Ph. D. Thesis Presentation

PU 기반 솔리드 유한요소의 개발

Development of the partition of unity based solid finite elements free from the linear dependence problem

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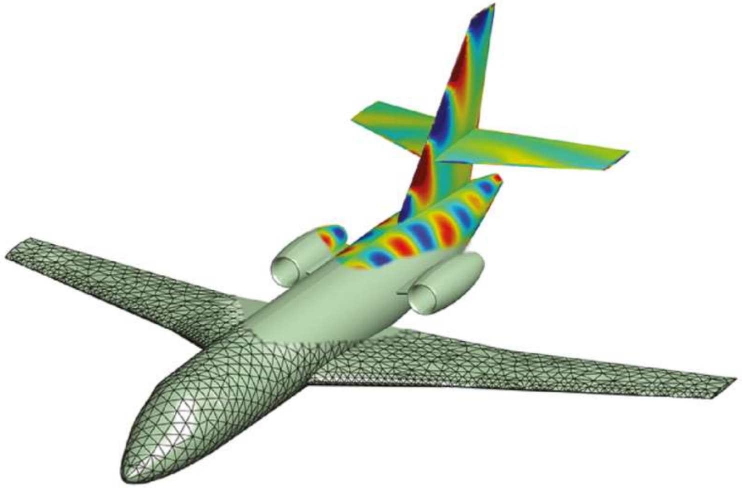
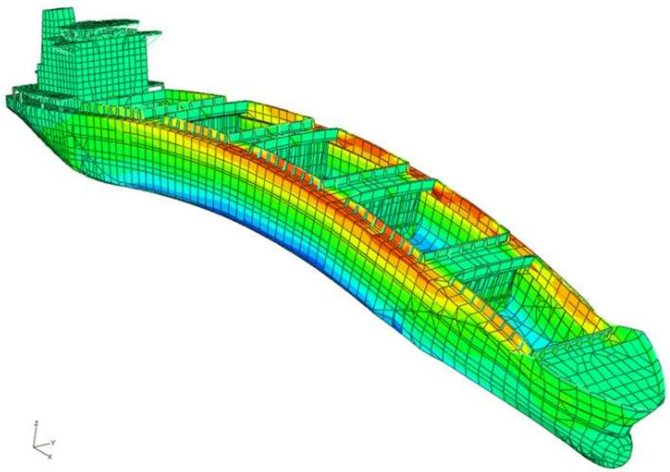
Contents

