elements; Plate & Shell Elements

- **Dai et al. (2007)**
  - An n-sided polygonal cell-based S-FEM was proposed for solid mechanics problems.
- **Nguyen-Thanh et al. (2008)**
  - A cell-based S-FEM for shell analysis was proposed.
- **Cui et al. (2009)**
  - An edge-based S-FEM for shell analysis was proposed.
- **Nguyen-Thoi et al. (2011)**
  - An n-sided polygonal edge-based S-FEM was proposed for solid mechanics problems.
- **Sohn and Im (2013)**
  - Variable-node plate and shell elements with smoothed integration was proposed.
- **Shin and Lee (2015)**
  - A strain-smoothed 3-node triangular flat shell element with drilling DOFs was proposed.
- **Nguyen-Hoang et al. (2016)**
  - A combined scheme of edge and node-based S-FEMs for shell analysis was proposed.
- **Lee et al. (2017)**
  - An n-sided polyhedral edge/node-based S-FEMs was proposed for solid mechanics problems.

2007

2017

History

3. Extension to Other Physics & Analysis

- **He et al. (2009)**
  - An edge-based S-FEM was proposed for 3D acoustic problems.
- **Bordas et al. (2010)**
  - Strain smoothing in XFEM was proposed.
- **Sohn et al. (2013)**
  - A new carving technique combined with smoothed integration was proposed.
- **Wang et al. (2015)**
  - A stable node-based smoothed finite element method for acoustic problems.
- **Jin et al. (2016)**
  - Polyhedral type variable-node elements was proposed for 3D contact analysis.
- **Eric et al. (2016)**
  - An S-FEM for analysis of multi-layered systems in biomaterials was proposed.
- **Onishi et al. (2017)**
  - An F-bar aided edge-based S-FEM was proposed.
- **Hamrani et al. (2017)**
  - A cell-based isogeometric analysis for 2D mechanics problems was proposed.
- **He (2019)**
  - A CS-FEM for the numerical simulation of viscoelastic fluid flow was proposed.

2009

2019

1. Introduction

2. Research background

3. Research topics

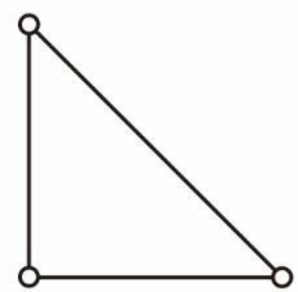
- ✓ Topic 1: Strain-smoothed linear 2D & 3D solid elements
- ✓ Topic 2: Strain-smoothed 4-node quadrilateral 2D solid element
- ✓ Topic 3: Improving the membrane behavior of the MITC3+ shell element
- ✓ Topic 4: Acoustic radiation analysis using the strain-smoothed triangular element.
- ✓ Topic 5: Variational formulation for the strain-smoothed element (SSE) method.

4. Conclusions & Future works

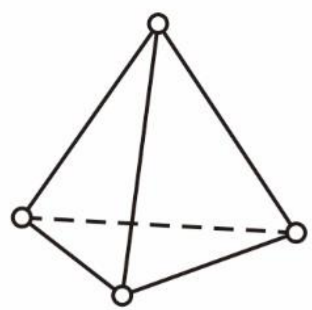
3. Research Topics

Topic 1

Strain-smoothed linear 2D & 3D solid elements



3-node triangular
2D solid element

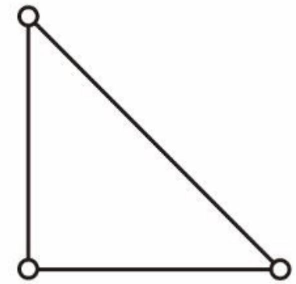


4-node tetrahedral
3D solid element

The 3-node triangular element

❖ Geometry and displacement interpolations

- $\mathbf{x} = \sum_{i=1}^{3} h_i(r,s)\mathbf{x}_i$ with $\mathbf{x}_i = [x_i \quad y_i]^T$.
- $\mathbf{u} = \sum_{i=1}^{3} h_i(r,s)\mathbf{u}_i$ with $\mathbf{u}_i = [u_i \quad v_i]^T$.
- Shape functions: $h_1 = 1 - r - s, \quad h_2 = r, \quad h_3 = s.$



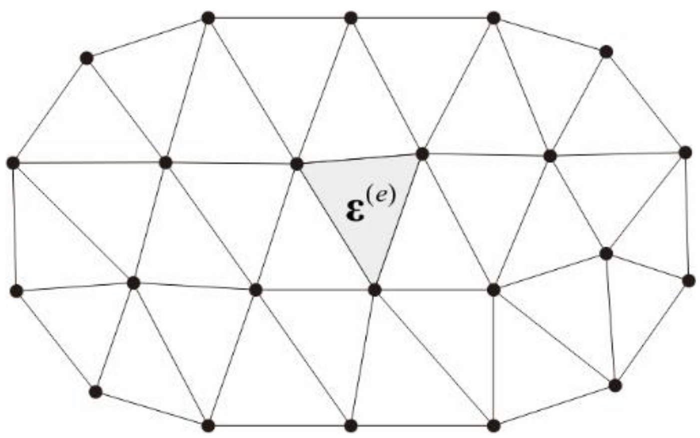
3-node triangular
2D solid element

❖ Strain field

- $\boldsymbol{\varepsilon}^{(e)} = \mathbf{B}^{(e)}\mathbf{u}^{(e)}$ with $\mathbf{u}^{(e)} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]^T$,

$$\mathbf{B}^{(e)} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3].$$

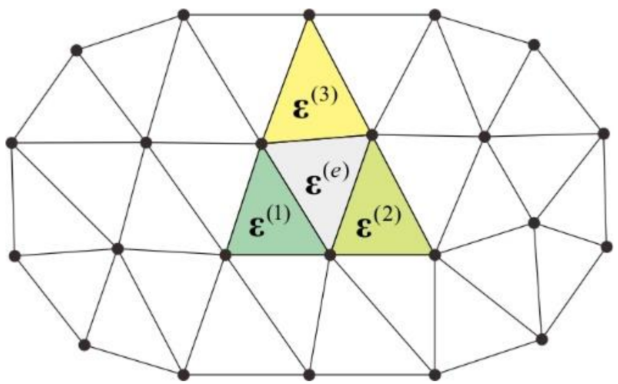
- The 3-node triangular element has a constant strain field.



- Lee C, Lee PS. A new strain smoothing method for triangular and tetrahedral finite elements. Comput Methods Appl Mech Eng 2018;341:939–955.

The 3-node element with the SSE method

❖ **The strain-smoothed element (SSE) method for the 3-node element**



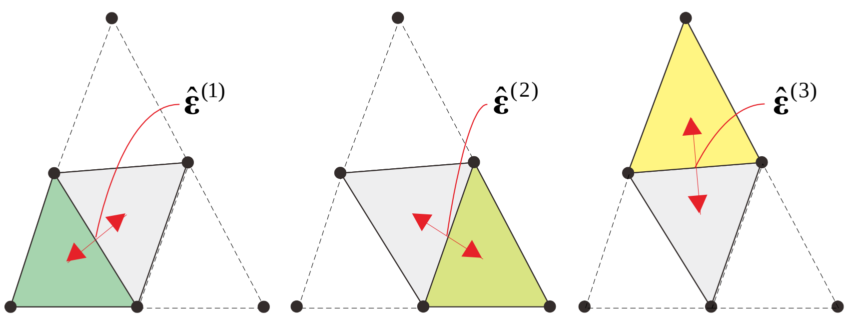
$\boldsymbol{\varepsilon}^{(e)}$: Strain of the target element.

$\boldsymbol{\varepsilon}^{(k)}$: Strain of the k^{th} neighboring element.

$A^{(e)}$: Area of the target element.

$A^{(k)}$: Area of the k^{th} neighboring element.

Step 1 of 2	Strain smoothing between the target element and each neighboring element.



- If neighboring element exists through k^{th} edge,

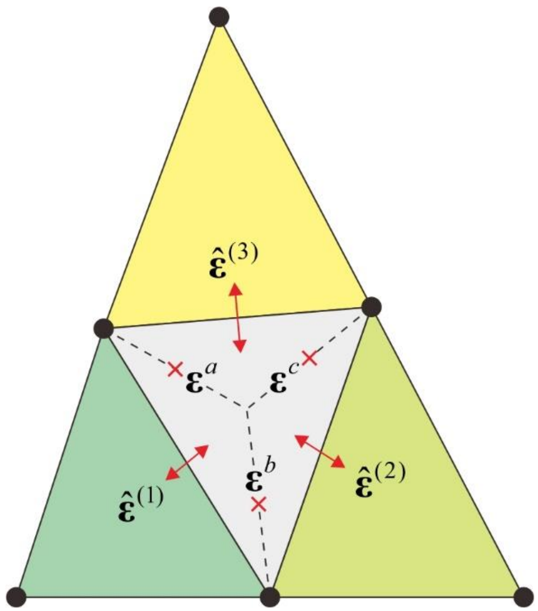
$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A^{(e)} + A^{(k)}}\left(A^{(e)}\boldsymbol{\varepsilon}^{(e)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right).$$

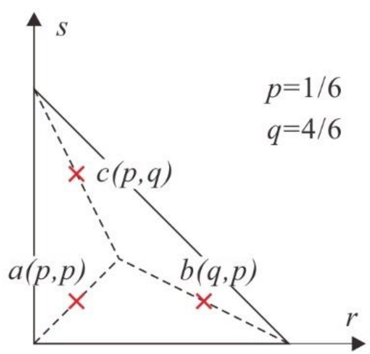
- If there is no neighboring element,

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \boldsymbol{\varepsilon}^{(e)}.$$

The 3-node element with the SSE method

Step 2 of 2 | **Construction of the smoothed strain field through Gauss points.**





- 1st strain smoothing (<u>in previous page</u>)

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A^{(e)} + A^{(k)}}\left(A^{(e)}\boldsymbol{\varepsilon}^{(e)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right).$$

- 2nd strain smoothing within elements

$$\boldsymbol{\varepsilon}^{a} = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(3)}),$$

$$\boldsymbol{\varepsilon}^{b} = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)}),$$

$$\boldsymbol{\varepsilon}^{c} = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)}).$$

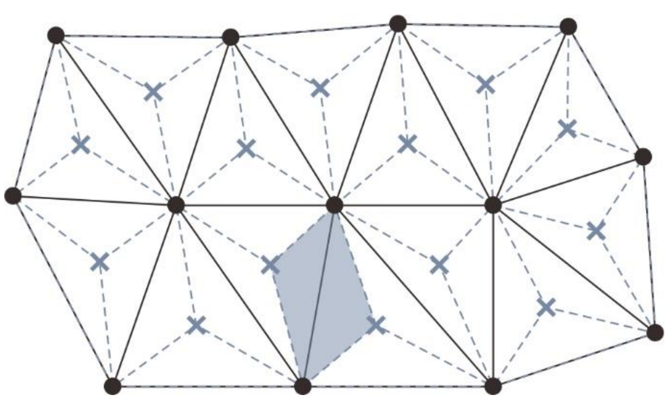
- Smoothed strain field

$$\bar{\boldsymbol{\varepsilon}}^{(e)} = \left[1 - \frac{1}{q-p}(r+s-2p)\right]\boldsymbol{\varepsilon}^{a} + \frac{r-p}{q-p}\boldsymbol{\varepsilon}^{b} + \frac{s-p}{q-p}\boldsymbol{\varepsilon}^{c}.$$

The Strain-Smoothed Element (SSE) Method

Previous strain smoothing methods

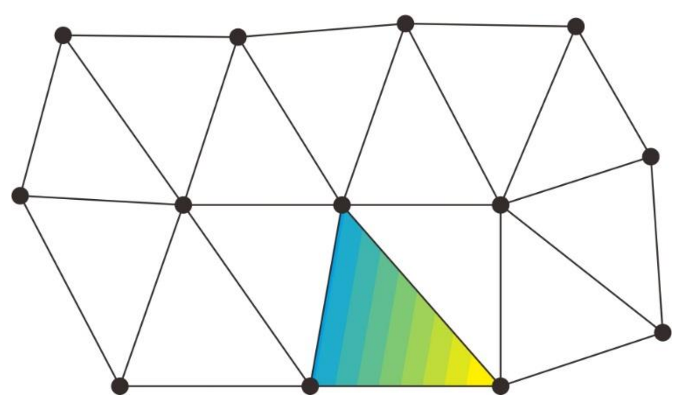
- ❖ Special smoothing domains.
- ❖ Constant strain fields.
- ❖ Some improvement in accuracy.



Edge-based smoothing domain

Strain-Smoothed Element (SSE) method

- ❖ Finite elements.
- ❖ (Bi-) linear strain fields.
- ❖ Very high accuracy.

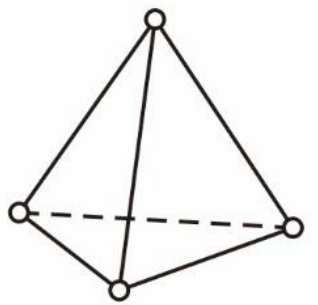


FEM domain (w/ SSE method)

The 4-node tetrahedral element

❖ Geometry and displacement interpolations

- $\mathbf{x} = \sum_{i=1}^{4} h_i(r,s,t)\mathbf{x}_i$ with $\mathbf{x}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^T$.
- $\mathbf{u} = \sum_{i=1}^{4} h_i(r,s,t)\mathbf{u}_i$ with $\mathbf{u}_i = \begin{bmatrix} u_i & v_i & w_i \end{bmatrix}^T$.
- Shape functions: $h_1 = 1 - r - s - t,\quad h_2 = r,\quad h_3 = s,\quad h_4 = t.$



4-node tetrahedral
3D solid element

❖ Strain field

- $\boldsymbol{\varepsilon}^{(e)} = \mathbf{B}^{(e)}\mathbf{u}^{(e)}$ with $\mathbf{u}^{(e)} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4]^T$,

$$\mathbf{B}^{(e)} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}.$$

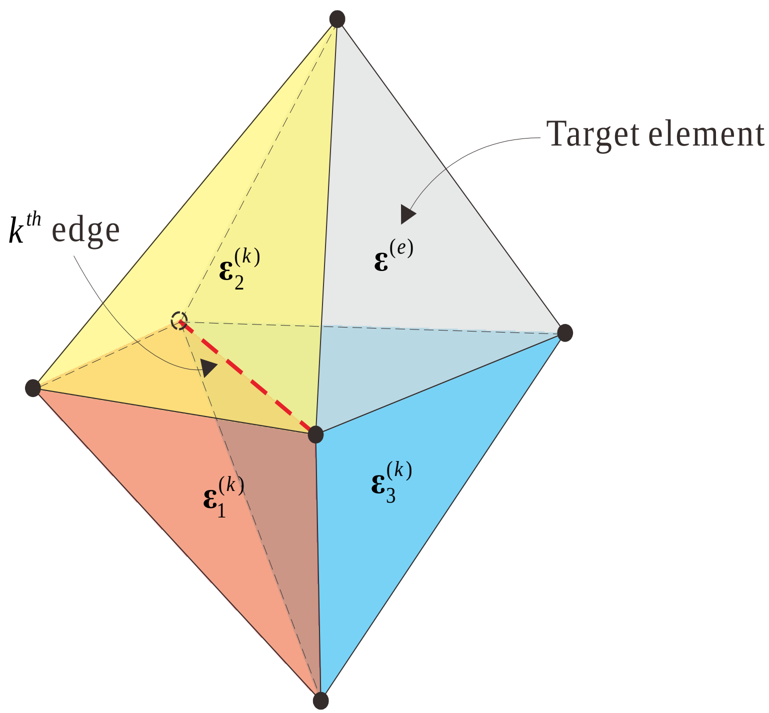
- The 4-node tetrahedral element has a constant strain field.

• Lee C, Lee PS. A new strain smoothing method for triangular and tetrahedral finite elements. Comput Methods Appl Mech Eng 2018;341:939–955.

The 4-node element with the SSE method

❖ The strain-smoothed element (SSE) method for the 4-node element

Step 1 of 2	Strain smoothing between the target element and each neighboring element.



$\boldsymbol{\varepsilon}^{(e)}$: Strain of the target element.

$\boldsymbol{\varepsilon}_i^{(k)}$: Strain of the i^{th} element neighboring the k^{th} edge of the target element.

- If neighboring element exists through k^{th} edge,

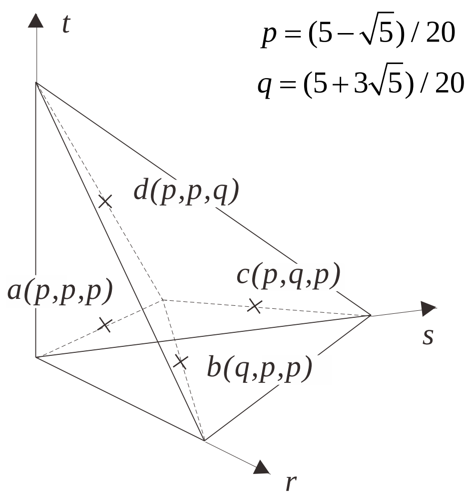
$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{V^{(e)} + \sum_{i=1}^{n_k} V_i^{(k)}} \left(V^{(e)}\boldsymbol{\varepsilon}^{(e)} + \sum_{i=1}^{n_k} V_i^{(k)}\boldsymbol{\varepsilon}_i^{(k)}\right).$$

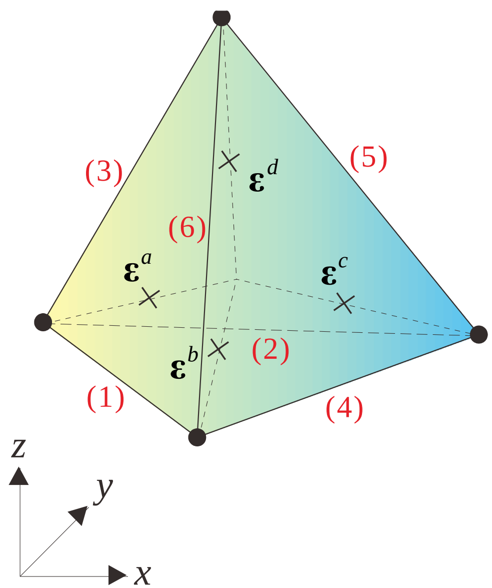
- If there is no neighboring element,

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \boldsymbol{\varepsilon}^{(e)}.$$

The 4-node element with the SSE method

Step 2 of 2 | **Construction of the smoothed strain field through Gauss points.**





- Strain smoothing within elements

$$\boldsymbol{\varepsilon}^a = \frac{1}{5}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)} + \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{(e)}),$$

$$\boldsymbol{\varepsilon}^b = \frac{1}{5}(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(4)} + \hat{\boldsymbol{\varepsilon}}^{(6)} + \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{(e)}),$$

$$\boldsymbol{\varepsilon}^c = \frac{1}{5}(\hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(4)} + \hat{\boldsymbol{\varepsilon}}^{(5)} + \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{(e)}),$$

$$\boldsymbol{\varepsilon}^d = \frac{1}{5}(\hat{\boldsymbol{\varepsilon}}^{(3)} + \hat{\boldsymbol{\varepsilon}}^{(5)} + \hat{\boldsymbol{\varepsilon}}^{(6)} + \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{(e)}),$$

$$\hat{\boldsymbol{\varepsilon}} = \frac{1}{6}\sum_{k=1}^{6}\hat{\boldsymbol{\varepsilon}}^{(k)}.$$

- Smoothed strain field

$$\bar{\boldsymbol{\varepsilon}}^{(e)} = \left[1 - \frac{1}{q-p}(r+s+t-3p)\right]\boldsymbol{\varepsilon}^a + \frac{r-p}{q-p}\boldsymbol{\varepsilon}^b + \frac{s-p}{q-p}\boldsymbol{\varepsilon}^c + \frac{t-p}{q-p}\boldsymbol{\varepsilon}^d.$$

Basic numerical tests

❖ Zero energy mode test

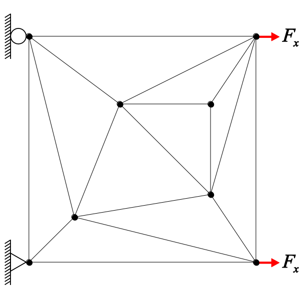
- The number of zero eigenvalues of the stiffness matrix of unsupported elements is counted.
- <u>The 2D and 3D solid elements should have three and six zero eigenvalues, respectively.</u>

❖ Isotropic element test

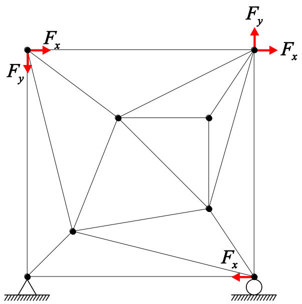
- The finite elements should give the same results regardless of the node numbering sequences used.