1. Introduction

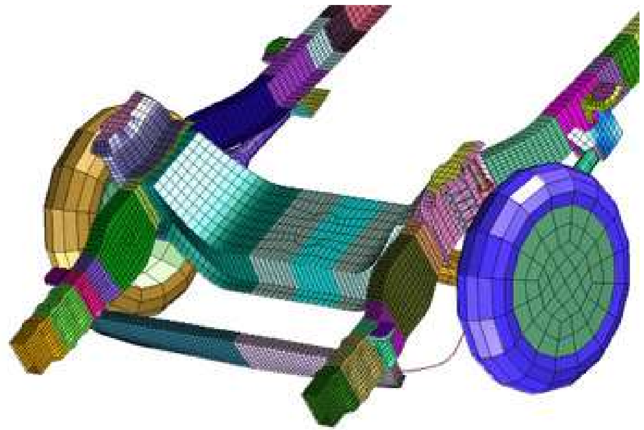
FEM in engineering fields



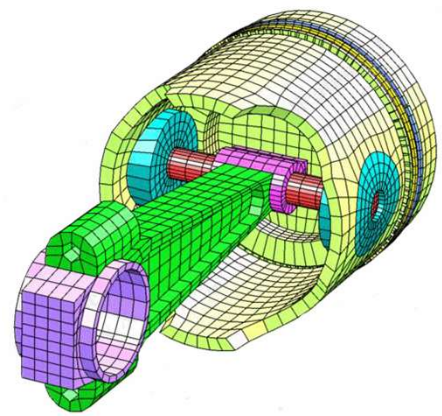
Aircraft



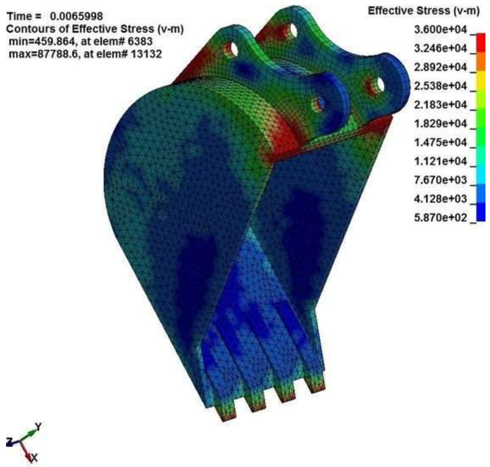
Ship building



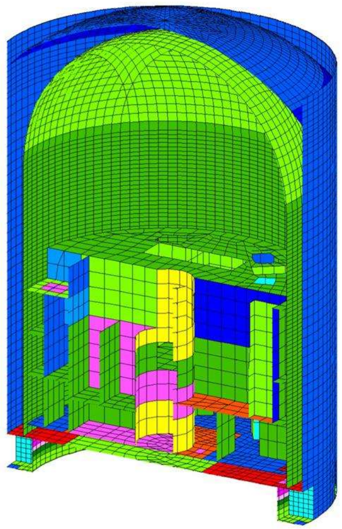
Automobile



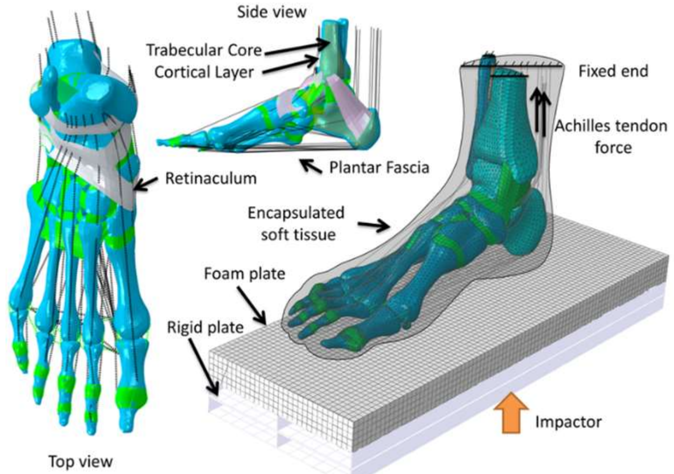
Mechanical component



Construction equipment



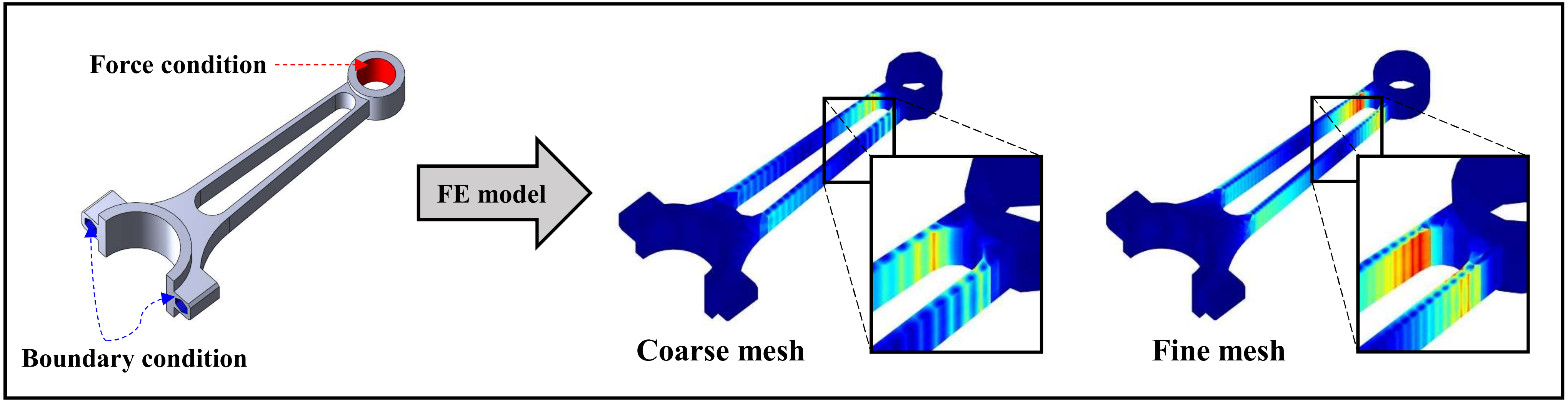
Reactor building



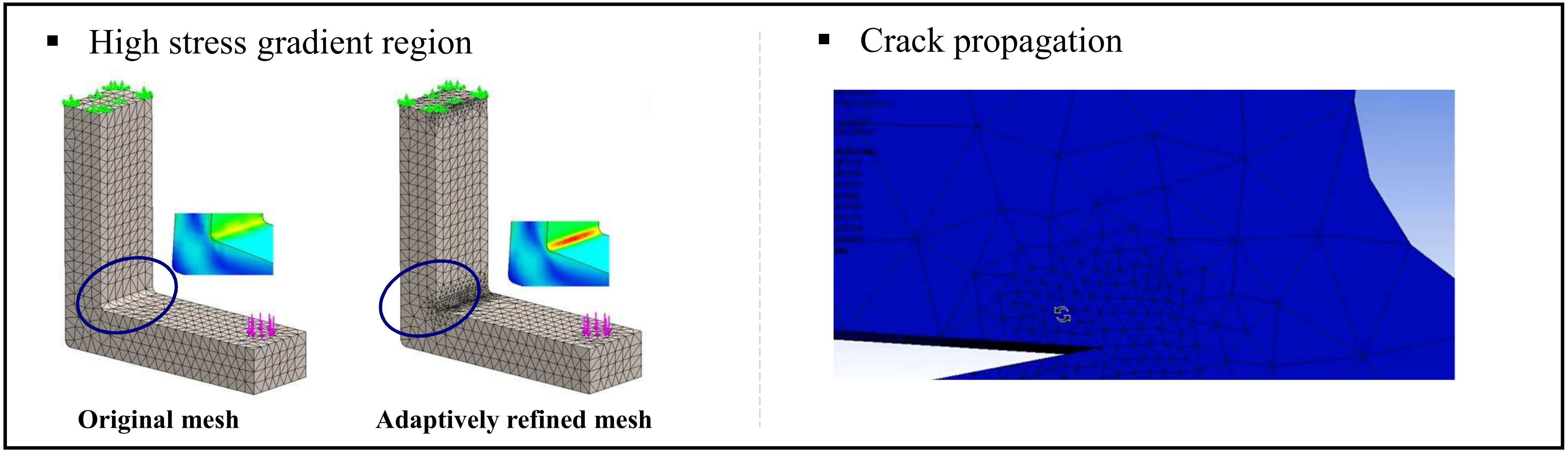
Healthcare

Motivation of PU based FEM

❖ The solution **accuracy depends on the meshes used.**



❖ To obtain reliable solution, **mesh refinement is frequently required.**



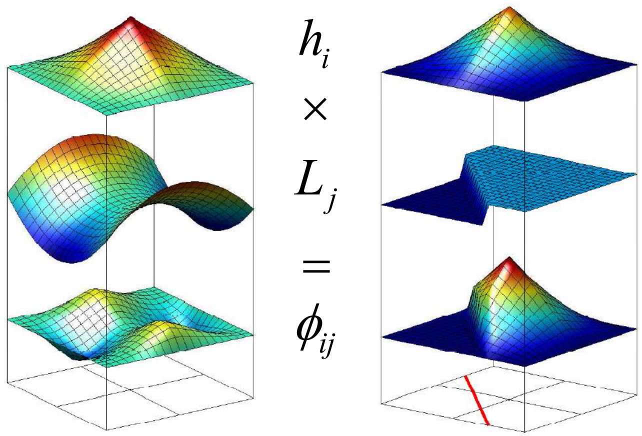
PU based FEM concepts

❖ Enriched FE interpolation

$$u_h(\mathbf{x}) = \sum_{i=1}^{n} h_i \bar{u}_i + \sum_{i=1}^{n} h_i \sum_{j=1}^{m} L_j(\mathbf{x}) \hat{u}_{ij}$$

$$= \underbrace{\boxed{\sum_{i=1}^{n} h_i \bar{u}_i}}_{\text{Standard FE interpolation}} + \underbrace{\boxed{\sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \hat{u}_{ij}}}_{\text{Enrichment}}$$