❖ Patch tests

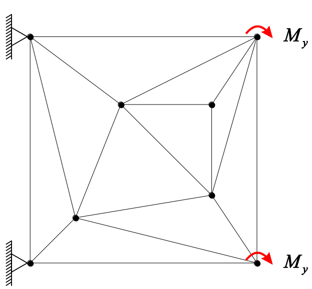
- The minimum number of DOFs is constrained to prevent rigid body motions.
- Proper loadings are applied to produce a constant stress field.
- To satisfy the patch tests, <u>a constant stress value should be obtained at every point on elements.</u>



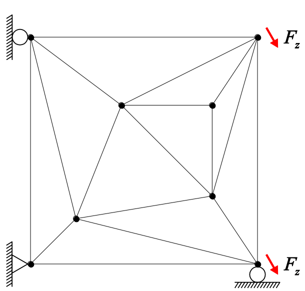
Membrane patch test I



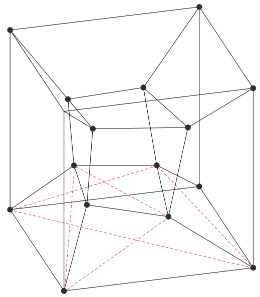
Membrane patch test II



Bending patch test



Shearing patch test



Mesh for 3D patch test

Numerical examples (2D)

❖ Finite elements considered

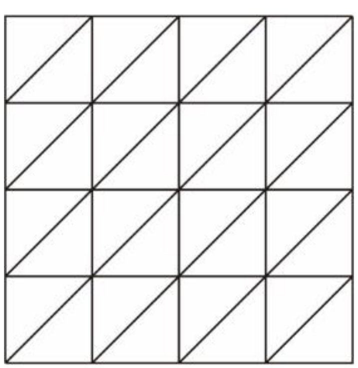
- **Standard FEM** : 3-node triangular element
- **ES-FEM** : 3-node triangular element with the edge-based S-FEM
- **SSE (proposed)** : 3-node triangular element with the SSE method

❖ Evaluation method

- Convergence curves obtained using the energy norm $E_e^2 = \dfrac{\left| \|\mathbf{u}_{ref}\|_e^2 - \|\mathbf{u}_h\|_e^2 \right|}{\|\mathbf{u}_{ref}\|_e^2}$ with $\|\mathbf{u}\|_e^2 = \int_\Omega \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega.$
- Displacements and stresses.

❖ Reference solution

❖ Reference solutions are calculated using a 64×64 regular mesh of 9-node 2D solid elements.



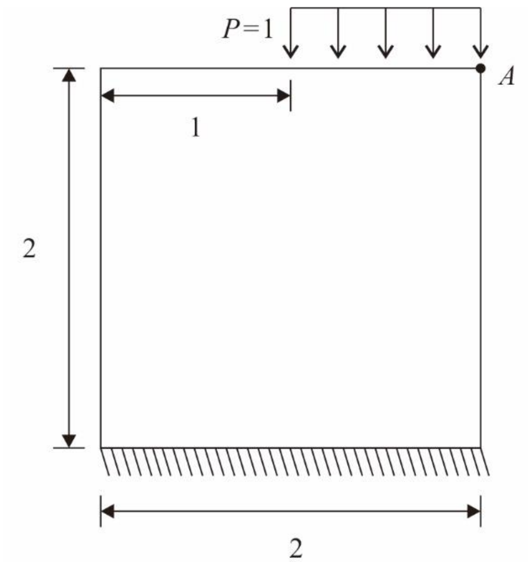
A mesh of $N\times N$ elements ($N = 4$)

N	Test elements	Reference
2	18	50
4	50	162
8	162	578
16	578	2,178
	....	
64	8,450	33,282

Degrees of freedom (DOFs) per N

Numerical examples (2D)

1) 2D block problem

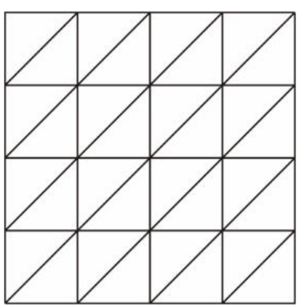


- **Force**

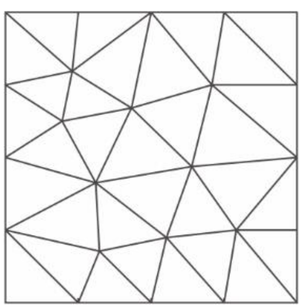
  Distributed compression force $P = 1$.
- **Boundary condition**

  Bottom edge is clamped.
- **Material property (plane stress condition)**

  $E = 3\times10^7, \quad \nu = 0.3, \quad \rho = 1\times10^7.$



Regular mesh



Distorted mesh

- **Regular meshes** ($N \times N$ elements)

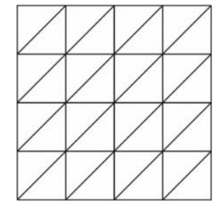
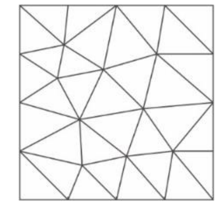
  N = 2, 4, 8, 16.
- **Distorted meshes** (N_e elements)

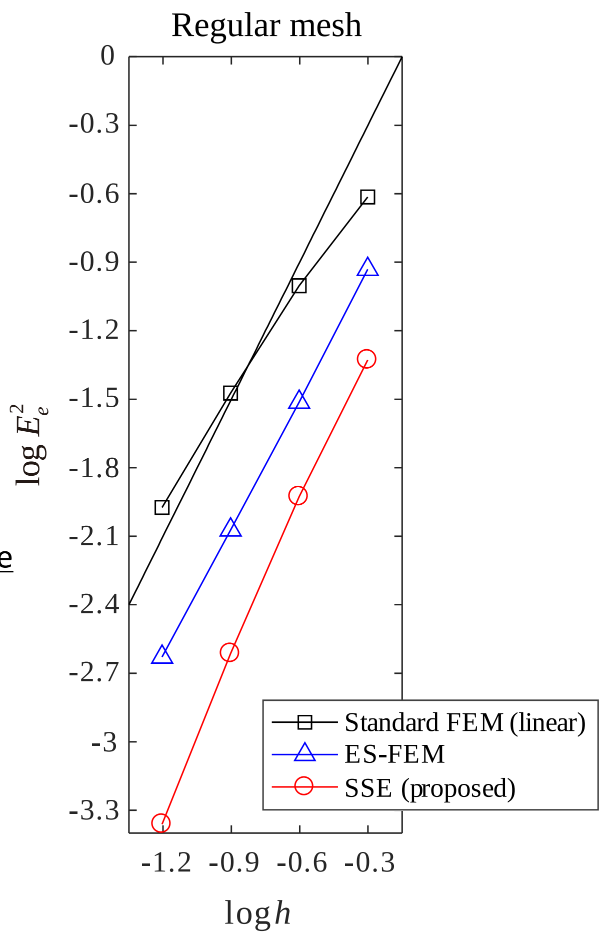
  N_e = 6, 32, 128, 500.
- The distorted meshes are acquired through the commercial software ANSYS.

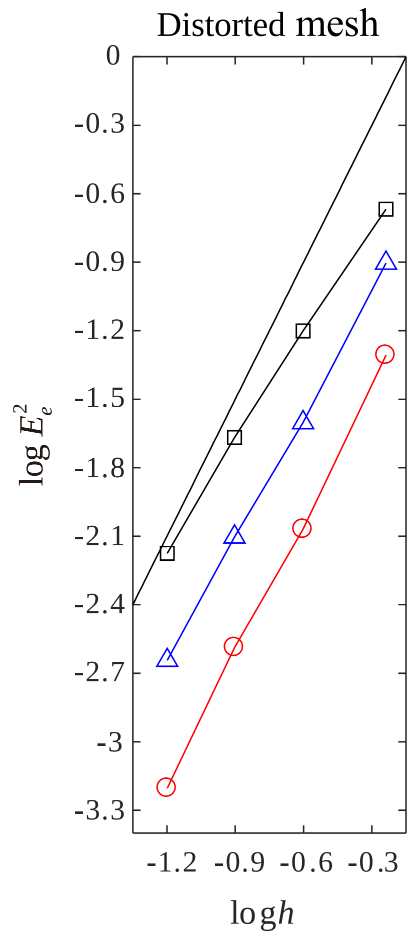
Numerical examples (2D)

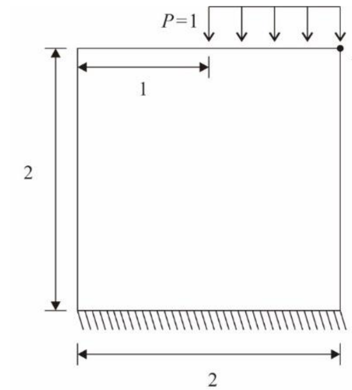
1) 2D block problem

- **Convergence curves.**







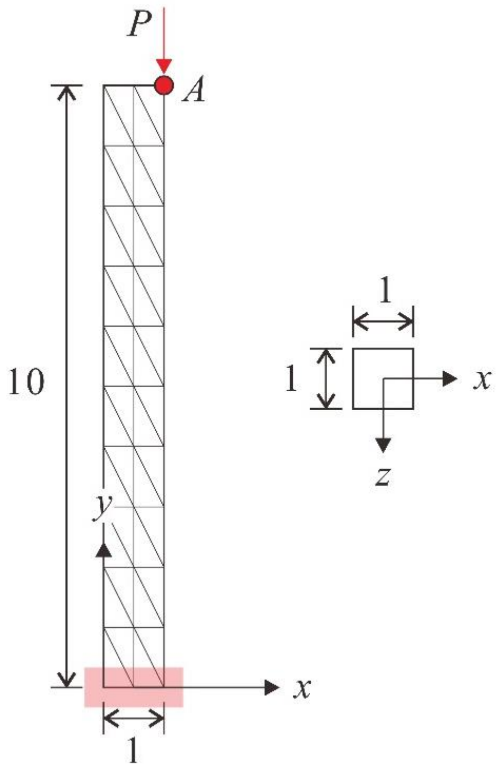
<u>More accurate</u>

<u>More elements</u>

$h=1/N$

Numerical examples (2D)

2) Column under a compressive load problem



- **Force**

  Compressive load $P_{\max} = 5\times10^{3}$.

- **Boundary condition**

  Bottom edge is clamped.

- **Material property (plane stress condition)**

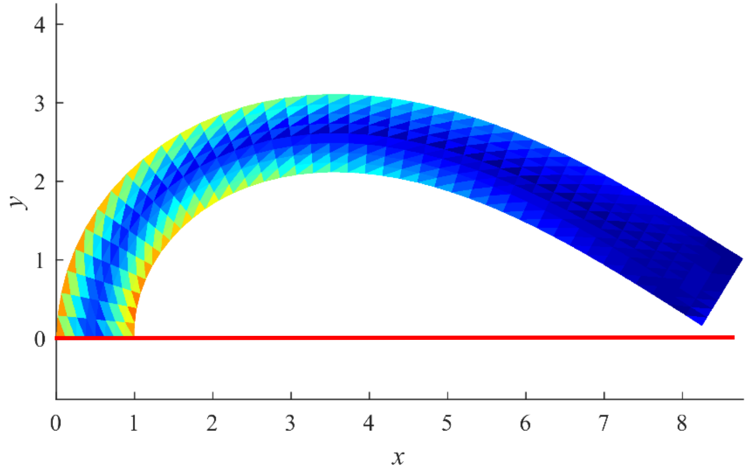
  $E = 10^{6}, \quad \nu = 0.$

- **Regular meshes** ($N \times 5N$ elements)

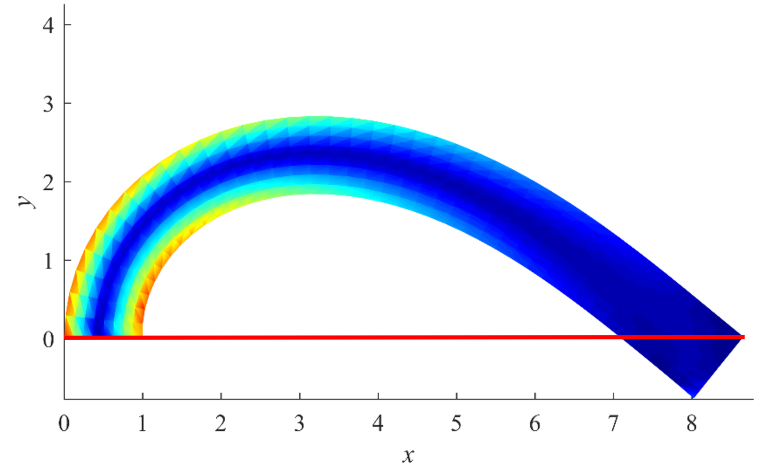
  N = 8, 16.

2) Column under a compressive load problem

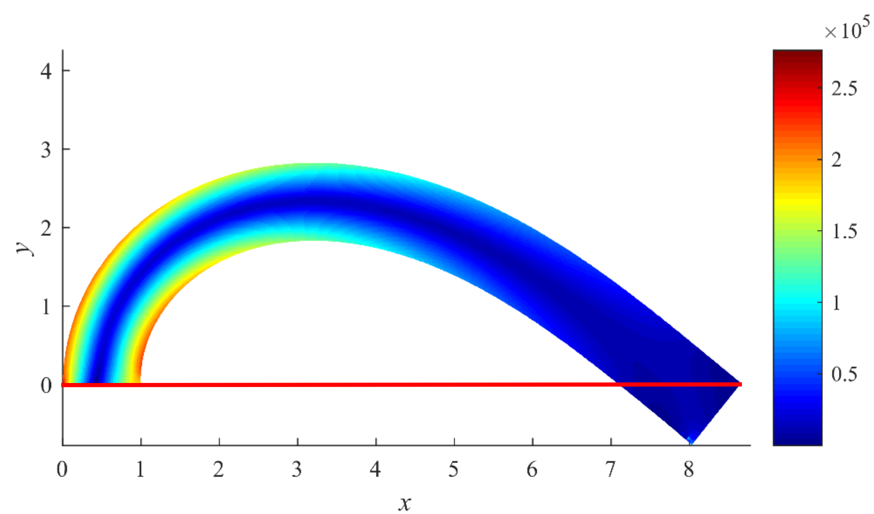
- **von Mises stress distributions for the regular mesh** **(N = 8).**



Standard FEM



SSE (proposed)



Reference

Numerical examples (3D)

❖ Finite elements considered

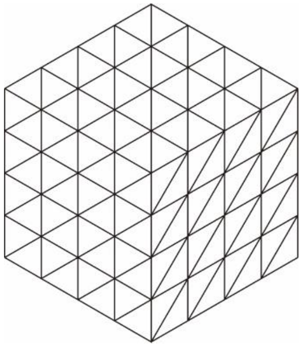
- **Standard FEM** : 4-node tetrahedral element
- **FS-FEM** : 4-node tetrahedral element with the face-based S-FEM
- **ES-FEM** : 4-node tetrahedral element with the edge-based S-FEM
- **SSE (proposed)** : 4-node tetrahedral element with the SSE method

❖ Evaluation method

- Convergence curves obtained using the energy norm $E_e^2 = \dfrac{\left| \|\mathbf{u}_{ref}\|_e^2 - \|\mathbf{u}_h\|_e^2 \right|}{\|\mathbf{u}_{ref}\|_e^2}$ with $\|\mathbf{u}\|_e^2 = \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega.$
- Displacements and stresses.

❖ Reference solution

- Reference solutions are calculated using a 16×16×16 regular mesh of 27-node 3D solid elements.



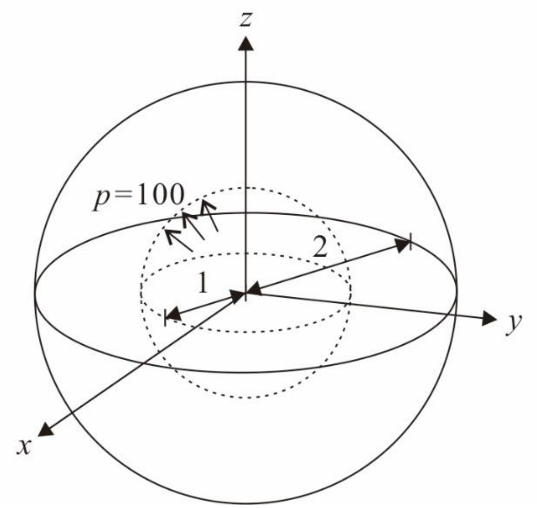
A mesh of $N \times N \times N$ elements ($N = 4$)

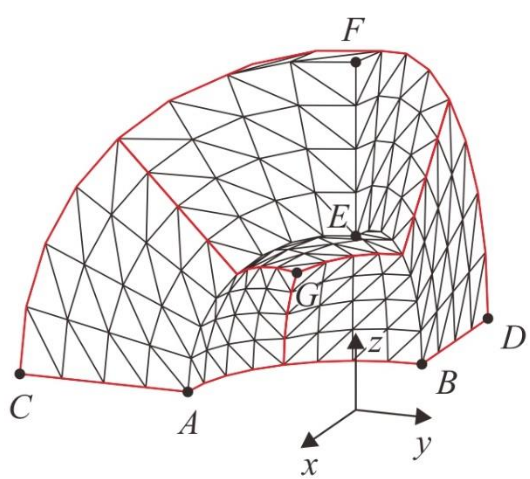
N	Test elements	Reference
2	81	375
4	375	2,187
8	2,187	14,739
16	14,739	107,811

Degrees of freedom (DOFs) per N

Numerical examples (3D)

1) Lame problem





- **Force**

  Internal pressure $p = 1$.
- **Boundary condition**

  Symmetric boundary conditions are imposed.
- **Material property**

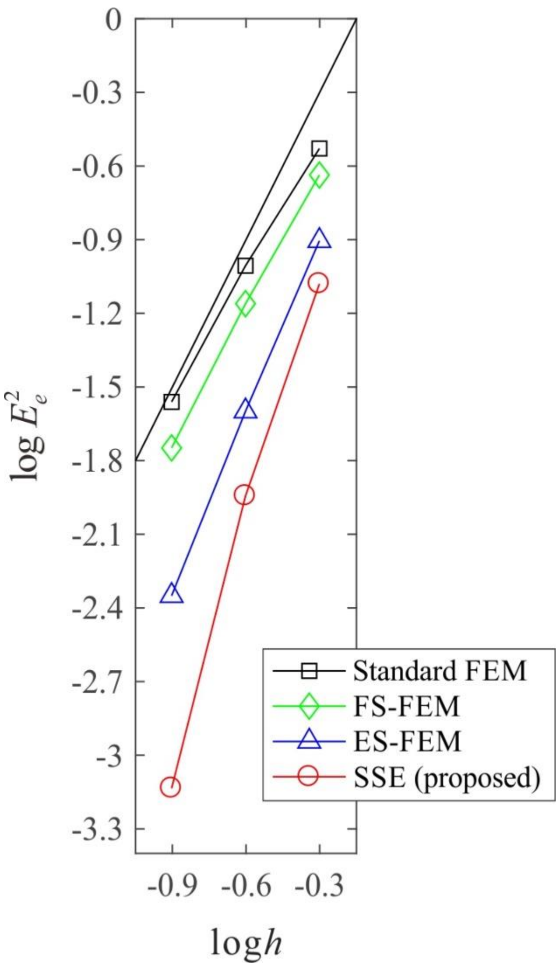
  $E = 1\times 10^{3}, \quad \nu = 0.3.$
- **Regular meshes** ($N\times N\times N$ elements)

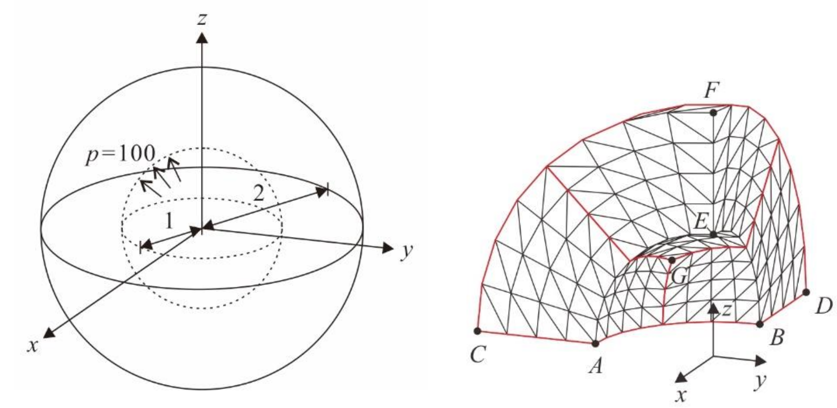
  $N =$2, 4, 8.

Numerical examples (3D)

1) Lame problem

- **Convergence curves.**





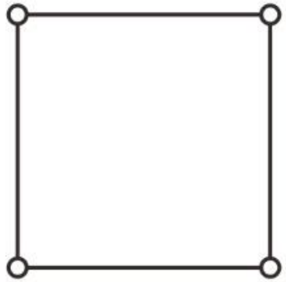
- **von Mises stress at point *G*.**

N	Standard FEM	FS-FEM	ES-FEM	SSE (proposed)
2	106.2858 (38.00)	125.3865 (26.86)	150.2877 (12.33)	166.7563 (2.73)
4	132.7286 (22.58)	143.6493 (16.20)	157.3288 (8.22)	166.3370 (2.97)
8	149.4181 (12.84)	155.2355 (9.45)	162.2212 (5.37)	166.6823 (2.77)
16	159.4205 (7.00)	162.5707 (5.17)	167.5228 (2.28)	169.5621 (1.09)

- Reference solution: 171.4286
- The values in () indicate relative errors (%).

Topic 2

Strain-smoothed 4-node quadrilateral 2D solid element



4-node quadrilateral
2D solid element

The 4-node quadrilateral element

- Smoothing operation

$$\tilde{\boldsymbol{\varepsilon}}_k = \int_{\Omega^{(k)}} \boldsymbol{\varepsilon}(\mathbf{x})\Phi_k(\mathbf{x})d\Omega.$$

✓ valid for strains with constant values.