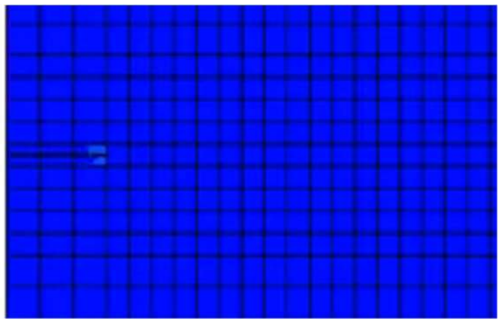
- Enriched FEM
- GFEM (**G**eneralized **F**inite **E**lement **M**ethod)
- XFEM (e**X**teneded **F**inite **E**lement **M**ethod)



Polynomial function

Discontinuous function

- $\bar{u}_i$: standard DOFs
- $\hat{u}_{ij}$: enriched DOFs
- h_i : Partition of unity function (shape function)
- $L_j(\mathbf{x})$: cover function (enrichment function)
  - ✓ Crack problem : Singular and discontinuous functions
  - ✓ Wave propagation problem : Harmonic function
  - ✓ Element performance improvement : Polynomial

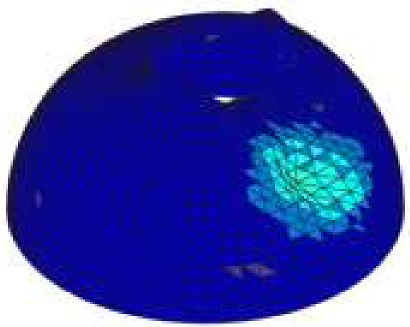


Crack propagation simulation without mesh refinement

Duarte. (2012), Lecture note of Delft University

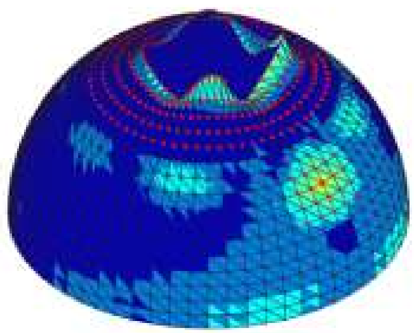
PU based FEM advantages

- **Various functions suitable for each problem can be applied as the cover function.**
- **Higher order solution can be obtained by applying polynomial cover functions to existing linear element meshes.**
  - Order of interpolation can be chosen arbitrary
- **The enrichment functions can be applied to local area adaptively.**
  - Improve the solution accuracy without local mesh refinement
  - Can be combined with mesh refinement

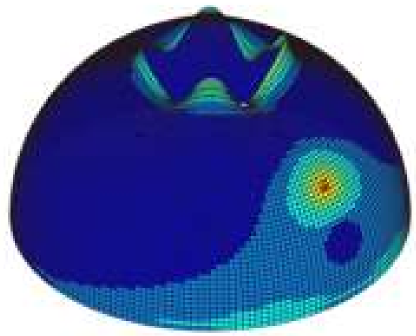


MITC3 shell FE

No mesh refinement
Adaptively enriched



Adaptively enriched MITC3



Reference

Jeon HM, Lee PS, Bathe KJ. (2014), *The MITC3 shell finite element enriched by interpolation covers.*

2. LD problem & History

Linear dependence problem

Functions in the enriched FE interpolation become linearly dependent.

- When **partition of unity** and **cover function** consist of **polynomial.**

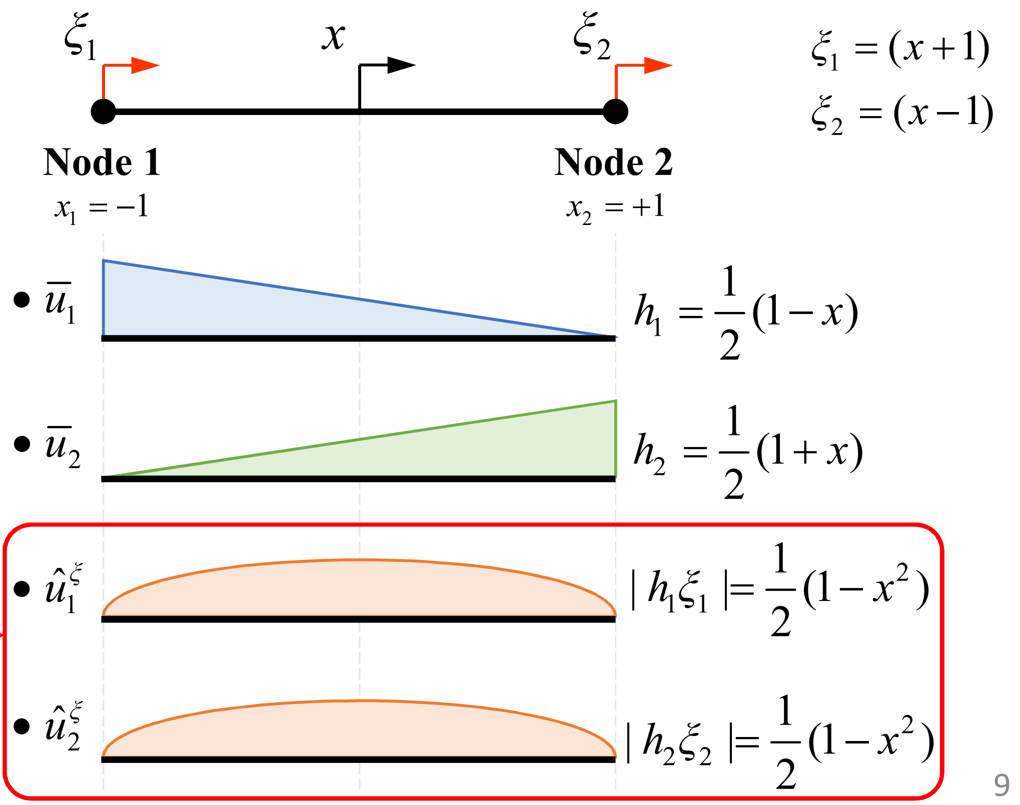
The Linear Dependence(LD) problem occurs.