❖ Standard 4-node quadrilateral element

✓ **Geometry and displacement interpolations**

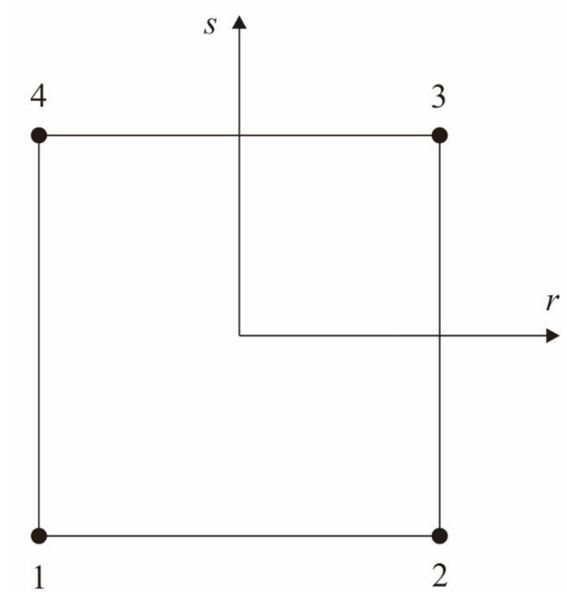
- $\mathbf{x} = \sum_{i=1}^{4} \hat{h}_i(r,s)\mathbf{x}_i$ with $\mathbf{x}_i = [x_i \quad y_i]^T$.
- $\mathbf{u} = \sum_{i=1}^{4} \hat{h}_i(r,s)\mathbf{u}_i$ with $\mathbf{u}_i = [u_i \quad v_i]^T$.

✓ **Strain field**

- $\boldsymbol{\varepsilon}^{(m)} = \nabla \mathbf{u}^{(m)}$ with $\mathbf{u}^{(m)} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4]^T$.
- $\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\mathbf{u}^{(m)}$ with $\mathbf{B}^{(m)} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \mathbf{B}_4]$.
- The 4-node 2D element has a non-constant strain field due to *rs* term in shape functions .

✓ **Standard bilinear shape functions:**

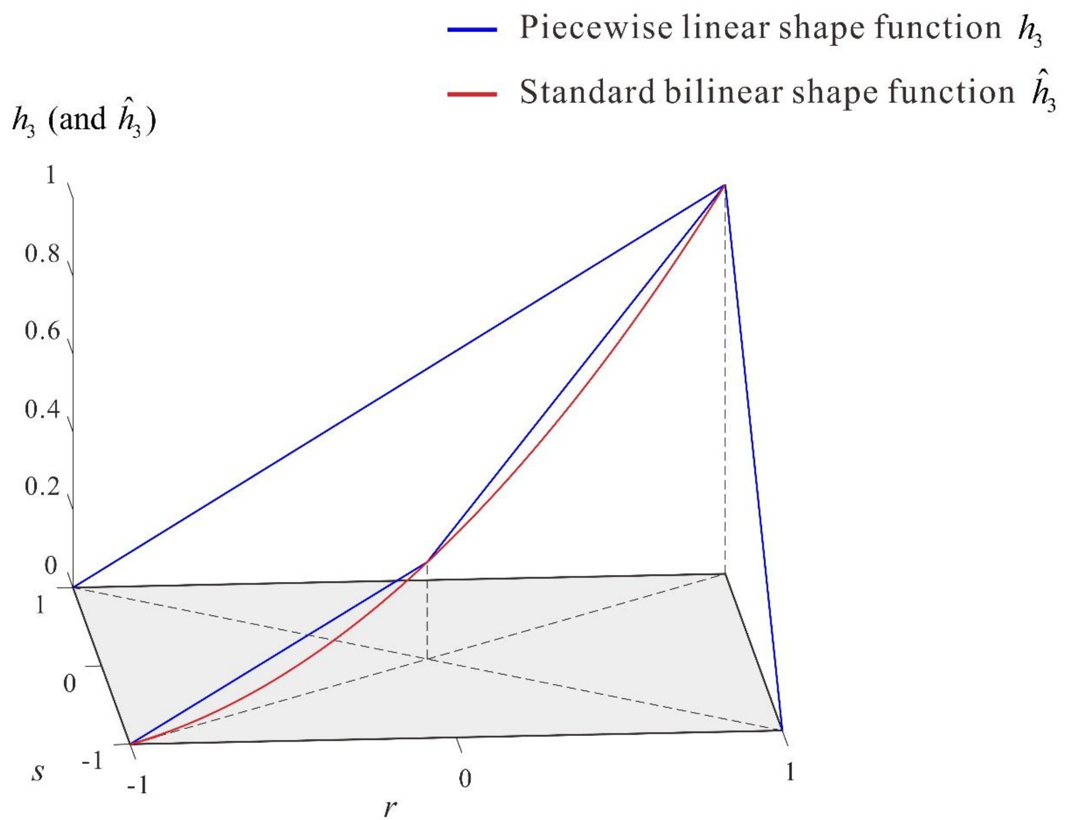
$$\hat{h}_1 = (1-r)(1-s)/4, \quad \hat{h}_2 = (1+r)(1-s)/4,$$
$$\hat{h}_3 = (1+r)(1+s)/4, \quad \hat{h}_4 = (1-r)(1+s)/4.$$



The strain smoothing method cannot be applied to standard 4-node element.

The 4-node element

❖ **Comparison of the piecewise linear & standard bilinear shape functions**



- The two shape functions corresponding to node 3 along element edges and a diagonal $r = s$ are depicted.
- The two shape functions show different variations along the diagonal.

The 4-node element

- Smoothing operation

$$\tilde{\boldsymbol{\varepsilon}}_k = \int_{\Omega^{(k)}} \boldsymbol{\varepsilon}(\mathbf{x})\Phi_k(\mathbf{x})d\Omega.$$

✓ valid for strains with constant values.

❖ Modified 4-node quadrilateral element

- The element domain is subdivide into four non-overlapping triangular domains (from T1 to T4).

✓ **Geometry and displacement interpolations**

- $\mathbf{x} = \sum_{i=1}^{4} h_i(r,s)\mathbf{x}_i$ with $\mathbf{x}_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^T$.
- $\mathbf{u} = \sum_{i=1}^{4} h_i(r,s)\mathbf{u}_i$ with $\mathbf{u}_i = \begin{bmatrix} u_i & v_i \end{bmatrix}^T$.

✓ **Strain field**

- ${}^k\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\mathbf{u}^{(m)}$ with $k = 1, 2, 3, 4$,

$${}^k\mathbf{B}^{(m)} = \begin{bmatrix} {}^k\mathbf{B}_1 & {}^k\mathbf{B}_2 & {}^k\mathbf{B}_3 & {}^k\mathbf{B}_4 \end{bmatrix},$$

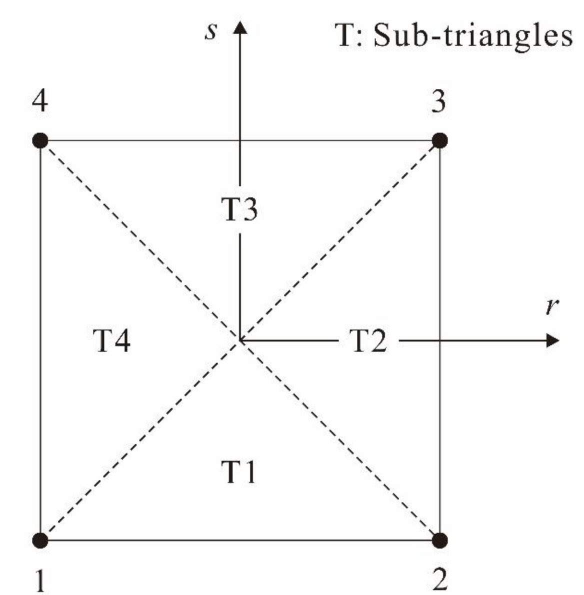
$$\mathbf{u}^{(m)} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4]^T.$$

- The 4-node 2D element has piecewise constant strain fields defined for sub-triangles.

✓ **Piecewise linear shape functions (on T1):**

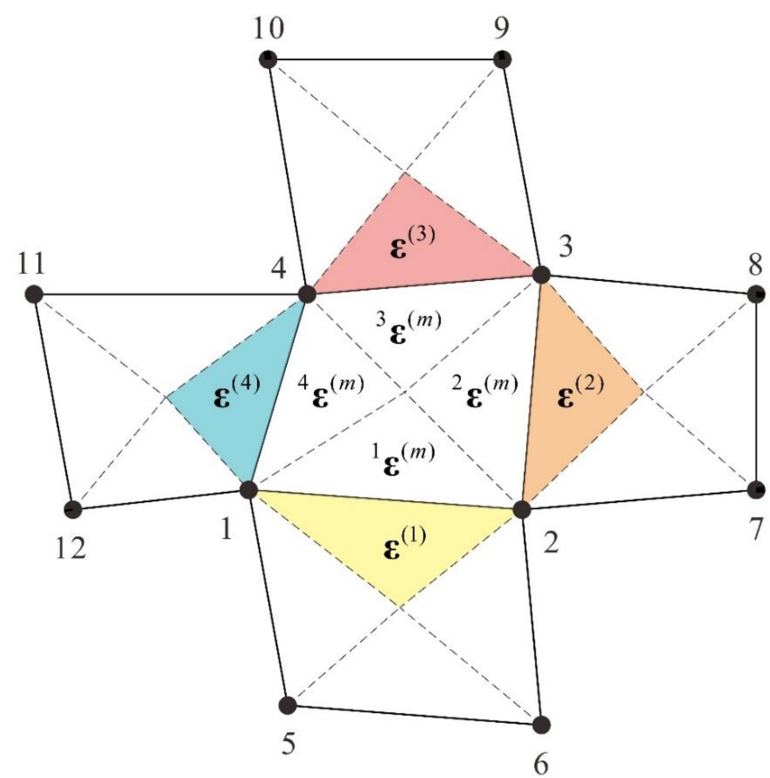
$$h_1 = (1-2r-s)/4, \quad h_2 = (1+2r-s)/4,$$

$$h_3 = (1+s)/4, \quad h_4 = (1+s)/4.$$



❖ The SSE method for the 4-node element

Step 1 of 2	Strain smoothing between the target element and each neighboring element.



${}^{k}\boldsymbol{\varepsilon}^{(m)}$: Strain of the k^{th} sub-triangle of the the target element.

$\boldsymbol{\varepsilon}^{(k)}$: Strain of the sub-triangle (belonging to neighboring element) through the k^{th} edge of the target element.

- If neighboring element exists through k^{th} edge,

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = \frac{1}{A_k^{(m)} + A^{(k)}}\left(A_k^{(m)}\, {}^{k}\boldsymbol{\varepsilon}^{(m)} + A^{(k)}\boldsymbol{\varepsilon}^{(k)}\right).$$

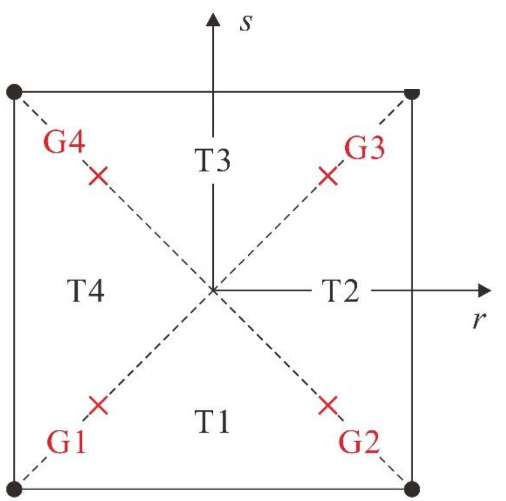
- If there is no neighboring element,

$$\hat{\boldsymbol{\varepsilon}}^{(k)} = {}^{k}\boldsymbol{\varepsilon}^{(m)}.$$

The 4-node element with the SSE method

Step 2 of 2 — **Construction of the smoothed strain field through Gauss points.**

T: Sub-triangles, G: Gauss points



- Strain smoothing within elements

$$\bar{\boldsymbol{\varepsilon}}_1 = \frac{1}{A_4^{(m)} + A_1^{(m)}}\left(A_4^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(4)} + A_1^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(1)}\right) \quad \text{for G1},$$

$$\bar{\boldsymbol{\varepsilon}}_2 = \frac{1}{A_1^{(m)} + A_2^{(m)}}\left(A_1^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(1)} + A_2^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(2)}\right) \quad \text{for G2},$$

$$\bar{\boldsymbol{\varepsilon}}_3 = \frac{1}{A_2^{(m)} + A_3^{(m)}}\left(A_2^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(2)} + A_3^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(3)}\right) \quad \text{for G3},$$

$$\bar{\boldsymbol{\varepsilon}}_4 = \frac{1}{A_3^{(m)} + A_4^{(m)}}\left(A_3^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(3)} + A_4^{(m)}\,\hat{\boldsymbol{\varepsilon}}^{(4)}\right) \quad \text{for G4}.$$

- Smoothed strain field

$$\bar{\boldsymbol{\varepsilon}}^{(m)} = \sum_{i=1}^{4} \bar{h}_i(r,s)\bar{\boldsymbol{\varepsilon}}_i \qquad \text{with} \qquad \bar{h}_i(r,s) = \frac{3}{4}\left(\frac{1}{\sqrt{3}} - \eta_i r\right)\left(\frac{1}{\sqrt{3}} - \zeta_i r\right),$$

$$\begin{bmatrix}\eta_1 & \eta_2 & \eta_3 & \eta_4\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}, \qquad \begin{bmatrix}\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\end{bmatrix} = \begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix}.$$

Numerical examples (2D)

❖ Finite elements considered

- **Q4** : 4-node quadrilateral element
- **ES-Q4** : 4-node quadrilateral element with the edge-based S-FEM
- **ICM-Q4** : incompatible modes 4-node quadrilateral element
- **SSE-Q4 (proposed)** : strain-smoothed 4-node quadrilateral element (SSE method)

❖ Evaluation method

- Convergence curves obtained using the energy norm $E_e^2 = \frac{\left| \|\mathbf{u}_{ref}\|_e^2 - \|\mathbf{u}_h\|_e^2 \right|}{\|\mathbf{u}_{ref}\|_e^2}$ with $\|\mathbf{u}\|_e^2 = \int_\Omega \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega.$
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❖ Reference solution

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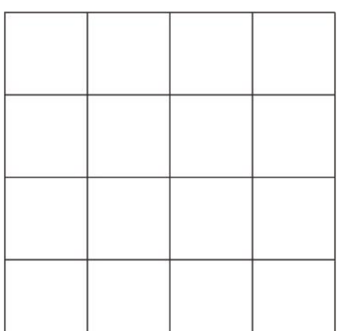

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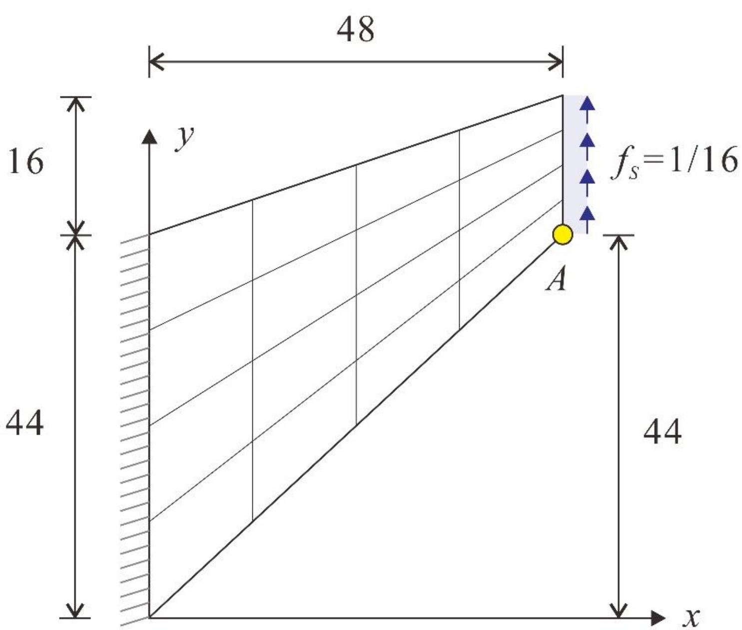
8	162	578
16	578	2,178
64	8,450	33,282

Degrees of freedom (DOFs) per N

1) Cook's skew beam problem



- **Force**

  Distributed shearing force $f_s = 1/16$.

- **Boundary condition**

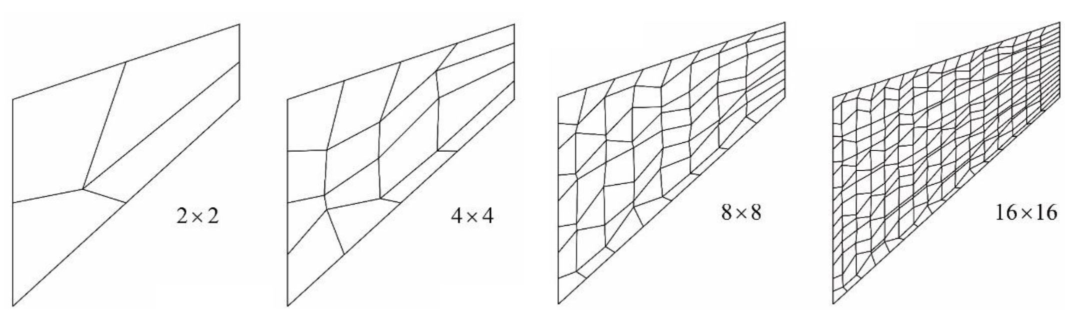
  Left edge is clamped.

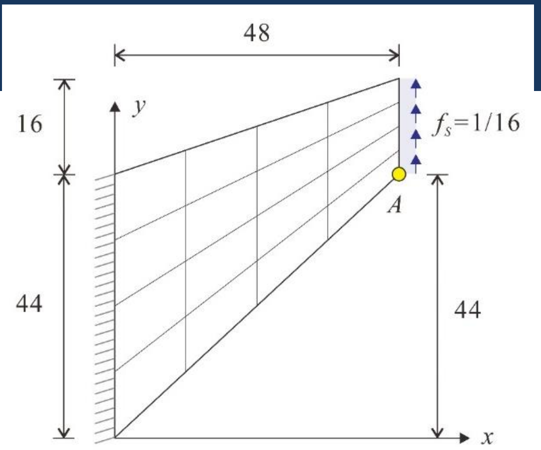
- **Material property (plane stress condition)**

  $E = 3 \times 10^7, \quad \nu = 0.3.$

- **Regular and distorted meshes** ($N \times N$ elements)

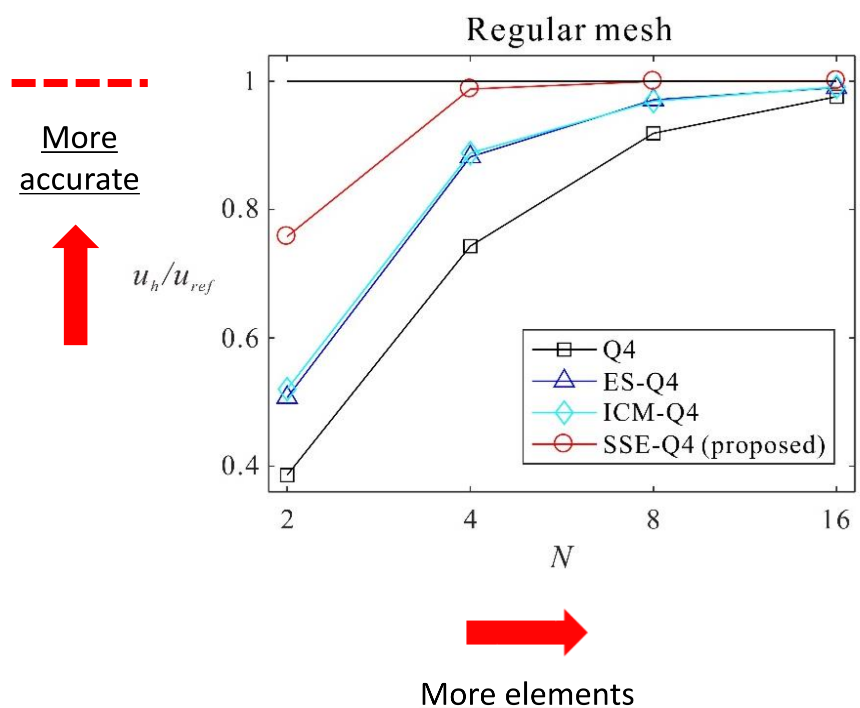
  $N =$2, 4, 8, 16.