- **Singular stiffness matrix.**

❖ 1D bar example

$$u_h(\mathbf{x}) = \underbrace{\sum_{i=1}^{2} h_i \bar{u}_i}_{\text{Standard FE interpolation}} + \underbrace{\sum_{i=1}^{2} h_i \sum_{j=1}^{1} L_j(\mathbf{x}) \hat{u}_i^j}_{\text{Enrichment}}$$

- Cover function : $L_j(\mathbf{x}) = \xi_j$ (linear polynomial)
- Total DOFs = $2\,(\bar{u}_1\,,\,\bar{u}_2\,) + 2\,(\hat{u}_1^{\xi}\,,\,\hat{u}_2^{\xi}\,) = 4$
- Linearly dependent DOFs : $\hat{u}_1^{\xi}$ and $\hat{u}_2^{\xi}$
- Rank deficiency(RD) = 1(rigid body) + **1(LD)**



Linear dependence problem

Functions in the enriched FE interpolation become linearly dependent.

- When **partition of unity** and **cover function** consist of **polynomial.**

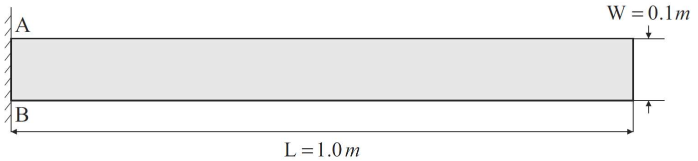
⇩

The Linear Dependence(LD) problem occurs.

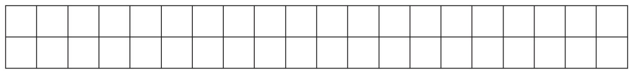
- **Singular stiffness matrix.**

❖ 2D cantilever beam – free vibration analysis

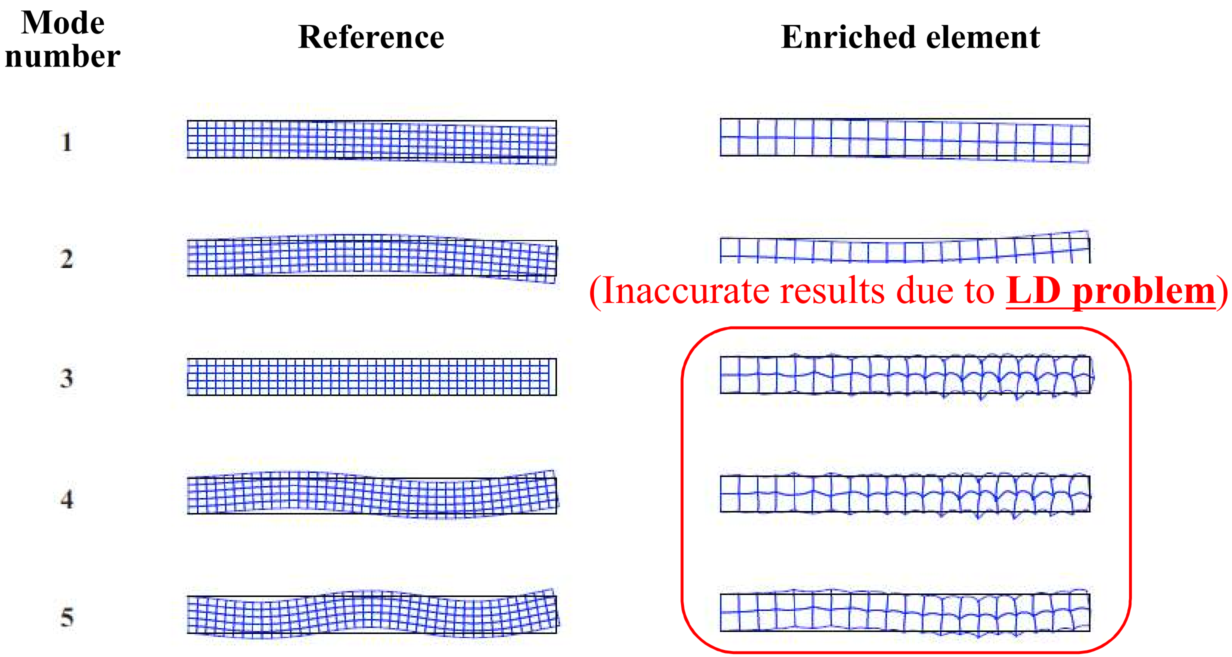
- **Cantilever beam problem**



- **Finite element model**



- **Free vibration modes**

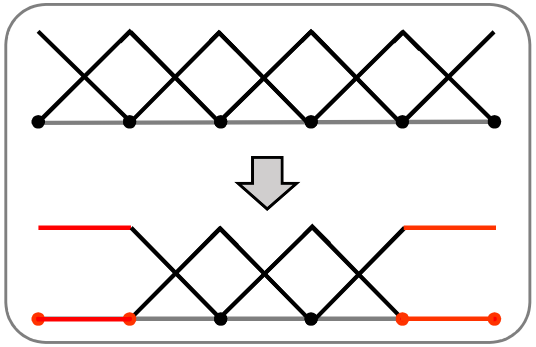


Brief history of LD problem

1990s

- Babuska and Melenk (1997)
  - **A pioneer of PU based FEM**
  - First report of LD problem in 1D
  - **Design PU function to avoid LD** problem (1D)

Modification of PU function



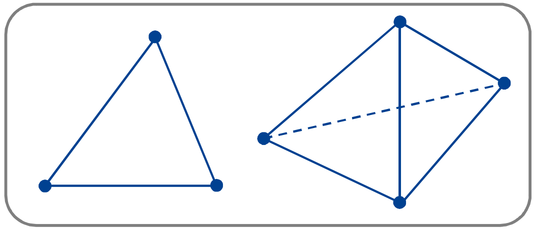
- Oden et al. (1998)
  - **Elimination linear polynomial term** in cover function

$$\{\cancel{\xi}, \cancel{\eta}, \xi^2, \xi\eta, \eta^2\}$$

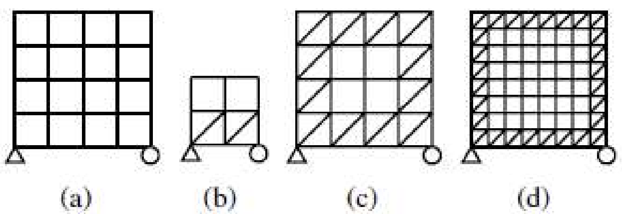
2000s

- Strouboulis et al. (2000)
  - Report that Oden’s work is not enough to avoid LD problem
  - Adapt special equation solver

3-node & 4-node elements



- Tian et al. (2006)
  - **Suppressing enriched DOFs at essential boundary**
  - Effective for 2D **3-node triangular** and 3D **4-node tetrahedral** elements
  - **Modeling method**



(a), (b) : LD problem occurs
(c), (d) : LD problem removed

Brief history of LD problem

2010s

- Oh et al. (2008)
  - **Flat top PU function** (2D)
  - Expansion of earlier work (1D, 1997)
  - Hong and Lee (2013), An et al. (2014)
  - Not easy to construct, artificial constant

- An et al. (2011, 2012)
  - **Prediction of RD in regular mesh**
  - 2D **3-node triangular** and **4-node quadrilateral** elements
  - 3D **4-node tetrahedral** and **8-node hexahedral** elements

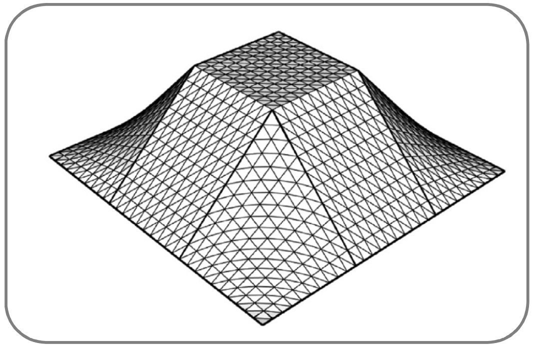
- Ham and Bathe (2012)
  - Harmonic enrichment function
  - **Modification of mass matrix** to avoid LD problem

$$\mathbf{M} = (1-\alpha)\mathbf{M}_{consistent} + \alpha\mathbf{M}_{lumped}$$

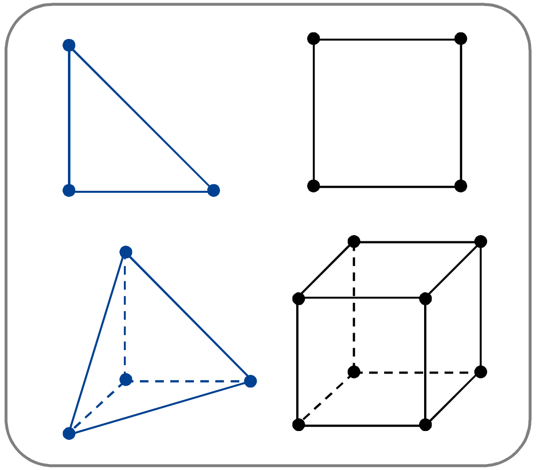
- Kim and Bathe (2014)
  - A scheme to improve finite element solution by use of cover functions
  - **Enriched 2D 3-node triangular and 3D 4-node tetrahedral elements**
  - Suppressing enriched DOFs at essential boundary (Tian's work, 2006)

- Jeon et al. (2014, 2018)
  - **Enriched 3-node triangular shell elements** (cover function : linear polynomial)
  - Suppressing enriched DOFs at essential boundary (Tian's work, 2006)