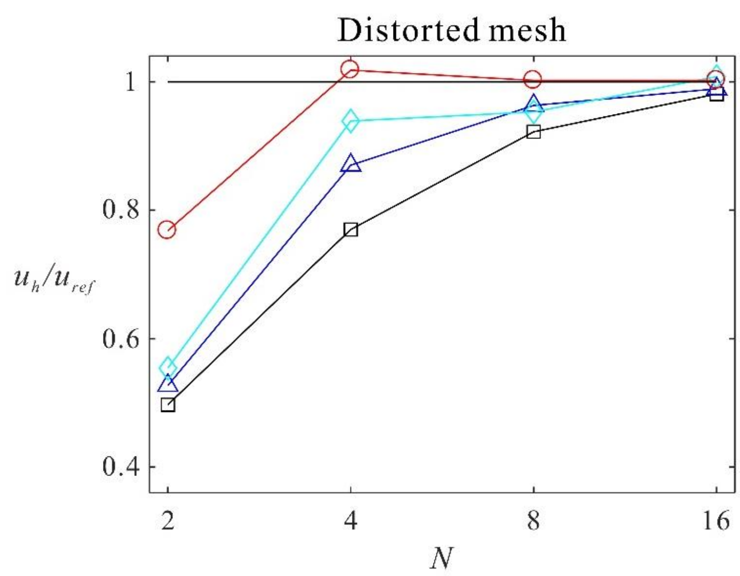




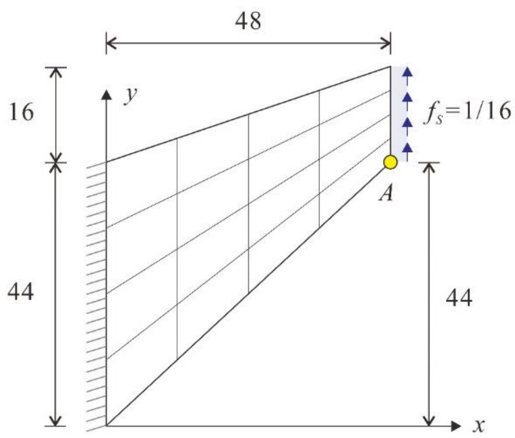
1) Cook's skew beam problem

- **Normalized horizontal displacements (u_h / u_{ref}) at point *A*.**





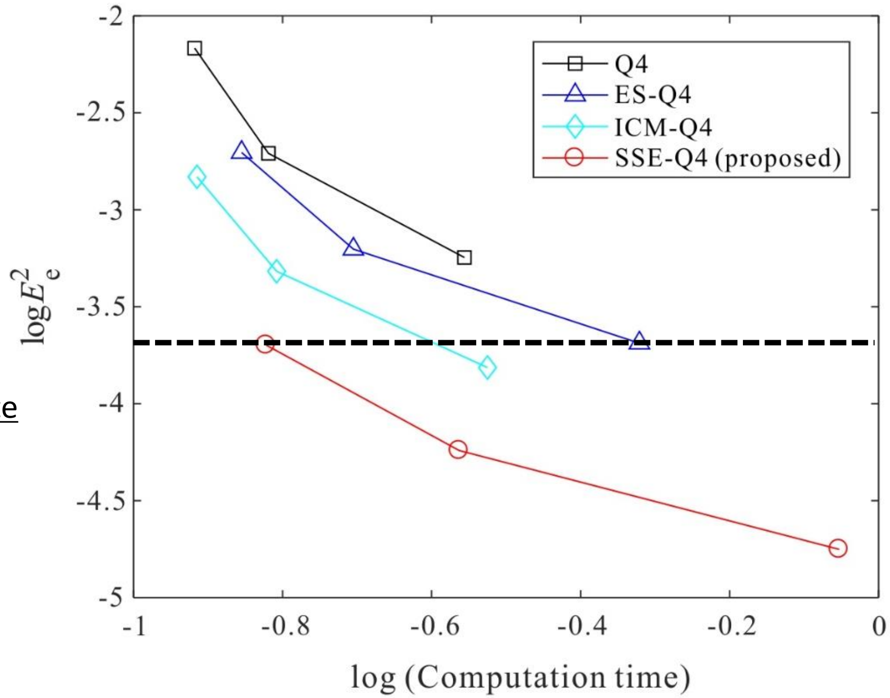
Numerical examples (2D)



1) Cook's skew beam problem

- **Computational efficiency curves.**

More accurate

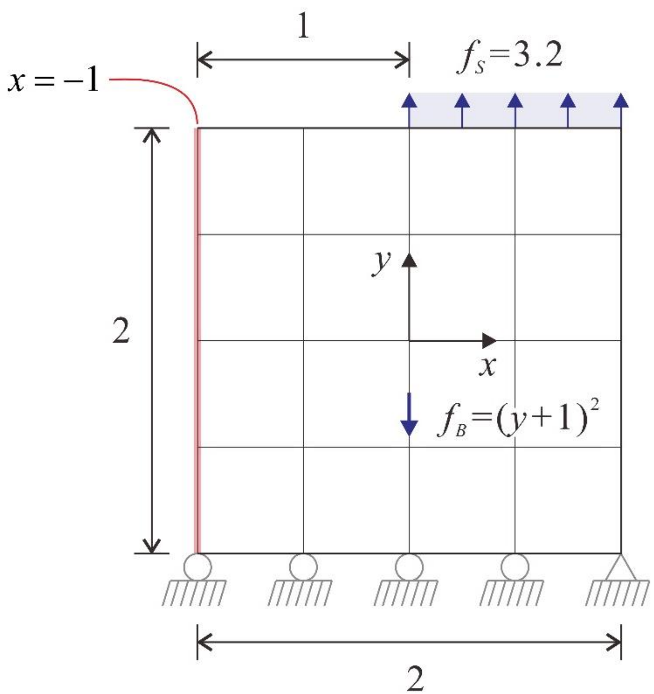


More computation time

- Computation times taken from obtaining stiffness matrices to solving linear equations are measured.
- Computations are performed in a PC with Intel Core i7-6700, 3.40GHz CPU and 64GB RAM.
- The CSR format is used for storing matrices and Intel MKL PARDISO is used for solving linear equations.

Numerical examples (2D)

2) Block under complex forces problem



- **Force**

  Compressive body force $f_B = (y+1)^2$ and

  eccentric tensile traction $f_s = 3.2$.

- **Boundary condition**

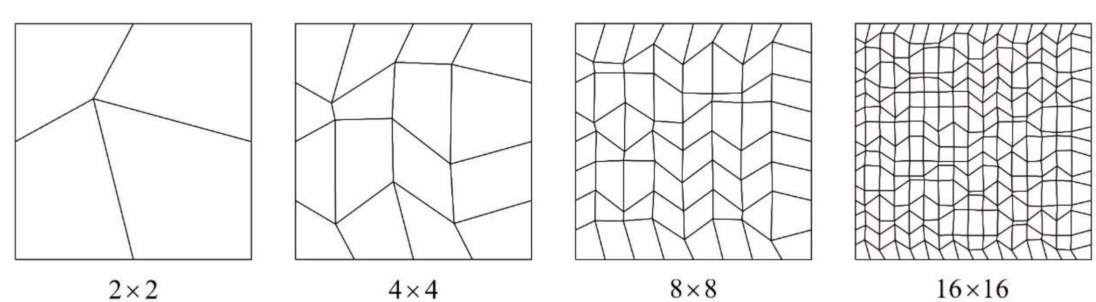
  The block is supported along its bottom.

- **Material property (plane stress condition)**

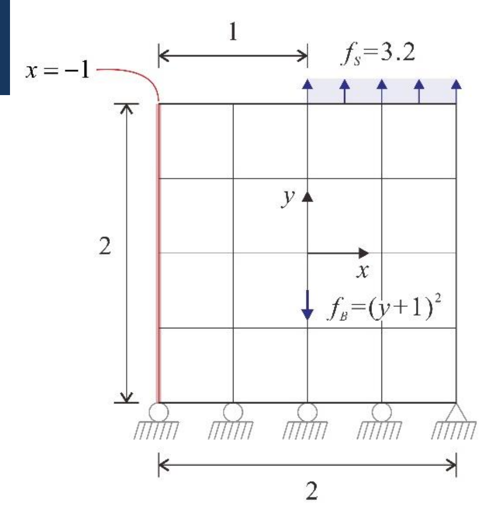
  $E = 3\times10^7$, $\nu = 0.25$.

- **Regular and distorted meshes** ($N \times N$ elements)

  $N =$2, 4, 8, 16.

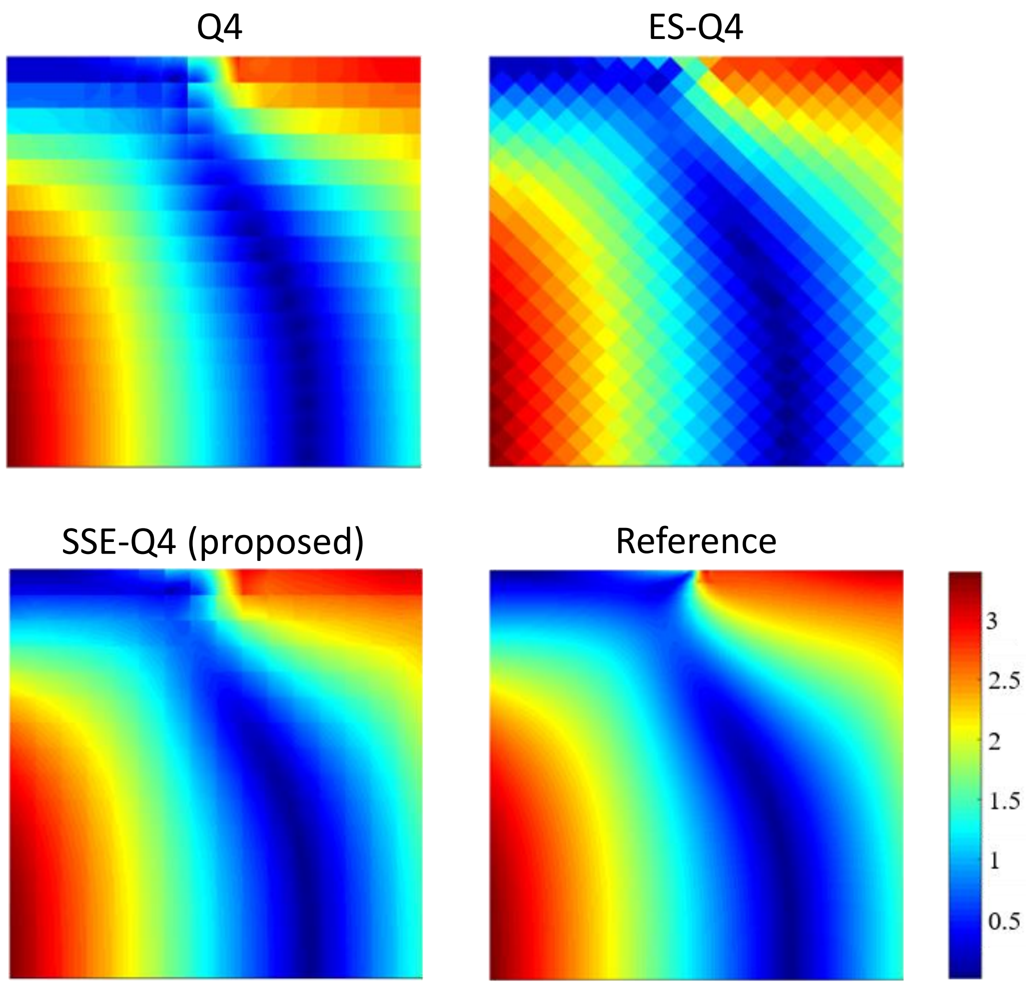


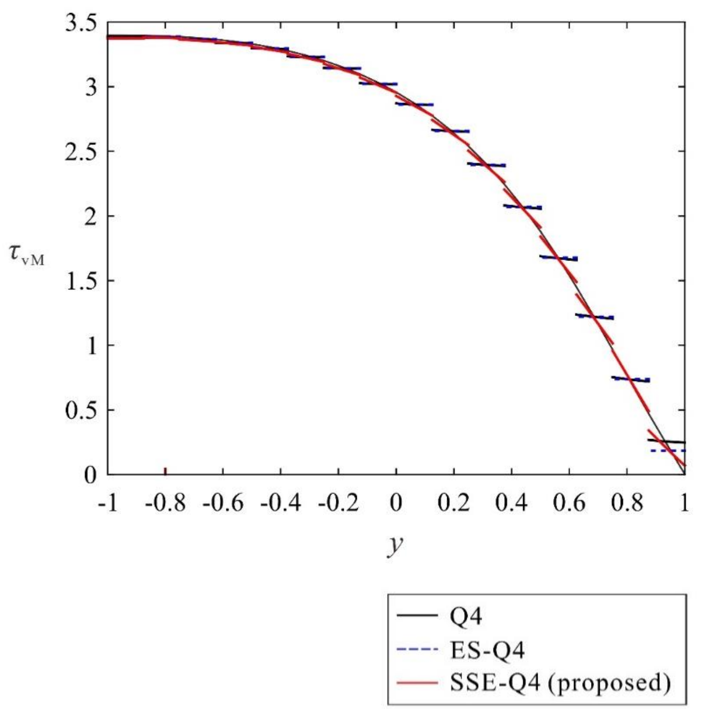
Numerical examples (2D)



2) Block under complex forces problem

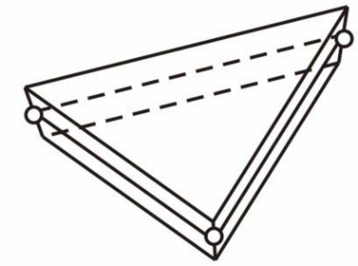
- von Mises stress distributions for the regular mesh **(*N* = 16).**





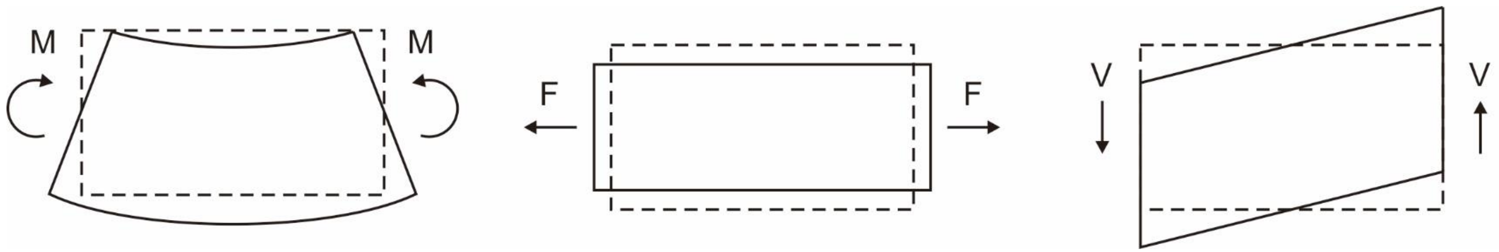
Topic 3

Strain-smoothed MITC3+ shell element for geometric nonlinear analysis



3-node triangular shell element

❖ Shell behaviors



Bending Membrane Transverse shear

❖ Asymptotic behaviors

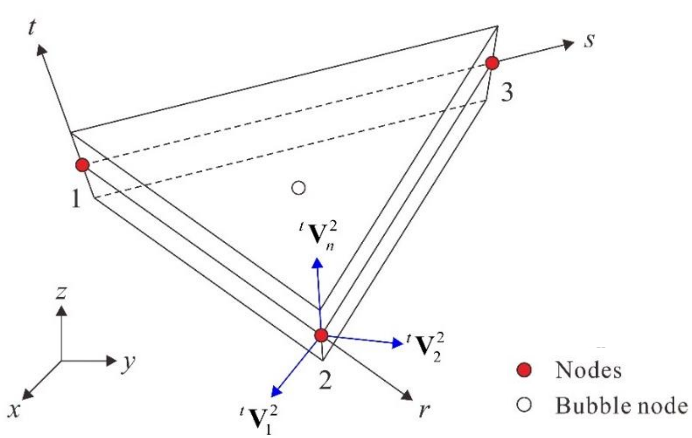
- The behaviors of shell converges to a specific limit state as the thickness becomes small.
- Three different asymptotic categories:
  - ✓ Bending-dominated, membrane-dominated, mixed behaviors.

❖ Locking

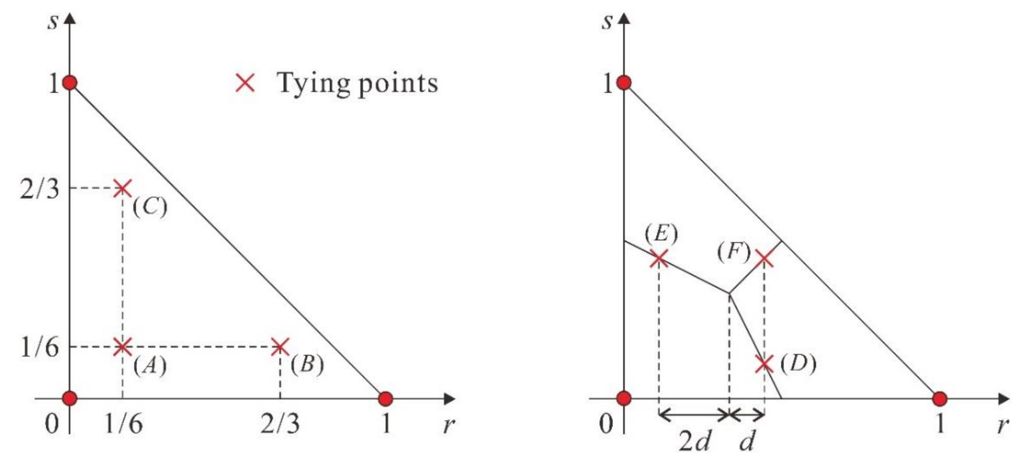
- The accuracy of the solution deteriorates rapidly as the thickness becomes small.
- It happens when the finite elements discretization cannot accurately represent pure bending displacement fields.
- Membrane locking, shear locking.

❖ The MITC3+ shell finite element.

- The shell element has an internal bubble node at element center.
- The bubble node only has two rotational DOFs with a cubic bubble function.
- Assumed covariant transverse shear strain fields are employed to alleviate shear locking.
- Its excellent bending behavior is demonstrated through various linear and nonlinear analyses.



Geometry



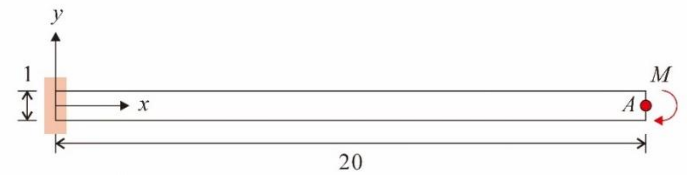
Tying points for the assumed shear strain fields

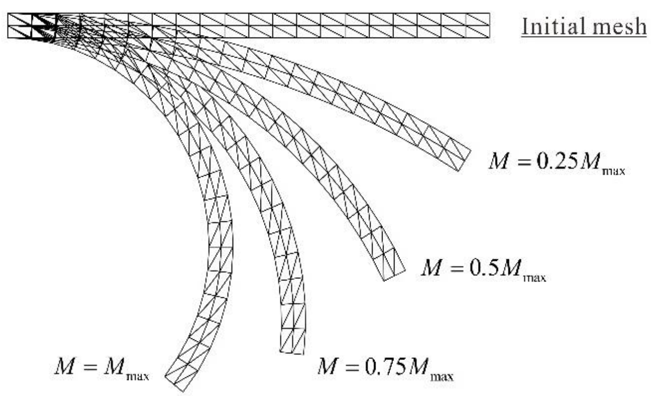
- Lee Y, Lee PS, Bathe KJ. The MITC3+ shell element and its performance. Comput Struct 2014;138:12–23.

The MITC3+ shell element

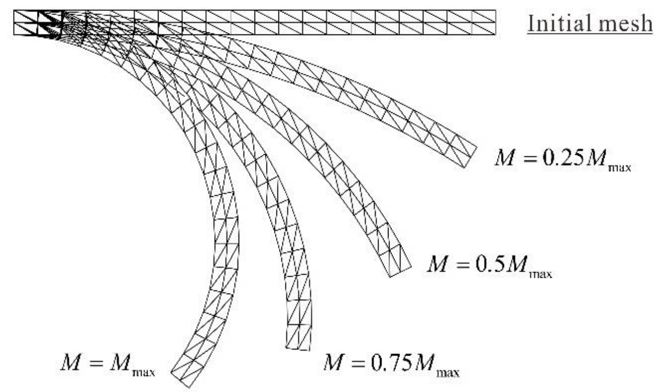
❖ **In membrane problem**

- DISP3 = MITC3+

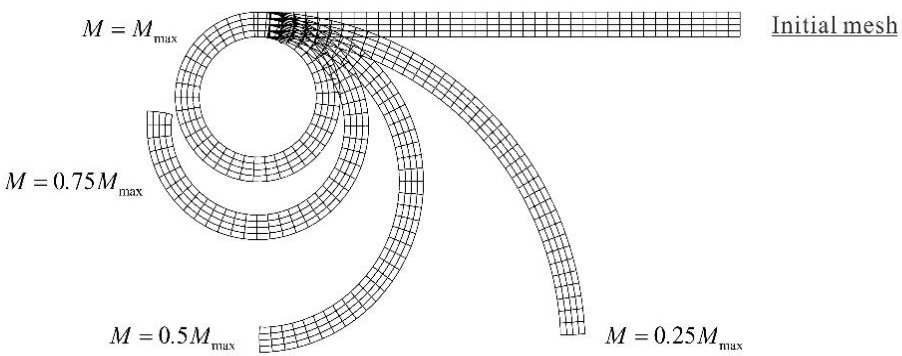




DISP3



MITC3+



Reference

DISP3:
displacement-based 3-node shell element (no treatment for alleviating shear locking).

MITC3+:
3-node shell element with the MITC method.

Reference is obtained using the **MITC9** shell elements.

The MITC3+ shell element

❖ Geometry & displacement interpolations

- ${}^{t}\mathbf{x}(r,s,\xi) = {}^{t}\mathbf{x}_m + \xi\, {}^{t}\mathbf{x}_b$ with ${}^{t}\mathbf{x}_m = \sum_{i=1}^{3} h_i(r,s)\, {}^{t}\mathbf{x}_i$, ${}^{t}\mathbf{x}_b = \frac{1}{2}\sum_{i=1}^{4} a_i f_i(r,s)\, {}^{t}\mathbf{V}_n^i$.

- Shape functions: $h_1 = 1 - r - s,\quad h_2 = r,\quad h_3 = s,$

$$f_1 = h_1 - \frac{1}{3} f_4,\quad f_2 = h_2 - \frac{1}{3} f_4,\quad f_3 = h_3 - \frac{1}{3} f_4,\quad f_4 = 27rs(1-r-s).$$

- $\mathbf{u}(r,s,\xi) = \mathbf{u}_m + \xi(\mathbf{u}_{b1} + \mathbf{u}_{b2})$

with $\mathbf{u}_m = \sum_{i=1}^{3} h_i(r,s)\mathbf{u}_i,$

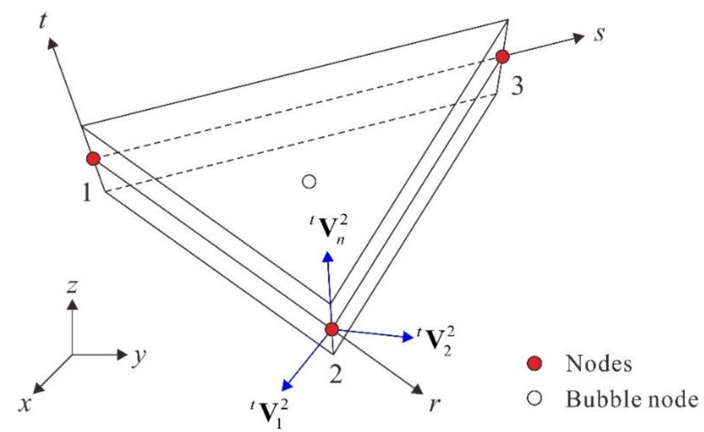
$$\mathbf{u}_{b1} = \frac{1}{2}\sum_{i=1}^{4} a_i f_i(r,s)\left(-\alpha_i\, {}^{t}\mathbf{V}_2^i + \beta_i\, {}^{t}\mathbf{V}_1^i\right),$$

$$\mathbf{u}_{b2} = -\frac{1}{4}\sum_{i=1}^{4} a_i f_i(r,s)\left[\left(\alpha_i^2 + \beta_i^2\right) {}^{t}\mathbf{V}_n^i\right].$$

- $\mathbf{u}_l = \mathbf{u}_m + \xi\mathbf{u}_{b1},\quad \mathbf{u}_q = \xi\mathbf{u}_{b2}.$

${}^{t}\mathbf{x}$: position vector in the configuration at time t.