2D Flat-top PU function

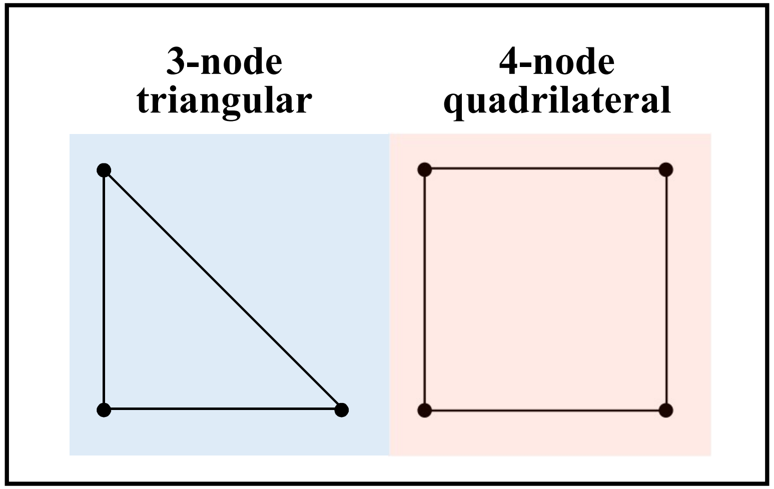


2D and 3D elements



Research purpose

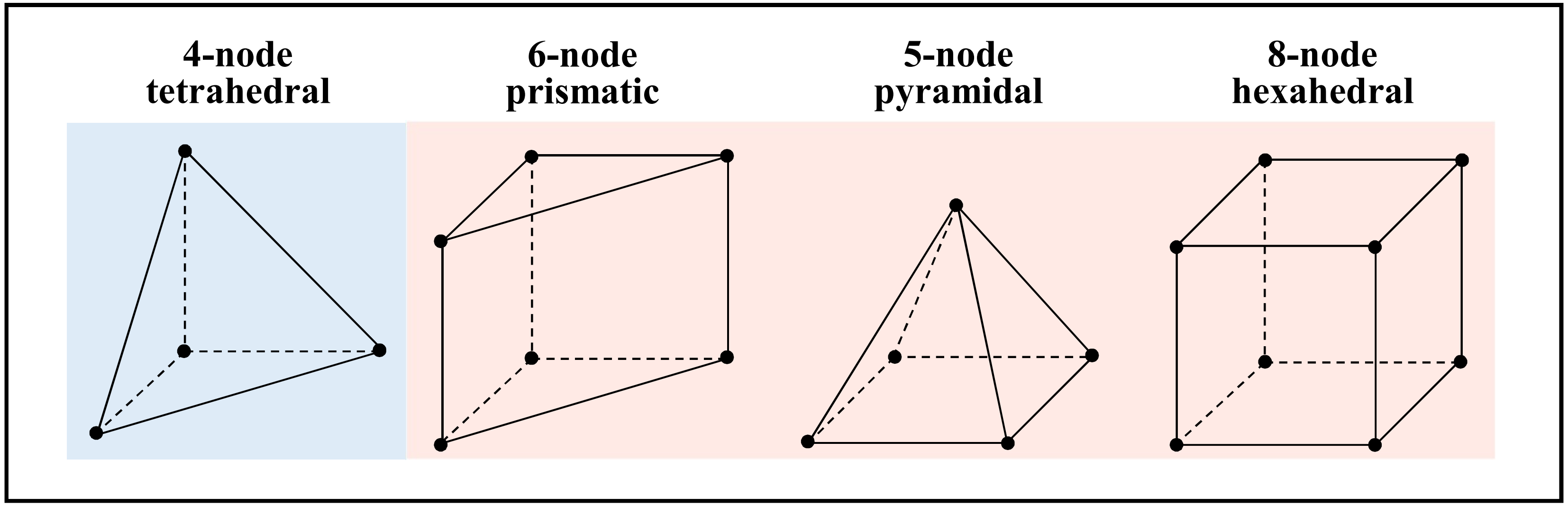
❖ 2D solid finite elements



➢ **Tian's approach**
(suppressing enriched DOFs at essential boundary)

• 2D 3-node triangular element • 3D 4-node tetrahedral element		**Resolved**
• 2D 4-node quadrilateral element • 3D 5-node pyramidal element • 3D 6-node prismatic element • 3D 8-node hexahedral element		**Still remain**

❖ 3D solid finite elements



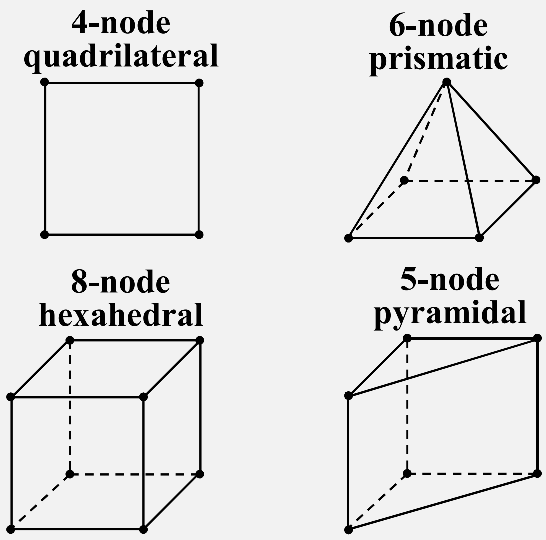
Research purpose

❖ Resolution the linear dependence problem

- Topic 1-1. Development of the enriched 4-node 2D solid finite element
- Topic 1-2. Development of the enriched 3D solid finite elements
  - ✓ Suppression of the enriched DOFs at essential boundary
  - ✓ Application of new shape function to FE interpolations
  - ✓ Investigation on the performance of the new enriched elements

Requirements

- Resolving the LD problem (in a simple and effective way)
- Spatially isotropic behavior
- Pass the patch test and zero energy mode test
- Good convergence behavior
- Adaptive use of the cover function



Research purpose

❖ **Resolution the linear dependence problem**

- Topic 1-1. Development of the enriched 4-node 2D solid finite element
- Topic 1-2. Development of the enriched 3D solid finite elements
  - ✓ Suppression of the enriched DOFs at essential boundary
  - ✓ Application of new shape function to FE interpolations
  - ✓ Investigation on the performance of the new enriched elements

❖ **Adaptive use of cover functions**

- Topic 2. Automatic procedure to improve FE solutions
  - ✓ Error indicator that selects the order of cover functions for each node
  - ✓ Feasibility of automatically improving FE solutions

❖ **Enrichment scheme for smoothed finite element**

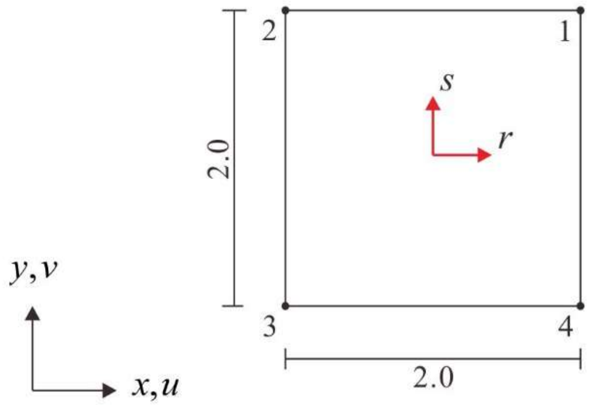
- Topic 3. Enriching strain-smoothed 3-node solid element
  - ✓ Polynomial enrichment for strain-smoothed element
  - ✓ Investigation on the performance of the enriched strain-smoothed element

3. Research topics

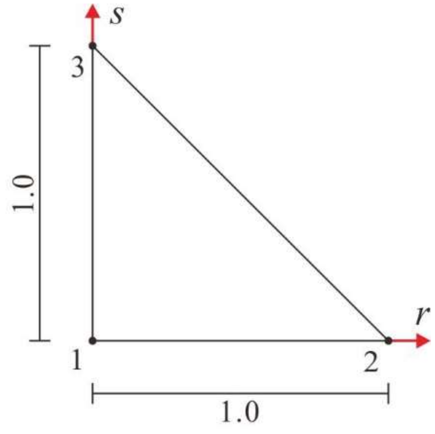
Topic 1-1. Enriched 4-node 2D solid element

Enriched 2D solid finite elements

4-node quadrilateral element $(n=4)$



3-node triangular element $(n=3)$



- **The enriched finite element interpolation of displacement** u

➢ $$u = \sum_{i=1}^{n} h_i \bar{u}_i + \sum_{i=1}^{n} \sum_{j=1}^{m} h_i L_j \hat{u}_i^j$$

Applying polynomial as the cover function

$$L_1 = \xi_i,\ L_2 = \eta_i,\ L_3 = \xi_i^2,\ \cdots \qquad \text{with} \quad \xi_i = \frac{(x - x_i)}{\chi_i},\ \eta_i = \frac{(y - y_i)}{\chi_i}$$