$\mathbf{u}$: incremental displacement vector from the configuration at time t to that at time $t + \Delta t$.



Geometry of the MITC3+ shell element

The MITC3+ shell element

❖ Green-Lagrange strain components

- $${}_0^t\varepsilon_{ij} = \frac{1}{2}\left({}^t\mathbf{g}_i \cdot {}^t\mathbf{g}_j - {}^0\mathbf{g}_i \cdot {}^0\mathbf{g}_j\right) \quad \text{with} \quad i, j = 1, 2, 3.$$

- In-plane strain components ($i, j = 1, 2$): $${}_0^t\varepsilon_{ij} = {}_0^t\varepsilon_{ij}^{m} + \xi\, {}_0^t\varepsilon_{ij}^{b1} + \xi^2\, {}_0^t\varepsilon_{ij}^{b2}$$

where $${}_0^t\varepsilon_{ij}^{m} = \frac{1}{2}\left({}^t\mathbf{x}_{m,i} \cdot {}^t\mathbf{x}_{m,j} - {}^0\mathbf{x}_{m,i} \cdot {}^0\mathbf{x}_{m,j}\right), \qquad {}_0^t\varepsilon_{ij}^{b2} = \frac{1}{2}\left({}^t\mathbf{x}_{b,i} \cdot {}^t\mathbf{x}_{b,j} - {}^0\mathbf{x}_{b,i} \cdot {}^0\mathbf{x}_{b,j}\right),$$

$${}_0^t\varepsilon_{ij}^{b1} = \frac{1}{2}\left[\left({}^t\mathbf{x}_{m,i} \cdot {}^t\mathbf{x}_{b,j} + {}^t\mathbf{x}_{m,j} \cdot {}^t\mathbf{x}_{b,i}\right) - \left({}^0\mathbf{x}_{m,i} \cdot {}^0\mathbf{x}_{b,j} + {}^0\mathbf{x}_{m,j} \cdot {}^0\mathbf{x}_{b,i}\right)\right],$$

and $${}^t\mathbf{x}_{m,i} = \frac{\partial\, {}^t\mathbf{x}_m}{\partial r_i}, \qquad {}^t\mathbf{x}_{b,i} = \frac{\partial\, {}^t\mathbf{x}_b}{\partial r_i}.$$

$${}^t\mathbf{g}_i = \frac{\partial\, {}^t\mathbf{x}}{\partial r_i}$$: covariant base vectors at time t.

$$\mathbf{u}_{,i} = \frac{\partial \mathbf{u}}{\partial r_i}$$: derivatives of incremental displacement vector.

❖ Incremental strain components

- $${}_0\varepsilon_{ij} = {}_0^{t+\Delta t}\varepsilon_{ij} - {}_0^t\varepsilon_{ij} = \frac{1}{2}(\mathbf{u}_{,i} \cdot {}^t\mathbf{g}_j + {}^t\mathbf{g}_i \cdot \mathbf{u}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{u}_{,j}) \quad \text{with} \quad i, j = 1, 2, 3.$$

- $${}_0\varepsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \quad \text{with} \quad {}_0e_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_l}{\partial r_i}\, {}^t\mathbf{g}_j + {}^t\mathbf{g}_i \frac{\partial \mathbf{u}_l}{\partial r_j}\right), \qquad {}_0\eta_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_l}{\partial r_i} \cdot \frac{\partial \mathbf{u}_l}{\partial r_j}\right) + \frac{1}{2}\left(\frac{\partial \mathbf{u}_q}{\partial r_i}\, {}^t\mathbf{g}_j + {}^t\mathbf{g}_i \frac{\partial \mathbf{u}_q}{\partial r_j}\right).$$

The MITC3+ shell element

❖ **Incremental strain components** (continued)

- In-plane strain components $(i, j = 1, 2)$: $${}_0\varepsilon_{ij} = {}_0\varepsilon_{ij}^{m} + \xi\, {}_0\varepsilon_{ij}^{b1} + \xi^2\, {}_0\varepsilon_{ij}^{b2}.$$

  - $${}_0\varepsilon_{ij}^{m} = {}_0e_{ij}^{m} + {}_0\eta_{ij}^{m}, \qquad {}_0\varepsilon_{ij}^{b1} = {}_0e_{ij}^{b1} + {}_0\eta_{ij}^{b1}, \qquad {}_0\varepsilon_{ij}^{b2} = {}_0e_{ij}^{b2} + {}_0\eta_{ij}^{b2}.$$

  - $${}_0e_{ij} = {}_0e_{ij}^{m} + \xi\, {}_0e_{ij}^{b1} + \xi^2\, {}_0e_{ij}^{b2}$$ with $${}_0e_{ij}^{m} = \frac{1}{2}\left({}^t\mathbf{x}_{m,i} \cdot \mathbf{u}_{m,j} + {}^t\mathbf{x}_{m,j} \cdot \mathbf{u}_{m,i} \right),$$ $${}_0e_{ij}^{b1} = \frac{1}{2}\left({}^t\mathbf{x}_{m,i} \cdot \mathbf{u}_{b1,j} + {}^t\mathbf{x}_{m,j} \cdot \mathbf{u}_{b1,i} + {}^t\mathbf{x}_{b,i} \cdot \mathbf{u}_{m,j} + {}^t\mathbf{x}_{b,j} \cdot \mathbf{u}_{m,i} \right),$$ $${}_0e_{ij}^{b2} = \frac{1}{2}\left({}^t\mathbf{x}_{b,i} \cdot \mathbf{u}_{b1,j} + {}^t\mathbf{x}_{b,j} \cdot \mathbf{u}_{b1,i} \right).$$

  - $${}_0\eta_{ij} = {}_0\eta_{ij}^{m} + \xi\, {}_0\eta_{ij}^{b1} + \xi^2\, {}_0\eta_{ij}^{b2}$$ with $${}_0\eta_{ij}^{m} = \frac{1}{2}\mathbf{u}_{m,i} \cdot \mathbf{u}_{m,j},$$ $${}_0\eta_{ij}^{b1} = \frac{1}{2}\left(\mathbf{u}_{m,i} \cdot \mathbf{u}_{b1,j} + \mathbf{u}_{m,j} \cdot \mathbf{u}_{b1,i} + {}^t\mathbf{x}_{m,i} \cdot \mathbf{u}_{b2,j} + {}^t\mathbf{x}_{m,j} \cdot \mathbf{u}_{b2,i} \right),$$ $${}_0\eta_{ij}^{b2} = \frac{1}{2}\left(\mathbf{u}_{b1,i} \cdot \mathbf{u}_{b1,j} + {}^t\mathbf{x}_{b,i} \cdot \mathbf{u}_{b2,j} + {}^t\mathbf{x}_{b,j} \cdot \mathbf{u}_{b2,i} \right).$$

The MITC3+ shell element

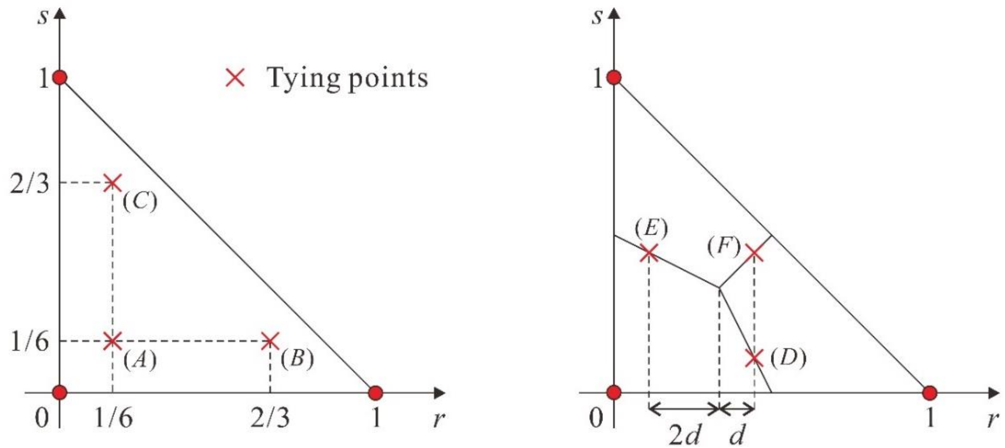
❖ Assumed transverse shear strain fields

- $_0\varepsilon_{i3}^{MITC3+} = {_0}e_{i3}^{MITC3+} + {_0}\eta_{i3}^{MITC3+}$ with $i, j = 1, 2, 3.$

where $$_0\varepsilon_{13}^{MITC3+} = \frac{2}{3}\left({_0}\varepsilon_{13}^{B} - \frac{1}{2}\, {_0}\varepsilon_{23}^{B} \right) + \frac{1}{3}\left({_0}\varepsilon_{13}^{A} + {_0}\varepsilon_{23}^{A} \right) + \frac{1}{3}\, {_0}\hat{c}(3s-1),$$

$$_0\varepsilon_{23}^{MITC3+} = \frac{2}{3}\left({_0}\varepsilon_{23}^{C} - \frac{1}{2}\, {_0}\varepsilon_{13}^{C} \right) + \frac{1}{3}\left({_0}\varepsilon_{13}^{A} + {_0}\varepsilon_{23}^{A} \right) + \frac{1}{3}\, {_0}\hat{c}(1-3r),$$

and $_0\hat{c} = {_0}\varepsilon_{13}^{F} - {_0}\varepsilon_{13}^{D} - {_0}\varepsilon_{23}^{F} + {_0}\varepsilon_{23}^{E}.$



Tying points for the assumed shear strain fields

The MITC3+ shell element with the SSE method

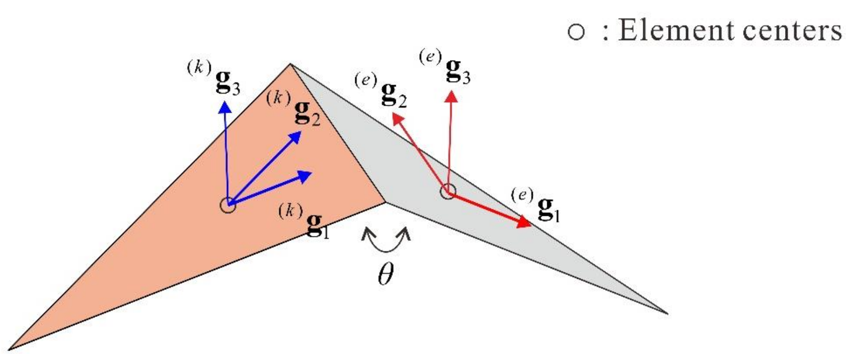
Step 1 of 4 — **Decomposition of strain components.**

- The covariant in-plane strain components ($i, j = 1, 2$) can be decomposed as follows

$$ {}_0\varepsilon_{ij} = {}_0\varepsilon_{ij}^{m} + \xi \, {}_0\varepsilon_{ij}^{b1} + \xi^2 \, {}_0\varepsilon_{ij}^{b2} \quad \text{with} \quad {}_0\varepsilon_{ij}^{m} = {}_0e_{ij}^{m} + {}_0\eta_{ij}^{m} $$

where
$$ {}_0e_{ij}^{m} = \frac{1}{2}\left({}^{t}\mathbf{x}_{m,i} \cdot \mathbf{u}_{m,j} + {}^{t}\mathbf{x}_{m,j} \cdot \mathbf{u}_{m,i} \right), \qquad {}_0\eta_{ij}^{m} = \frac{1}{2}\mathbf{u}_{m,i} \cdot \mathbf{u}_{m,j}. $$

Step 2 of 4 — **Matching the coordinate systems of strains of the target and neighboring elements.**



- Strain transformation

$$ {}_0\overline{\varepsilon}_{ij}^{m,(k)} = {}_0\varepsilon_{ln}^{m,(k)} \left({}^{(e)}\mathbf{g}_i \cdot {}^{(k)}\mathbf{g}^{l} \right)\left({}^{(e)}\mathbf{g}_j \cdot {}^{(k)}\mathbf{g}^{n} \right) $$

with $i, j, l, n = 1, 2$.

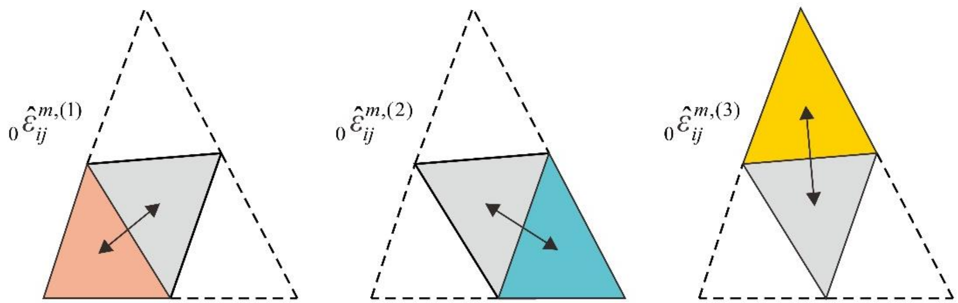
- Element centers ($r = s = 1/3$ and $\xi = 0$) are the reference points of the coordinate transformation.

- Effect of out-of-plane strains is neglected.

- Lee C, Lee PS. The strain-smoothed MITC3+ shell finite element. Comput Struct 2019;223.

The MITC3+ shell element with the SSE method

Step 3 of 4 | **Strain smoothing between the target element and each neighboring element.**



- Projected area

$$\bar{A}^{(k)} = (\mathbf{n}^{(e)} \cdot \mathbf{n}^{(k)}) A^{(k)} \quad \text{with} \quad \mathbf{n}^{(e)} = {}^{(e)}\mathbf{g}_3 / \left\| {}^{(e)}\mathbf{g}_3 \right\|, \quad \mathbf{n}^{(k)} = {}^{(k)}\mathbf{g}_3 / \left\| {}^{(k)}\mathbf{g}_3 \right\|.$$

- Smoothed strain

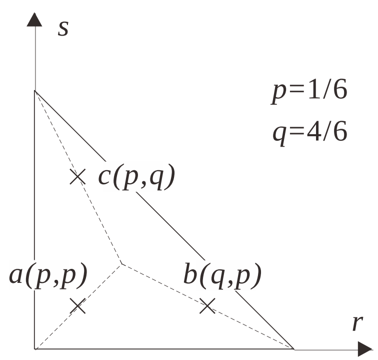
$${}_0\hat{\varepsilon}_{ij}^{m,(k)} = \frac{1}{A^{(e)} + \bar{A}^{(k)}} \left({}_0\varepsilon_{ij}^{m,(e)} A^{(e)} + {}_0\bar{\varepsilon}_{ij}^{m,(k)} \bar{A}^{(k)} \right) \quad \text{with} \quad i, j = 1, 2.$$

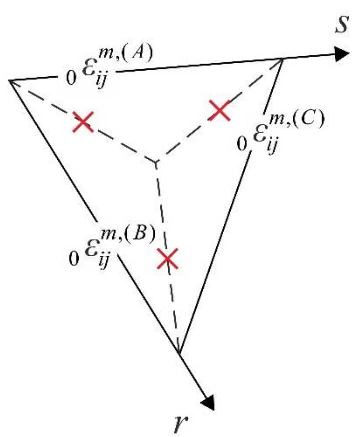
- If there is no neighboring element,

$${}_0\hat{\varepsilon}_{ij}^{m,(k)} = {}_0\varepsilon_{ij}^{m,(e)}.$$

The MITC3+ shell element with the SSE method

Step 4 of 4 | **Construction of the smoothed strain field through Gauss points.**





- Strain smoothing within elements

$$ {}_0\varepsilon_{ij}^{m,(A)} = \frac{1}{2}\left({}_0\hat{\varepsilon}_{ij}^{m,(3)} + {}_0\hat{\varepsilon}_{ij}^{m,(1)}\right), \quad {}_0\varepsilon_{ij}^{m,(B)} = \frac{1}{2}\left({}_0\hat{\varepsilon}_{ij}^{m,(1)} + {}_0\hat{\varepsilon}_{ij}^{m,(2)}\right), $$

$$ {}_0\varepsilon_{ij}^{m,(C)} = \frac{1}{2}\left({}_0\hat{\varepsilon}_{ij}^{m,(2)} + {}_0\hat{\varepsilon}_{ij}^{m,(3)}\right) \quad \text{with} \quad i, j = 1, 2. $$

- Smoothed covariant membrane strain field

$$ {}_0\varepsilon_{ij}^{m,SSE} = \left[1 - \frac{1}{q-p}(r+s-2p)\right] {}_0\varepsilon_{ij}^{m,(A)} + \frac{r-p}{q-p}\, {}_0\varepsilon_{ij}^{m,(B)} + \frac{s-p}{q-p}\, {}_0\varepsilon_{ij}^{m,(C)} $$

with $i, j = 1, 2.$

- The smoothed covariant membrane strain replaces the original covariant membrane strain.
- For the covariant transverse shear strains, we adopt the assumed strains of the MITC3+ shell element.

Numerical examples (Shell / Linear analysis)

❖ Finite elements considered (flat shell elements)

Element	Description
Allman	▪ A flat shell element that combines a triangular membrane element with Allman's drilling DOFs and the discrete Kirchhoff-Mindlin triangular (DKMT) plate element are combined. ▪ It requires 18 DOFs for an element.
ANDES (OPT)	▪ A flat shell element that combines the assumed natural deviatoric strain (ANDES) triangular membrane element with 3 drilling DOFs and optimal parameters and the DKMT plate element. ▪ It has 18 DOFs for an element.
Shin and Lee	▪ As a flat shell element, the edge-based strain smoothing method is applied to the ANDES formulation-based membrane element with 3 drilling DOFs, and the DKMT plate element is combined. ▪ New values of the free parameters in the ANDES formulation are introduced. ▪ It requires 18 DOFs for an element.

Numerical examples (Shell / Linear analysis)

❖ Finite elements considered (curved shell elements)

Element	Description
MITC3+	▪ A continuum mechanics based 3-node shell element with a bubble node. ▪ The bubble node has 2 rotational DOFs which can be condensed out on the element level. ▪ It has 15 DOFs for an element.
Enriched MITC3+	▪ The MITC3+ shell element enriched in membrane displacements by interpolation covers. ▪ 4 DOFs per node are added and thus the element has 27 DOFs for an element in total.

❖ Evaluation method

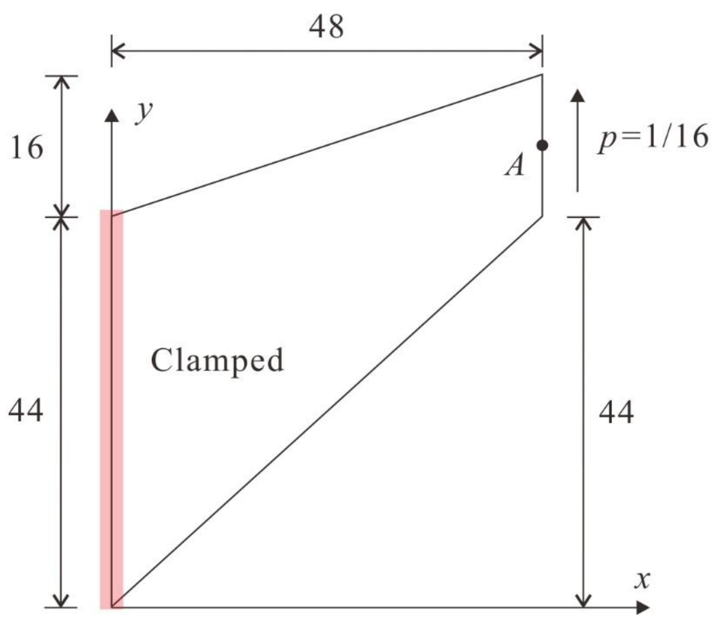
- Convergence curves obtained using the s-norm $E_h = \dfrac{\left\| \mathbf{u}_{ref} - \mathbf{u}_h \right\|_s^2}{\left\| \mathbf{u}_{ref} \right\|_s^2}$ with $\left\| \mathbf{u}_{ref} - \mathbf{u}_h \right\|_s^2 = \int_\Omega \Delta\boldsymbol{\varepsilon}^T \Delta\boldsymbol{\tau} d\Omega.$
- Displacements and stresses.

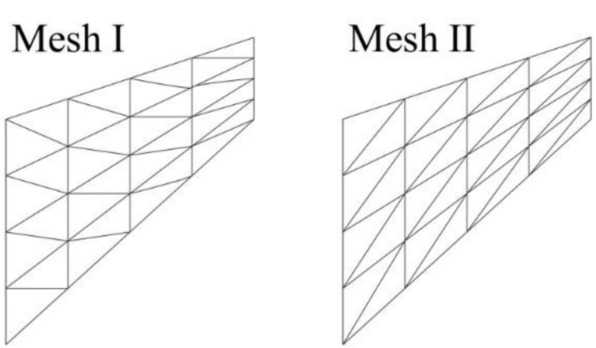
❖ Reference solution

- Reference solutions are calculated using a <u>64×64</u> regular mesh of the <u>MITC9 shell elements</u>.