➢ $$u = \sum_{i=1}^{n} h_i \bar{u}_i + \sum_{i=1}^{n} \hat{\mathbf{h}}_i \hat{\mathbf{u}}_i^u \quad \text{where} \quad \hat{\mathbf{h}}_i = h_i \begin{bmatrix} \xi_i & \eta_i & \xi_i^2 & \cdots \end{bmatrix},\ \hat{\mathbf{u}}_i^u = \begin{bmatrix} \hat{u}_i^{\xi} & \hat{u}_i^{\eta} & \hat{u}_i^{\xi^2} & \cdots \end{bmatrix}^{\mathrm{T}}$$

$$= \bar{u} + \hat{u}$$

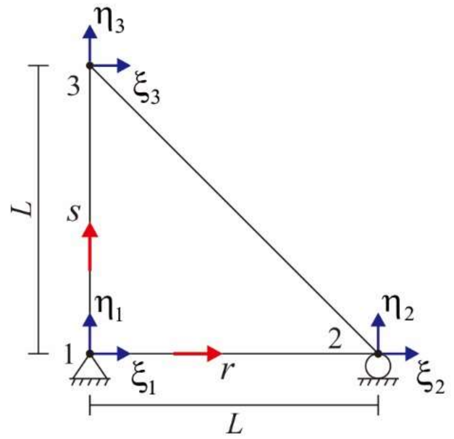
$\bar{u}$: Standard finite element interpolation

$\hat{u}$: Additional higher order interpolation

Investigation of the LD problem

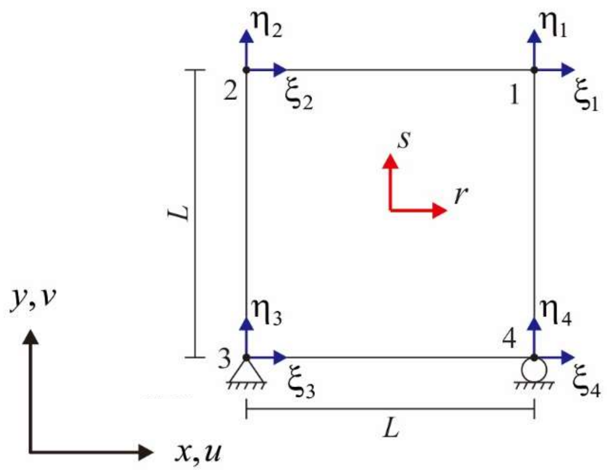
❖ A 4-node quadrilateral element and a 3-node triangular element

- Linear cover function ($\hat{\mathbf{h}}_i = h_i[\xi_i \quad \eta_i],\ \hat{\mathbf{u}}_i^u = [\hat{u}_i^{\xi} \quad \hat{u}_i^{\eta}]^{\mathrm{T}}$)
- Enriched DOFs are suppressed at essential boundary (Tian's approach)
- PU function (h_i) : 4-node quadrilateral(**bilinear shape function**), 3-node triangular(**linear shape function**)



➢ $u = (r)\bar{u}_2 + (s)\bar{u}_3 + (s)(r)\hat{u}_2^{\xi} + (s)(s-1)\hat{u}_1^{\eta}$

➢ **Functions are linear independent**



➢ $u = \alpha(1+r)(1+s)\bar{u}_1 - \beta(1-r^2)(1+s)\hat{u}_1^{\xi} - \beta(1+r)(1-s^2)\hat{u}_1^{\eta} + \alpha(1-r)(1+s)\bar{u}_2 - \beta(1-r^2)(1+s)\hat{u}_2^{\xi} - \beta(1-r)(1-s^2)\hat{u}_2^{\eta} + \alpha(1+r)(1-s)\bar{u}_4$

➢ **Linearly dependent functions** ($\hat{u}_1^{\xi}, \hat{u}_2^{\xi}$)

($\alpha,\ \beta$: constants)

Development of new enriched 4-node 2D element

❖ LD problem in 2D solid elements

- When the enriched DOFs are suppressed at essential boundary (simple and effective)
- **3-node triangular element** (linear shape function) : LD problem is **resolved.**
- **4-node quadrilateral element** (bilinear shape function) : LD problem **still exists.**

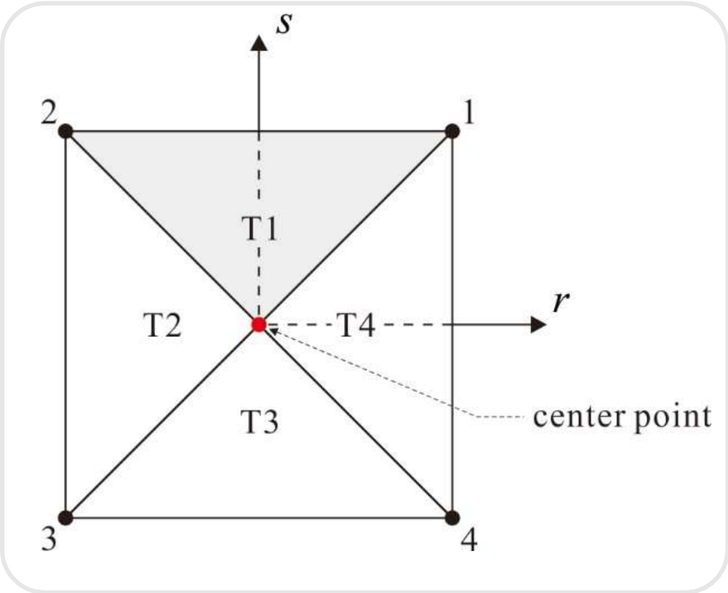
The key idea is to resolve the LD problem

➡ **Employ piecewise linear shape function for 4-node quadrilateral element**

❖ **Shape function requirements**

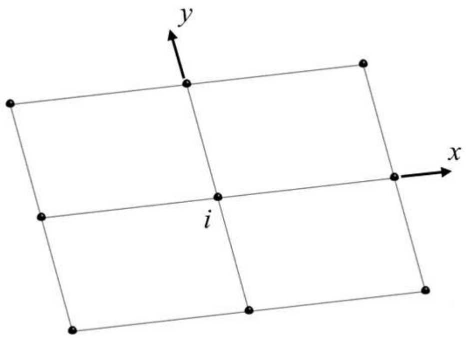
- Kronecker delta property: $\hat{h}_i(r_j, s_j) = \delta_{ij}$ with $i, j = 1, 2, 3, 4$
- Partition of unity: $\sum_{i=1}^{4} \hat{h}_i = 1$
- Compatibility: linear variation along edges of the element.
- Completeness: displacement interpolations able to represent rigid body modes and constant strain states.