Numerical examples (Shell / Linear analysis)

1) Cook's skew beam problem





- **Force**

  Distributed shearing force $p = 1/16$.

- **Boundary condition**

  Left edge is clamped.

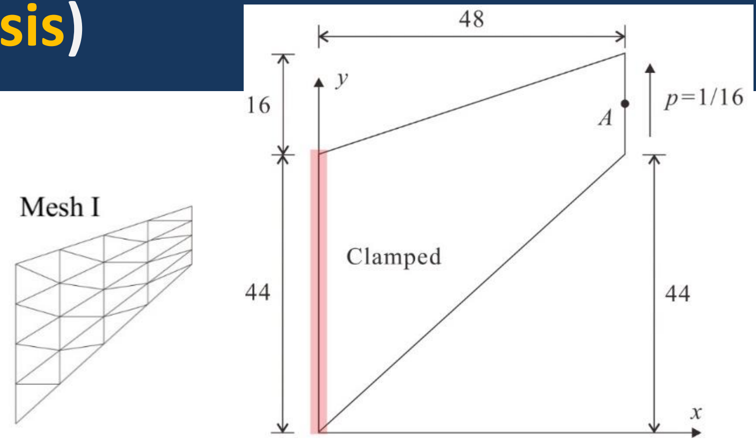
- **Material property**

  $E = 1, \quad \nu = 1/3.$

- **Two patterns of meshes** ($N \times N$ elements)

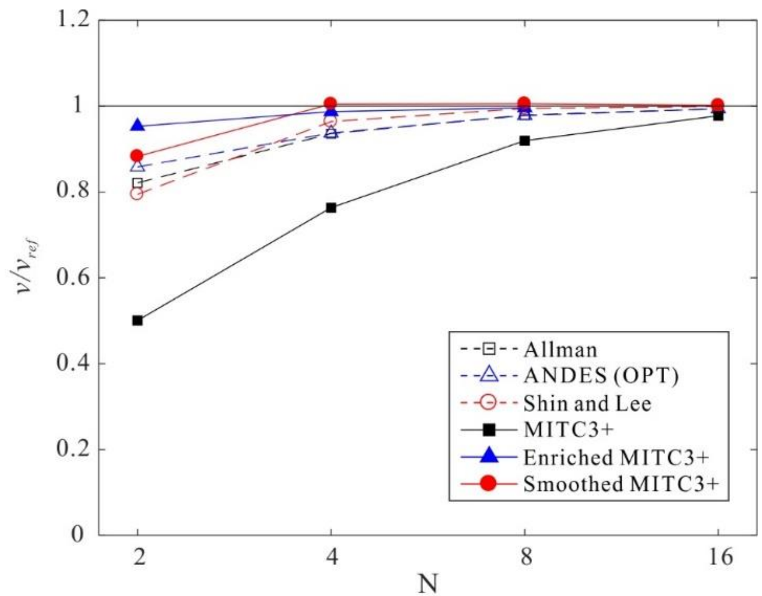
  N =2, 4, 8, 16, 32.

Numerical examples (Shell / Linear analysis)

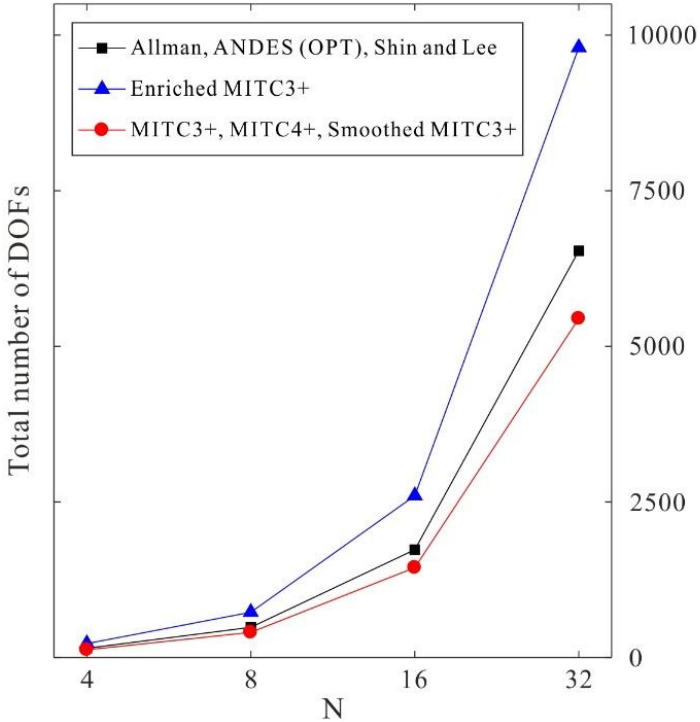


1) Cook's skew beam problem

- Normalized vertical displacements (v_h / v_{ref}) at point *A* for Mesh I.

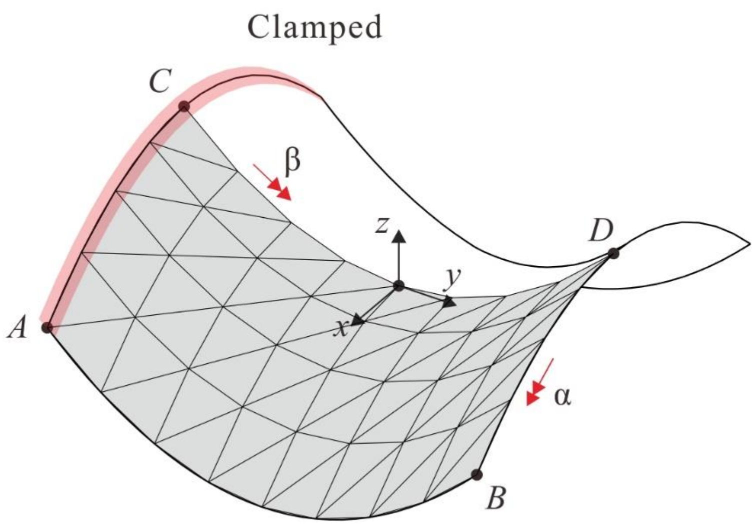


- **The total number of DOFs when increasing the *N*.**

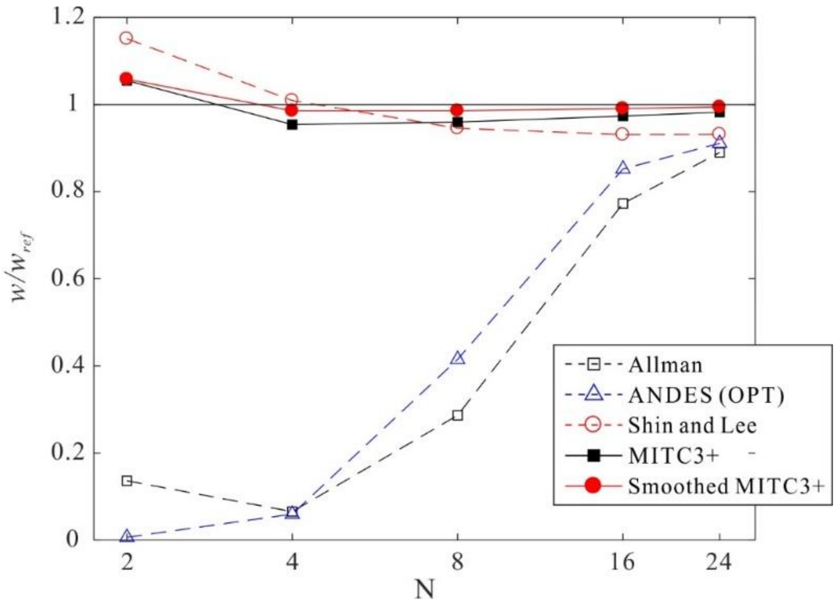


2) Hyperbolic paraboloid shell problem

- **Problem description**



- **Normalized vertical displacements (w_h / w_{ref}) at point *D*.**



- **Force**

  Self-weight loading $f_z = 8.$

- **Boundary condition**

  One end is clamped.

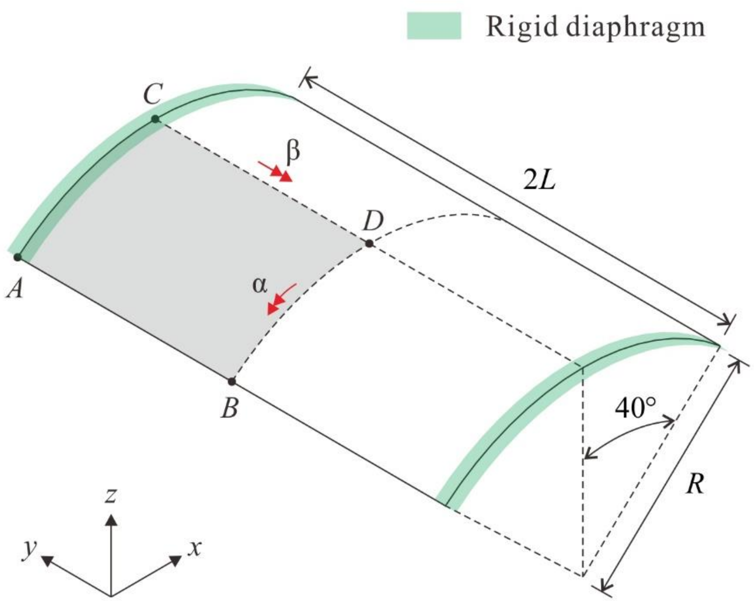
- **Material property & Thickness**

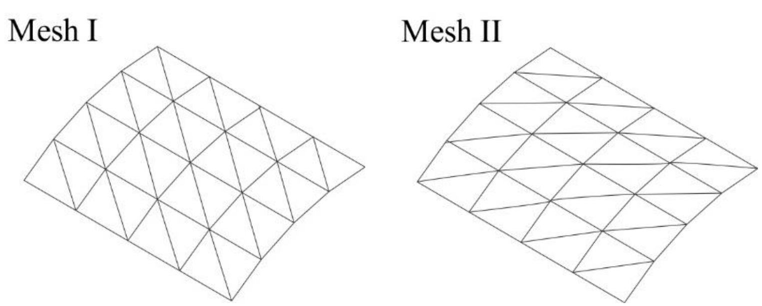
  $E = 2\times10^{11}, \quad \nu = 0.3, \quad t = 0.001.$

- **Regular meshes** ($N \times 2N$ elements)

  N = 2, 4, 8, 16, 24.

3) Scordelis-Lo roof shell problem





- **Force**

  Self-weight loading $f_z = 90.$

- **Boundary condition**

  The shell is supported by rigid diaphragms.

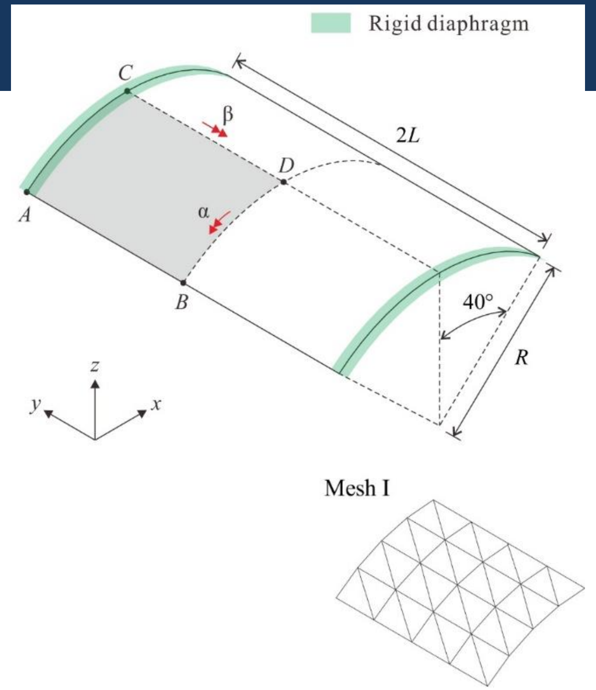
- **Material property & Thickness**

  $E = 4.32 \times 10^8, \quad \nu = 0, \quad t = 0.25, \; 0.025, \; 0.0025.$

- **Two patterns of meshes** ($N \times N$ elements)

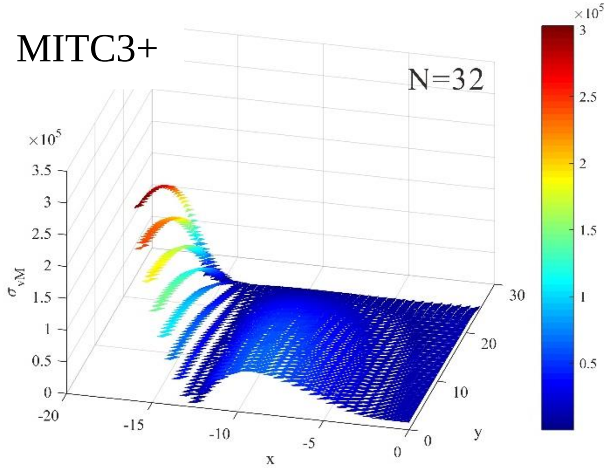
  $N =$ 4, 8, 16, 32.

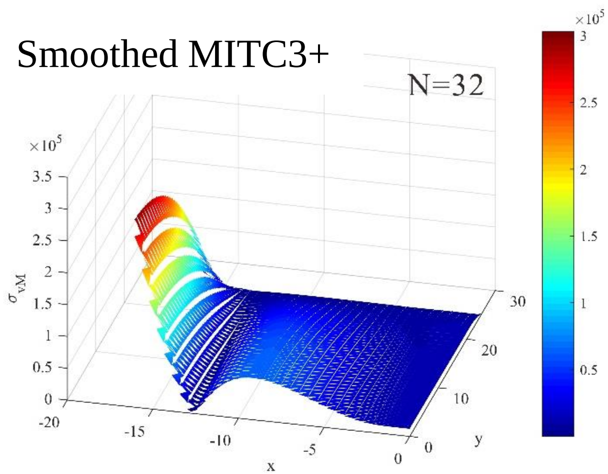
Numerical examples (Shell / Linear analysis)

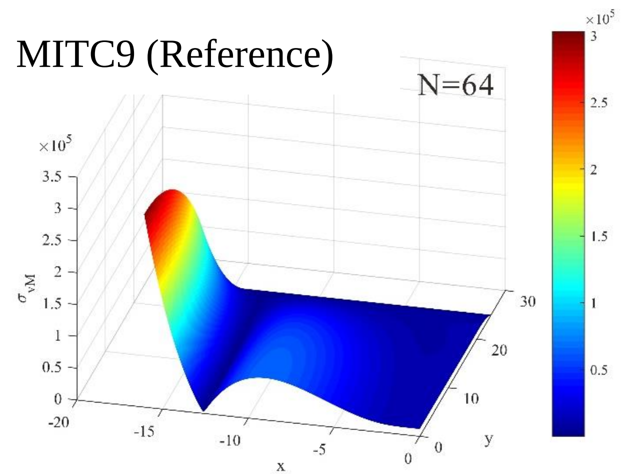


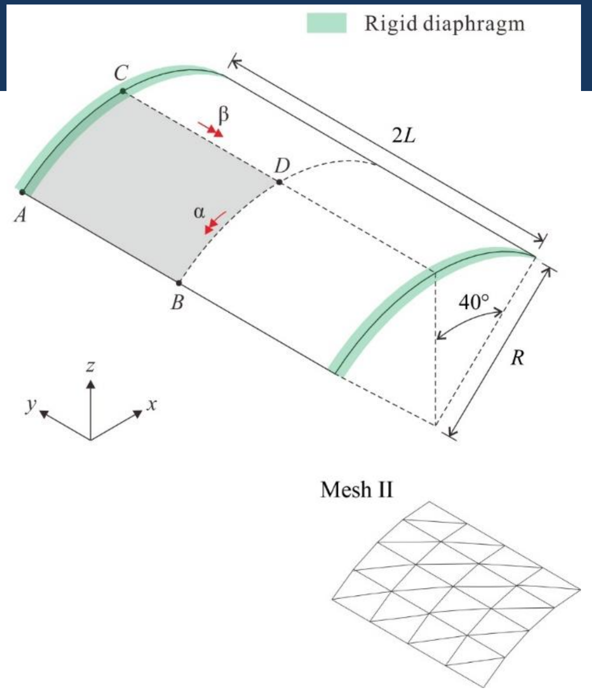
3) Scordelis-Lo roof shell problem

- **von Mises stress distributions for Mesh I when *t* = 0.25.**



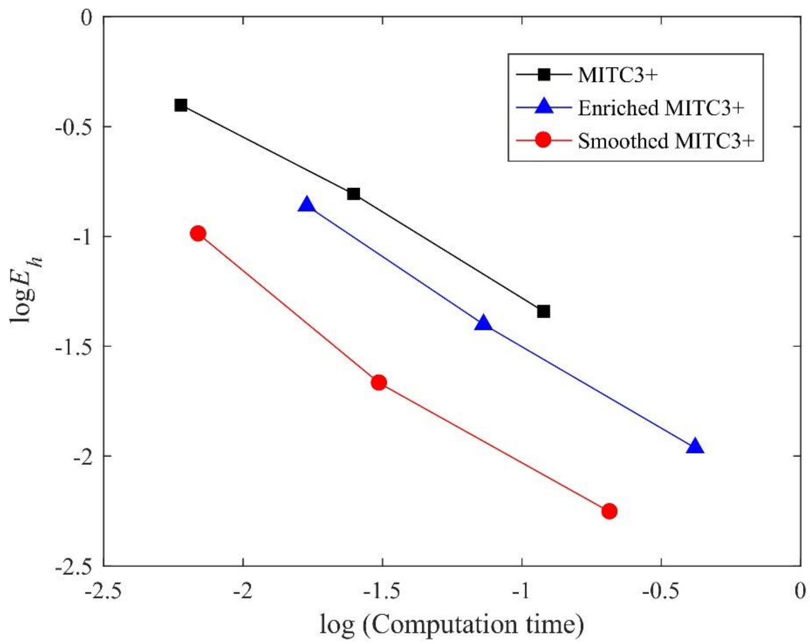






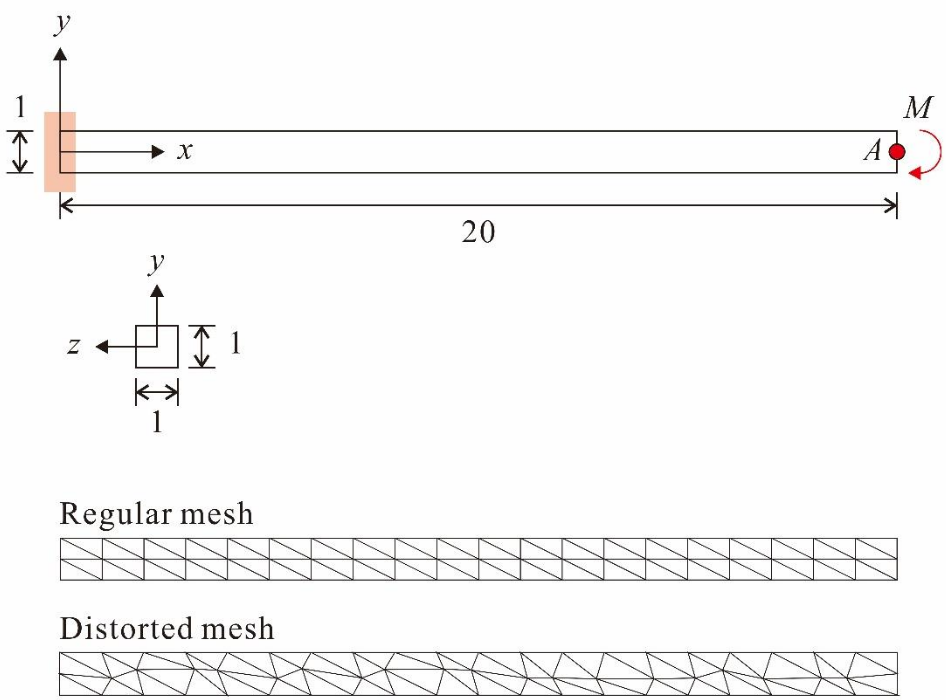
3) Scordelis-Lo roof shell problem

- **Computational efficiency curves for Mesh II when *t* = 0.025.**



- Computation times taken from obtaining stiffness matrices to solving linear equations are measured.
- Computations are performed in a PC with Intel Core i7-6700, 3.40GHz CPU and 64GB RAM.
- A symmetric skyline solver is used for solving linear equations.

1) Cantilever beam subjected to a tip moment problem



- **Force**

  Tip moment $M_{\max} = 10\pi.$

- **Boundary condition**

  Left edge is clamped.

- **Material property & Thickness**

  $E = 1.2 \times 10^{3}, \quad \nu = 0.2, \quad t = 1.$

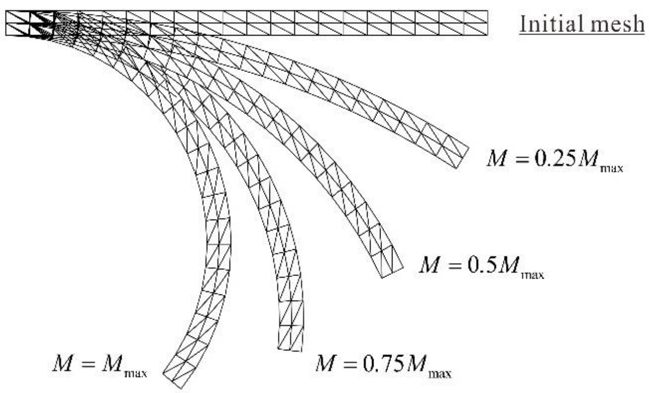
- **20 load steps**

- Regular and distorted 20×2 meshes of triangular elements.
- Regular 40×4 meshes of MITC9 elements for reference.

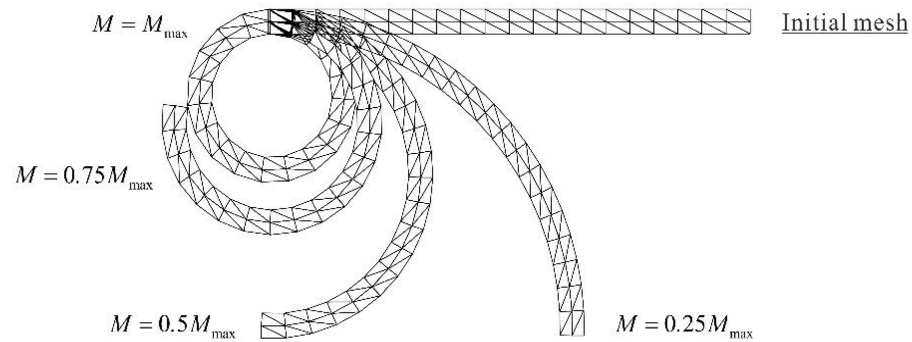
Numerical examples (Shell / Nonlinear analysis)

1) Cantilever beam subjected to a tip moment problem

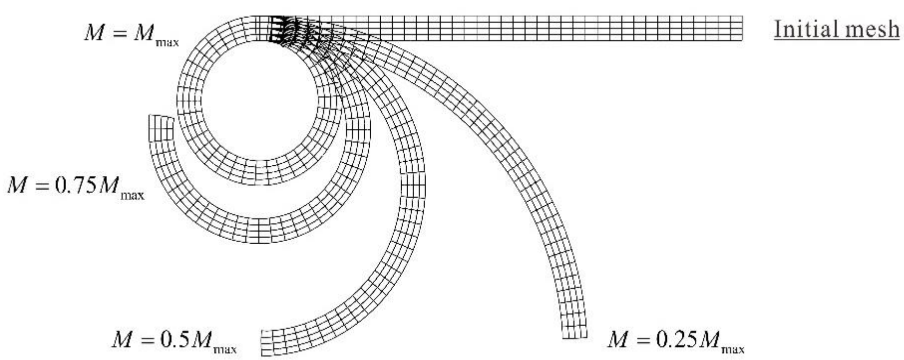
- **Deformed configurations at the load levels** $M / M_{max} = 0.25, 0.5, 0.75, 1.0$ **for regular mesh.**



MITC3+



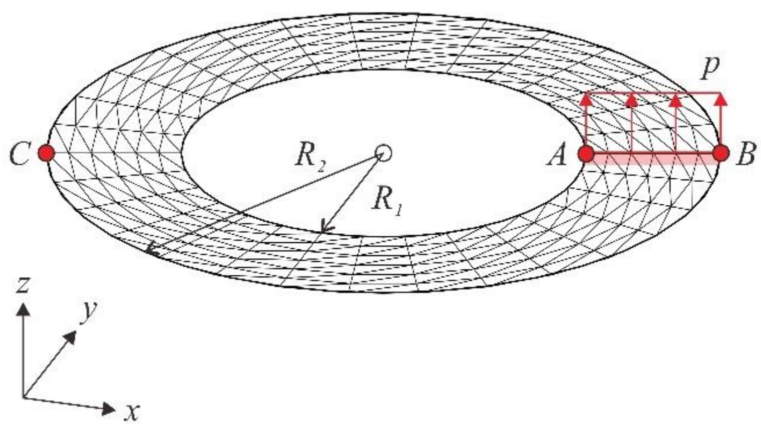
Smoothed MITC3+ (proposed)



Reference

Numerical examples (Shell / Nonlinear analysis)

2) Slit annular plate subjected to a lifting line force problem



- **Force**

  Shearing force $p_{\max} = 0.8.$

- **Boundary condition**

  One end is clamped.

- **Material property & Thickness**

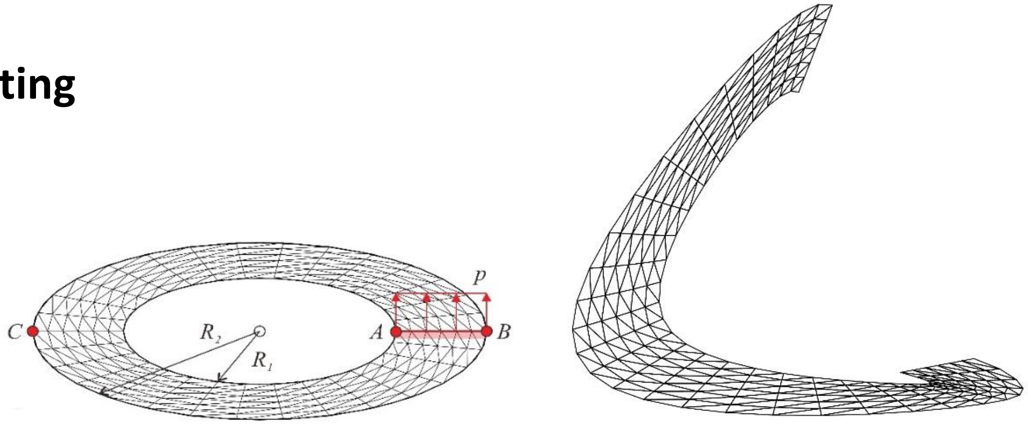
  $E = 2.1\times 10^{7}, \quad \nu = 0, \quad t = 0.03.$

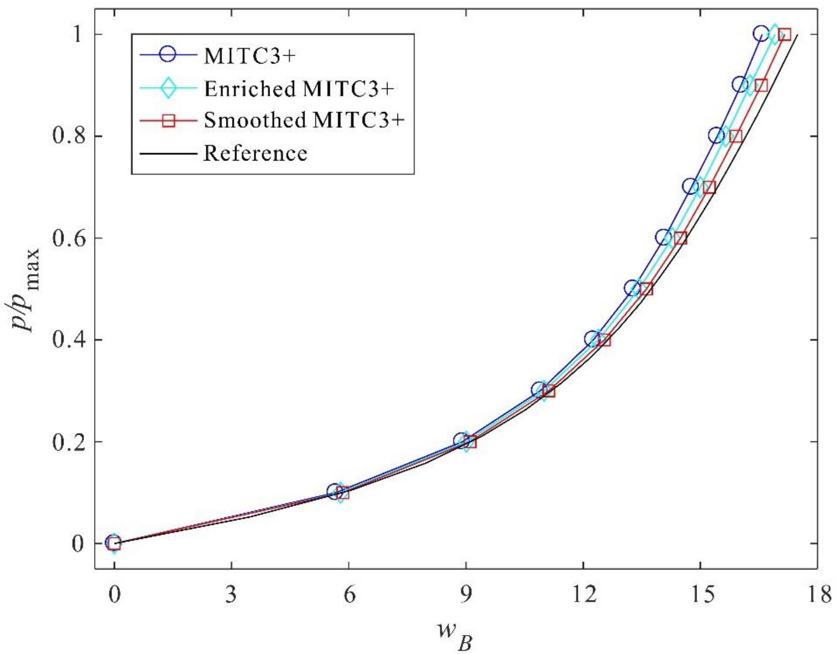
- **10 load steps**

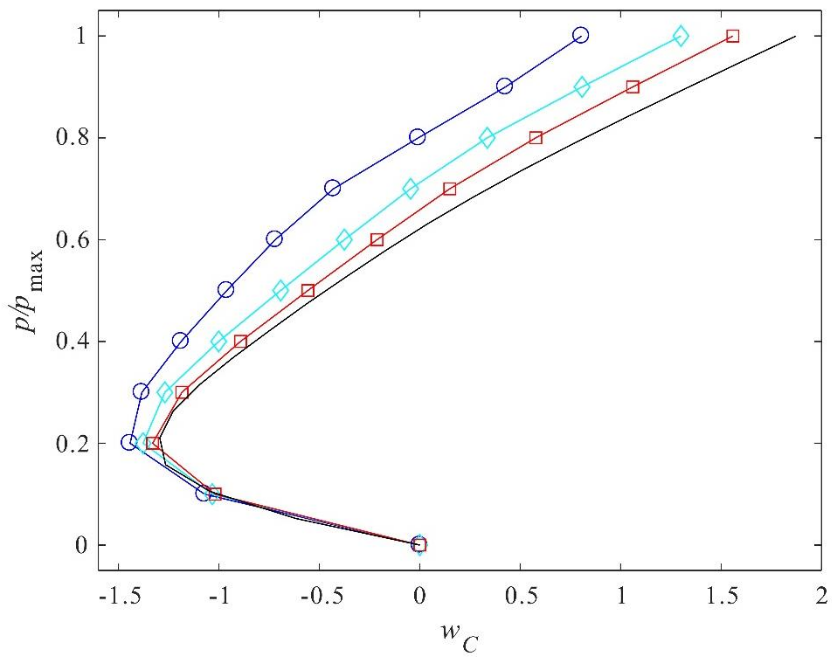
- 6×30 mesh of triangular elements.
- 12×60 mesh of MITC9 elements for reference.

2) Slit annular plate subjected to a lifting line force problem

- **Load-displacement curves (w_B and w_C).**

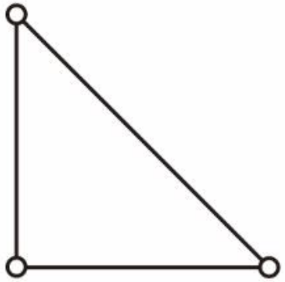






Topic 4

Acoustic radiation analysis using the strain-smoothed triangular element



3-node triangular
2D solid element

Acoustic radiation problem

❖ Formulation

- Helmholtz equation (reduced wave equation): $\Delta p + k^2 p = 0$.
- The Sommerfeld radiation condition for exterior acoustic problem: $\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial p}{\partial r} + jkp \right) = 0$.
- Galerkin weak form of this problem:

$$-\int_{\Omega} \nabla w \cdot \nabla p d\Omega + k^2 \int_{\Omega} w \cdot p d\Omega - j\rho\omega \int_{\Gamma_N} w \cdot v_n d\Omega - \int_{\Gamma_B} w \cdot M \cdot p d\Gamma = 0.$$

- Finite element formulation:

$$\left[\mathbf{K} - k^2\mathbf{M} + \mathbf{K}_S\right]\mathbf{p} = -j\rho\omega\mathbf{F},$$

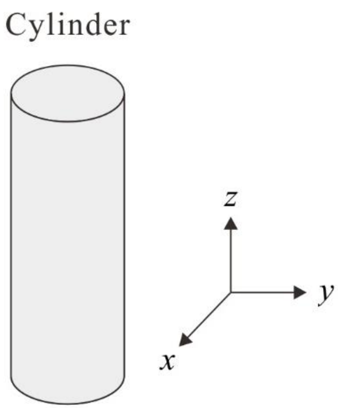
where

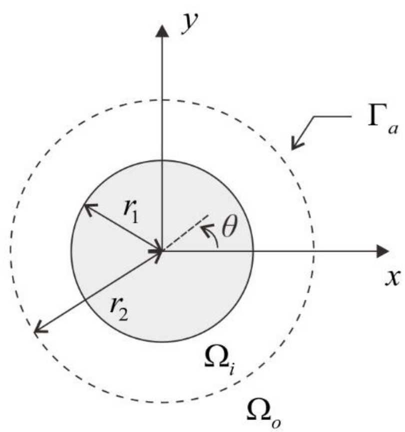
$$\mathbf{K} = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{B}^{(m)T}\mathbf{B}^{(m)} d\Omega, \quad \mathbf{M} = \sum_{m=1}^{e} \int_{\Omega^{(m)}} \mathbf{N}^T \cdot \mathbf{N}\, d\Omega, \quad \mathbf{F} = \sum_{m=1}^{e} \int_{\Gamma_N^{(m)}} \mathbf{N}^T \cdot \mathbf{v}_\mathbf{n} d\Gamma.$$

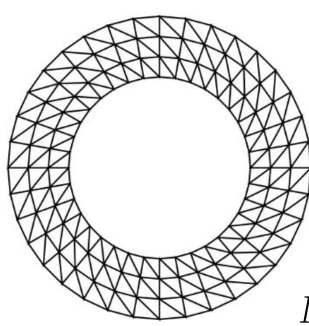
$\mathbf{B}^{(m)}$ is the differential operator matrix of an element.

Numerical examples (Acoustic radiation analysis)

1) Circumferentially harmonic radiation from a cylinder







$N \times 12N$

$(N = 3)$

- **Force**

  Pressure $p(\theta) = \cos(4\theta)$ at $r = r_1$.

- **Boundary condition**

  Dirichlet to Neumann (DtN) condition along Γ_a .

- **Material property**

  Density: $\rho = 1.225\ \mathrm{kg/m^3}$,

  Wave speed: $c = 340$ m/s.

  Wave numbers: $q = 10, 16, 22.$

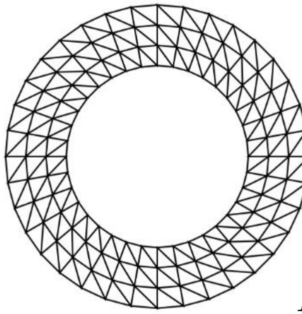
- **Regular meshes** ($N \times 12N$ elements)

  $N =$ 2, 3, 4, 10.

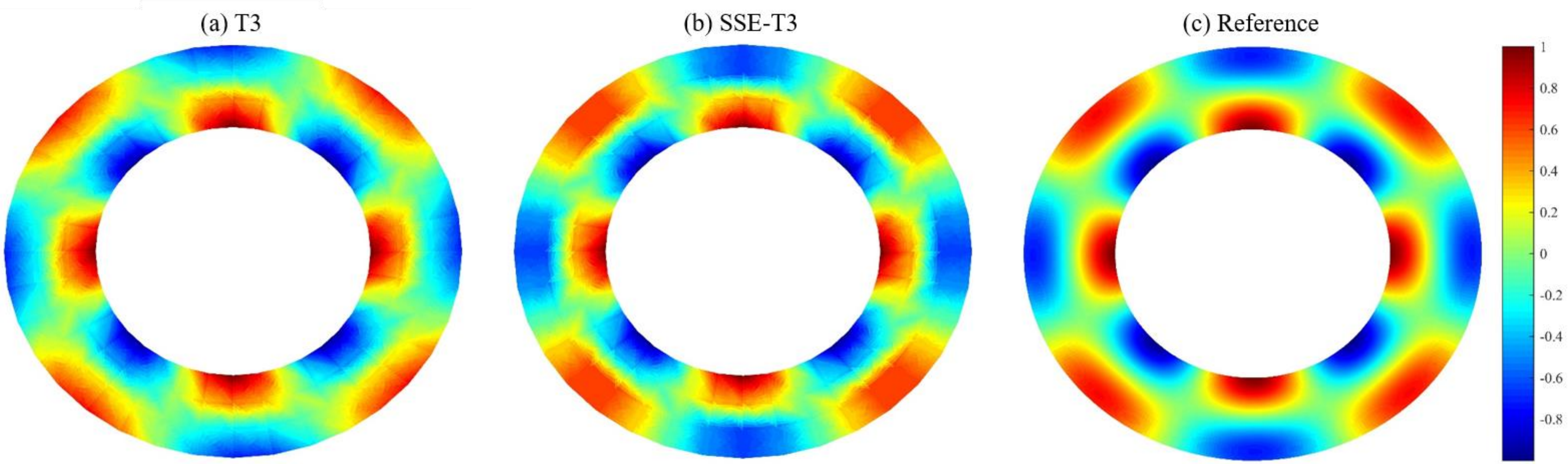
Numerical examples (Acoustic radiation analysis)

1) Circumferentially harmonic radiation from a cylinder

- **Pressure distributions for the mesh (N = 3).**



$N \times 12N$



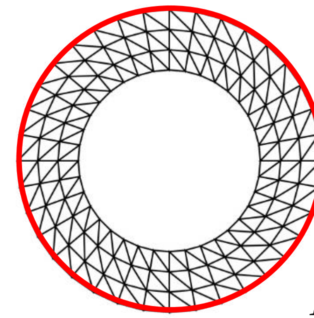
q (wave number) = **10**, 16, 22

- Analytical solution:

$$p(r,\theta) = \frac{H_4^{(1)}(kr)}{H_4^{(1)}(kr_1)}\cos(4\theta).$$

Numerical examples (Acoustic radiation analysis)

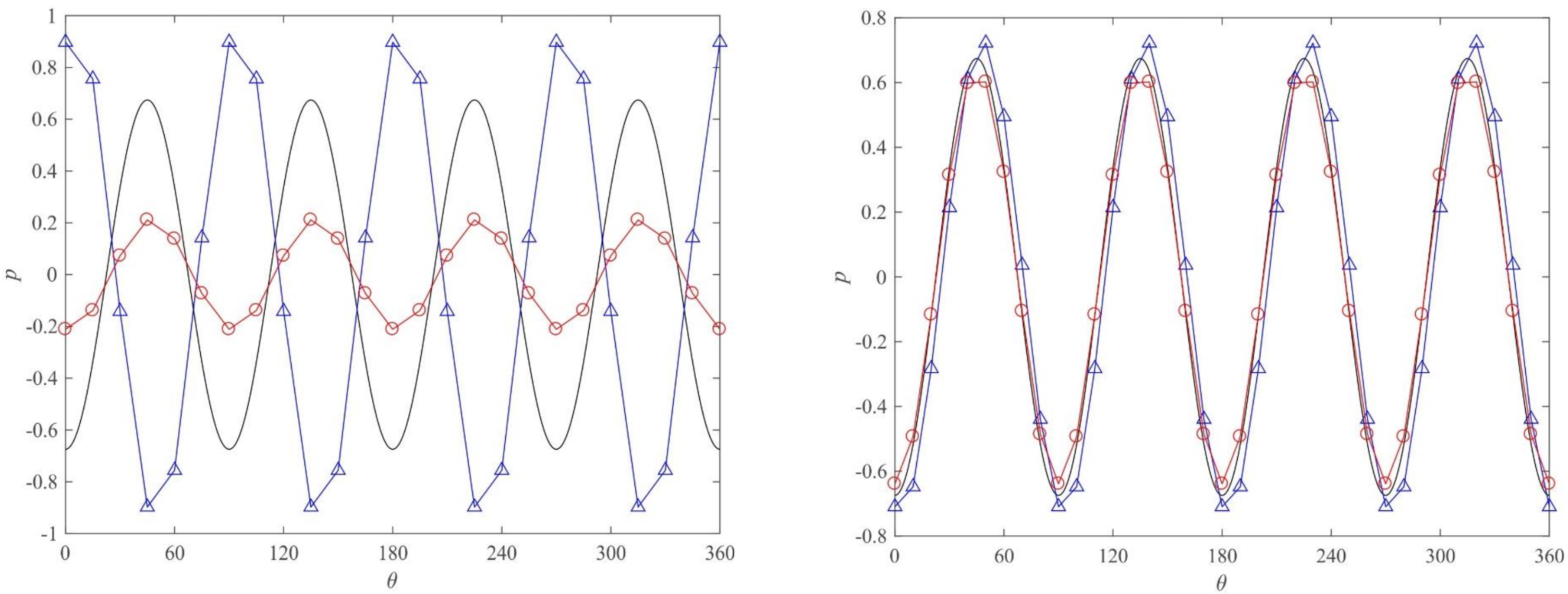
1) Circumferentially harmonic radiation from a cylinder

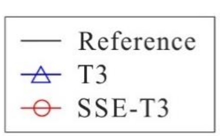


$N \times 12N$

- **Pressure distributions for the mesh (N = 2, 3).**

$N = 2$ $\qquad\qquad$ $N = 3$





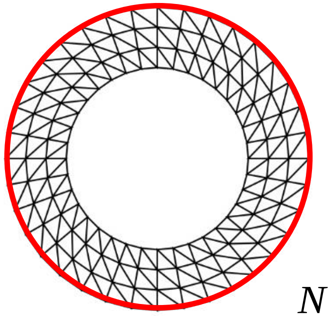
q (wave number) = **10**, 16, 22

- Analytical solution:

$$p(r,\theta) = \frac{H_4^{(1)}(kr)}{H_4^{(1)}(kr_1)}\cos(4\theta).$$

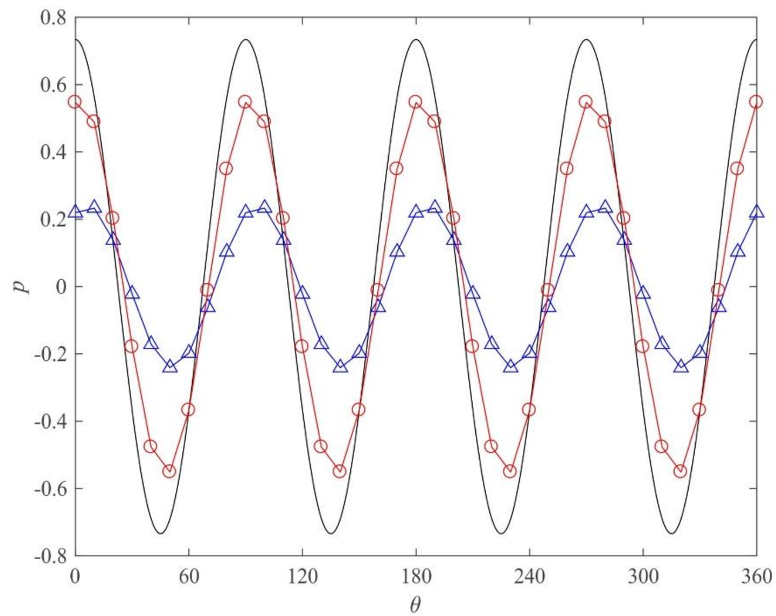
1) Circumferentially harmonic radiation from a cylinder

- **Pressure distributions for the mesh (N = 3, 4).**

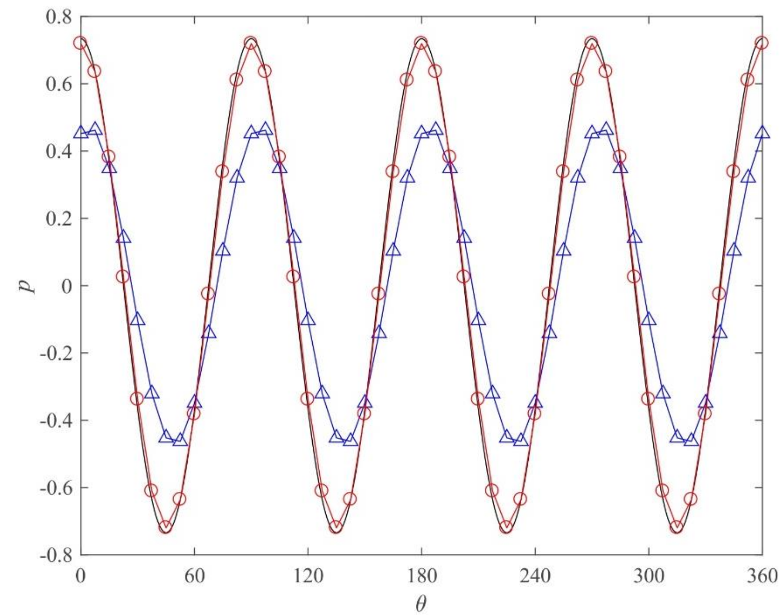


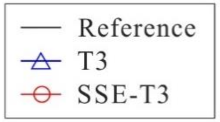
$N \times 12N$

$N = 3$



$N = 4$





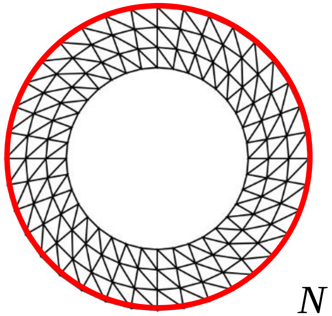
q (wave number) = 10, **16**, 22

- Analytical solution:

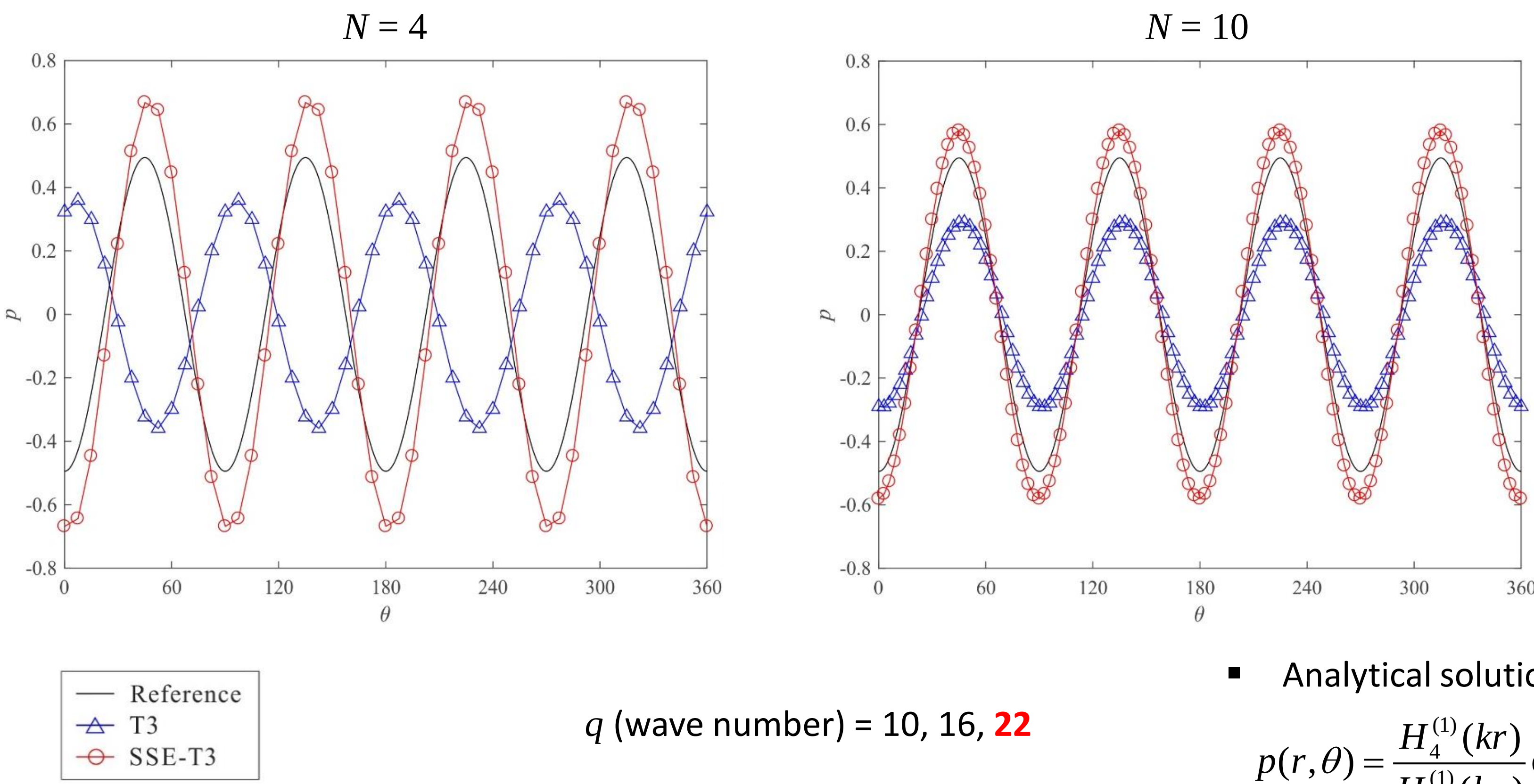
$$p(r,\theta) = \frac{H_4^{(1)}(kr)}{H_4^{(1)}(kr_1)}\cos(4\theta).$$

1) Circumferentially harmonic radiation from a cylinder

- **Pressure distributions for the mesh (N = 4, 10).**



$N \times 12N$



q (wave number) = 10, 16, **22**

- Analytical solution:

$$p(r,\theta)=\frac{H_4^{(1)}(kr)}{H_4^{(1)}(kr_1)}\cos(4\theta).$$

Topic 5

Variational formulation for the strain-smoothed element method

In collaboration with

Ph.D. Jongho Park,

Department of Mathematical Sciences, KAIST.

Variational formulation

❖ Abstract form

- The standard FEM for linear elasticity:

  find $\mathbf{u}_h \in V_h$ such that

  $a(\mathbf{u}_h, \mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in V_h$

  with $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{D}\boldsymbol{\varepsilon}[\mathbf{u}] : \boldsymbol{\varepsilon}[\mathbf{v}]\, d\Omega$.

- The SSE method for linear elasticity:

  find $\bar{\mathbf{u}}_h \in V_h$ such that

  $\bar{a}(\bar{\mathbf{u}}_h, \mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in V_h$

  with $\bar{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{D}\bar{\boldsymbol{\varepsilon}}[\mathbf{u}] : \bar{\boldsymbol{\varepsilon}}[\mathbf{v}]\, d\Omega$.

❖ Setting (mesh, functional space)

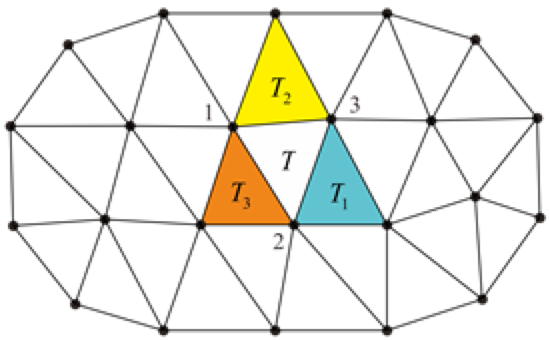
- $P_n(K)$: the collection of all polynomials of degree less than or equal to n (nonnegative integer) on a subregion K of Ω.
- $\mathcal{T}_h$: a geometrically conforming and quasi-uniform triangulation of Ω with the maximum element diameter $h > 0$.

- Discrete displacement space: $V_h = \left\{ \mathbf{u} \in V : \mathbf{u}|_T \in (P_1(T))^2 \text{ for all } T \in \mathcal{T}_h \right\}$.
- Discrete strain/stress space: $W_h = \left\{ \boldsymbol{\varepsilon} \in W : \boldsymbol{\varepsilon}|_T \in (P_0(T))^2 \text{ for all } T \in \mathcal{T}_h \right\}$.

Variational formulation

❖ Original approach of the SSE method

- SSE smoothing operator $S_h : W_h \to \overline{W}_h$.
- Discrete strain/stress space: $\overline{W}_h = \left\{ \bar{\boldsymbol{\varepsilon}} \in W : \bar{\boldsymbol{\varepsilon}}|_T \in (P_1(T))^3 \text{ for all } T \in \mathcal{T}_h \right\}$.



Take any element $T \in \mathcal{T}_h$.
There exist three elements T_1, T_2, and T_3 in $\mathcal{T}_h$ adjacent to T.

- Intermediate smoothed strains ($i = 1, 2, 3$): $\hat{\boldsymbol{\varepsilon}}^{(i)} = \dfrac{1}{|T \cup T_i|} \displaystyle\int_{T \cup T_i} \boldsymbol{\varepsilon}\, d\Omega$.
- The smoothed strains at three Gauss integration points ($i = 1, 2, 3$):

$$\bar{\boldsymbol{\varepsilon}}(Gi) = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(j)} + \hat{\boldsymbol{\varepsilon}}^{(k)}) \quad \text{with} \quad \{i, j, k\} = \{1, 2, 3\}.$$

- The smoothed strain field $\bar{\boldsymbol{\varepsilon}}$ is uniquely determined on T by <u>linear interpolation</u>.
- Bilinear form: $\bar{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{D} S_h \boldsymbol{\varepsilon}[\mathbf{u}] : S_h \boldsymbol{\varepsilon}[\mathbf{v}]\, d\Omega$, $\mathbf{u}, \mathbf{v} \in V_h$.

Variational formulation

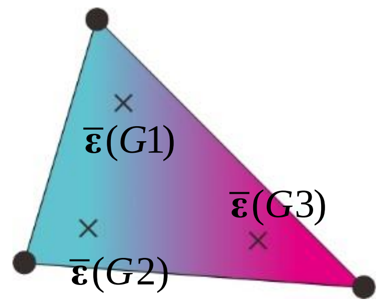
❖ Original approach

- The smoothed strains at three Gauss integration points ($i = 1, 2, 3$):

$$\bar{\boldsymbol{\varepsilon}}(Gi) = \frac{1}{2}(\hat{\boldsymbol{\varepsilon}}^{(j)} + \hat{\boldsymbol{\varepsilon}}^{(k)}) \quad \text{with} \quad \{i, j, k\} = \{1, 2, 3\}.$$

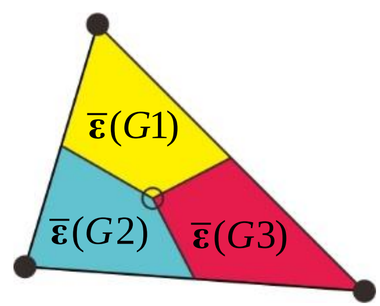
- The smoothed strain field $\bar{\boldsymbol{\varepsilon}}$ is uniquely determined on T by linear interpolation.

$$\bar{\boldsymbol{\varepsilon}}^{(e)} = \left[1 - \frac{1}{q-p}(r+s-2p)\right]\bar{\boldsymbol{\varepsilon}}(G1) + \frac{r-p}{q-p}\bar{\boldsymbol{\varepsilon}}(G2) + \frac{s-p}{q-p}\bar{\boldsymbol{\varepsilon}}(G3).$$

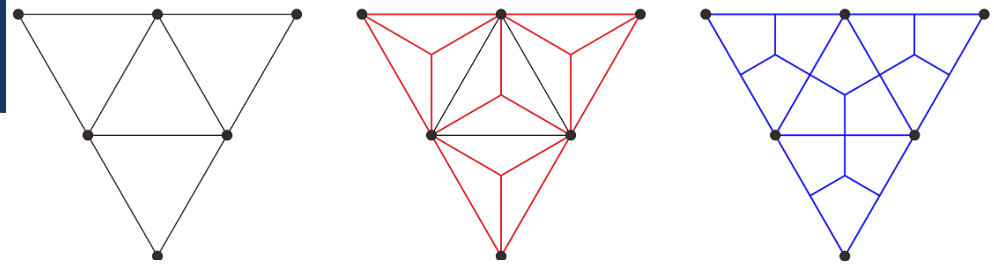


❖ Alternative view: twice-projected strain

- It does not imply modification of the method.
- The alternative view is used only in the process of establishing variational formulation and convergence theory.



Variational formulation



❖ Alternative view: twice-projected strain

- SSE smoothing operator $P_{k,h}: W \to W_{k,h}$
- $(P_{k,h}\boldsymbol{\varepsilon})(x) = \dfrac{1}{|T|}\int_T \boldsymbol{\varepsilon}\, d\Omega$, $\boldsymbol{\varepsilon} \in W$, $T \in \mathcal{T}_{k,h}$, $x \in T$.
- Discrete strain/stress space: $W_{k,h} = \left\{\boldsymbol{\varepsilon} \in W : \boldsymbol{\varepsilon}|_T \in (P_0(T))^3 \text{ for all } T \in \mathcal{T}_{k,h}\right\}$.
- Bilinear form: $\bar{a}(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{D} P_{2,h} P_{1,h} \boldsymbol{\varepsilon}[\mathbf{u}] : P_{2,h} P_{1,h} \boldsymbol{\varepsilon}[\mathbf{v}]\, d\Omega$, $\mathbf{u},\mathbf{v} \in V_h$.

- **Lemma 1.** *Let* $\mathbf{A}$ *be a* 3×3 *matrix. For* $k = 1, 2$, *the piecewise smoothing operator* $P_{k,h}$ *commutes with* $\mathbf{A}$, *i.e.,* $P_{k,h}(\mathbf{A}\boldsymbol{\varepsilon}) = \mathbf{A} P_{k,h}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} \in W$.
- **Lemma 2.** *For* $k = 1, 2$, *the piecewise smoothing operator* $P_{k,h}$ *is the* $(L^2(\Omega))^3$*-orthogonal projection onto* $W_{k,h}$, *i.e.,* $P_{k,h}^2 = P_{k,h}$ *and* $\int_\Omega P_{k,h}\boldsymbol{\varepsilon} : \boldsymbol{\delta}\, d\Omega = \int_\Omega \boldsymbol{\varepsilon} : P_{k,h}\boldsymbol{\delta}\, d\Omega$, $\boldsymbol{\varepsilon},\boldsymbol{\delta} \in W$.
- **Theorem 3.** *Two bilinear forms are identical, i.e., it satisfies that*

  $$\int_\Omega \mathbf{D} S_h \boldsymbol{\varepsilon}[\mathbf{u}] : S_h \boldsymbol{\varepsilon}[\mathbf{v}]\, d\Omega = \int_\Omega \mathbf{D} P_{2,h} P_{1,h} \boldsymbol{\varepsilon}[\mathbf{u}] : P_{2,h} P_{1,h} \boldsymbol{\varepsilon}[\mathbf{v}]\, d\Omega, \quad \mathbf{u},\mathbf{v} \in V_h.$$
- **Proposition 4.** *The SSE method has a unique solution.*

❖ Variational formulation

- Constrained minimization problem:

$$\min_{\mathbf{u}\in V,\boldsymbol{\sigma}_1\in W_1,\boldsymbol{\sigma}_2\in W_2}\left\{\frac{1}{2}\int_\Omega \boldsymbol{\sigma}_2:\mathbf{D}^{-1}\boldsymbol{\sigma}_2\,d\Omega - f(\mathbf{u})\right\}\ \text{subject to}\ \ \boldsymbol{\sigma}_1=\mathbf{DBu}\,,\ \ \boldsymbol{\sigma}_1=\boldsymbol{\sigma}_2\,.$$

- Saddle point problem (using the method of Lagrange multipliers):

$$\min_{\mathbf{u}\in V,\boldsymbol{\sigma}_1\in W_1,\boldsymbol{\sigma}_2\in W_2}\ \max_{\boldsymbol{\varepsilon}_1\in W_1,\boldsymbol{\varepsilon}_2\in W_2}\left\{\frac{1}{2}\int_\Omega \boldsymbol{\sigma}_2:\mathbf{D}^{-1}\boldsymbol{\sigma}_2\,d\Omega - f(\mathbf{u})+\int_\Omega(\mathbf{DBu}-\boldsymbol{\sigma}_1):\boldsymbol{\varepsilon}_1\,d\Omega+\int_\Omega(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2):\boldsymbol{\varepsilon}_2\,d\Omega\right\}.$$

- Variational problem: find $(\mathbf{u},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)\in V\times W_1\times W_2\times W_1\times W_2$ such that

$$\int_\Omega \mathbf{DBv}:\boldsymbol{\varepsilon}_1\,d\Omega+\int_\Omega \boldsymbol{\tau}_1:(-\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_2)d\Omega+\int_\Omega \boldsymbol{\tau}_2:(\mathbf{D}^{-1}\boldsymbol{\sigma}_2-\boldsymbol{\varepsilon}_2)d\Omega=f(\mathbf{v})\,,$$

$$\mathbf{v}\in V\,,\ \ \boldsymbol{\tau}_1\in W_1,\ \ \boldsymbol{\tau}_2\in W_2\,,$$

$$\int_\Omega(\mathbf{DBu}-\boldsymbol{\sigma}_1):\boldsymbol{\delta}_1\,d\Omega+\int_\Omega(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2):\boldsymbol{\delta}_2\,d\Omega=0,\ \ \boldsymbol{\delta}_1\in W_1,\ \ \boldsymbol{\delta}_2\in W_2\,.$$

- **Proposition 5.** The variational problem has a unique solution $(\mathbf{u},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)$ $\in V\times W_1\times W_2\times W_1\times W_2$.

- *Remark* 6. From Proposition 5, we observe that the Lagrange multipliers $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ in fact play a role of the strain field.

❖ Variational formulation

- Galerkin approximation: find $(\bar{\mathbf{u}}_h, \boldsymbol{\sigma}_{1,h}, \boldsymbol{\sigma}_{2,h}, \boldsymbol{\varepsilon}_{1,h}, \boldsymbol{\varepsilon}_{2,h}) \in V_h \times W_{1,h} \times W_{2,h} \times W_{1,h} \times W_{2,h}$ such that

$$\int_\Omega \mathbf{DBv} : \boldsymbol{\varepsilon}_{1,h}\, d\Omega + \int_\Omega \boldsymbol{\tau}_1 : (-\boldsymbol{\varepsilon}_{1,h} + \boldsymbol{\varepsilon}_{2,h})\, d\Omega + \int_\Omega \boldsymbol{\tau}_2 : (\mathbf{D}^{-1}\boldsymbol{\sigma}_{2,h} - \boldsymbol{\varepsilon}_{2,h})\, d\Omega = f(\mathbf{v}),$$

$$\mathbf{v} \in V_h, \quad \boldsymbol{\tau}_1 \in W_{1,h}, \quad \boldsymbol{\tau}_2 \in W_{2,h},$$

$$\int_\Omega (\mathbf{D}\mathbf{B}\bar{\mathbf{u}}_h - \boldsymbol{\sigma}_{1,h}) : \boldsymbol{\delta}_1\, d\Omega + \int_\Omega (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_{2,h}) : \boldsymbol{\delta}_2\, d\Omega = 0, \quad \boldsymbol{\delta}_1 \in W_{1,h}, \quad \boldsymbol{\delta}_2 \in W_{2,h}.$$

1) $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\tau}_2 = \mathbf{0} \;\rightarrow\; \boldsymbol{\varepsilon}_{1,h} = P_{1,h}\boldsymbol{\varepsilon}_{2,h}.$

2) $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\tau}_1 = \mathbf{0} \;\rightarrow\; \boldsymbol{\varepsilon}_{2,h} = \mathbf{D}^{-1}\boldsymbol{\sigma}_{2,h}.$

3) $\boldsymbol{\tau}_1 = \mathbf{0}$ and $\boldsymbol{\tau}_2 = \mathbf{0} \;\rightarrow\; \int_\Omega \mathbf{D} P_{2,h} P_{1,h} \mathbf{B}\bar{\mathbf{u}}_h : P_{2,h} P_{1,h} \mathbf{B}\mathbf{v}\, d\Omega = f(\mathbf{v}), \quad \mathbf{v} \in V_h.$

<u>Therefore, the SSE method can be derived from the proposed variational principle.</u>

A convergence theory for the SSE method will be studied based on the proposed variational formulation.

4. Conclusions & Future works

Conclusions

1. **The SSE method has been developed for the linear 2D & 3D solid elements.**
   - The methods require no additional DOFs and no special smoothing domains.
   - The elements provide more accurate solutions, especially more continuous strain/stress fields.

2. **The SSE has been extended to 4-node 2D solid elements and 3-node shell elements.**
   - The piecewise linear shape functions are used for the extension to the 4-node elements.
   - For the extension to the shell elements, the strain components are decomposed and transformed in a proper way, and then the SSE method is applied.

3. **The strain-smoothed elements are still effective in geometric nonlinear analysis.**

4. **The elements are also effective in acoustic radiation analysis.**

5. **The variational formulation of the SSE method has been established.**

Future works

1. **Convergence analysis using the proposed variational principle.**

2. **Incompressible and nearly incompressible materials.**

3. **Material nonlinear analyses.**

4. **Acoustic radiation/scattering analyses.**

5. **Dynamic analysis.**

감사합니다.