Piecewise linear shape functions

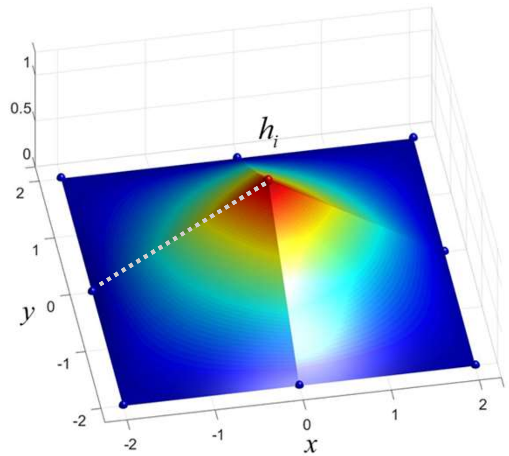


Triangular subdivision of the 4-node quadrilateral element

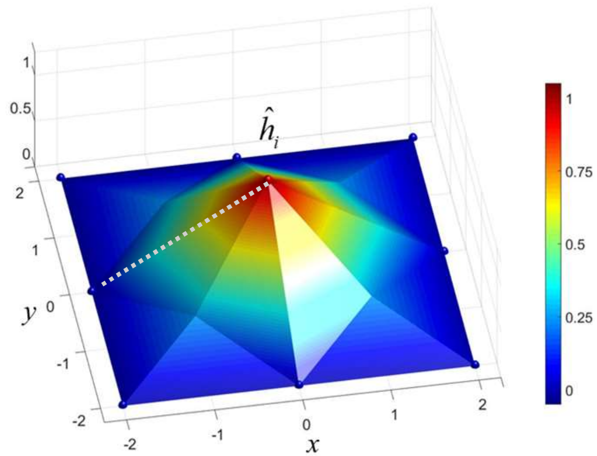
- **The linear shape function on each sub-domain**
  - $\boxed{\hat{h}_i(r,s) = \hat{a}_i + \hat{b}_i\, r + \hat{c}_i\, s}$
- Requirements of the linear shape function ($\hat{h}_1$) on the T1
  - $\hat{h}_1(1,1) = 1,\ \hat{h}_1(1,1) = 0$: Kronecker delta property at nodes
  - $\hat{h}_1(0,0) = 1/4$: Partition of unity at center point
- The linear shape function($\hat{h}_1$) on the T1
  - $\hat{h}_1 = (1 + 2r + s)/4$



Bilinear shape function

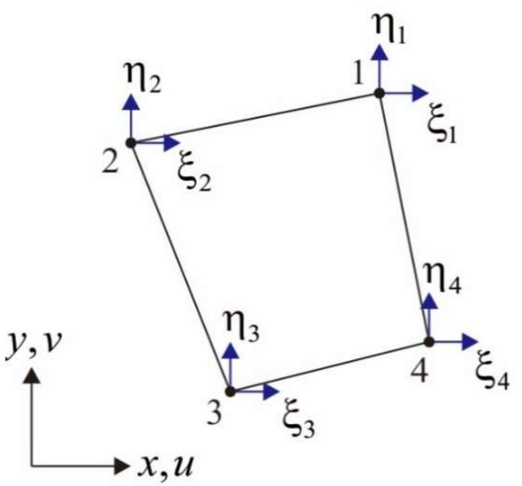


Piecewise linear shape function



New enriched 4-node 2D element

❖ Geometry and displacement interpolations

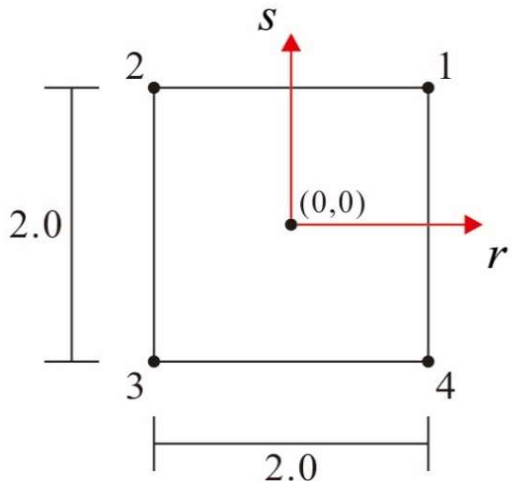


Global and nodal local coordinate systems

➢ $$\mathbf{x}(r,s)=\sum_{i=1}^{4}\hat{h}_i(r,s)\mathbf{x}_i \quad \text{with} \quad \mathbf{x}_i=\begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

➢ $$\mathbf{u}(r,s)=\overline{\mathbf{u}}+\hat{\mathbf{u}}=\sum_{i=1}^{4}\hat{h}_i(r,s)\overline{\mathbf{u}}_i+\sum_{i=1}^{4}\hat{\mathbf{H}}_i(r,s)\hat{\mathbf{u}}_i$$

$$\text{with} \quad \overline{\mathbf{u}}_i=\begin{bmatrix} \overline{u}_i \\ \overline{v}_i \end{bmatrix},\ \hat{\mathbf{u}}_i=\begin{bmatrix} \hat{\mathbf{u}}_i^u \\ \hat{\mathbf{u}}_i^v \end{bmatrix},\ \hat{\mathbf{H}}=\begin{bmatrix} \hat{\mathbf{h}}_i & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{h}}_i \end{bmatrix}$$



Natural coordinate system

- PU function ($\hat{h}_i$) : **the piecewise linear shape function**

- linear cover

$$\hat{\mathbf{h}}_i=\hat{h}_i[\xi_i \quad \eta_i]$$
$$\hat{\mathbf{u}}_i^u=[u_i^{\xi} \quad u_i^{\eta}]$$
$$\hat{\mathbf{u}}_i^v=[v_i^{\xi} \quad v_i^{\eta}]$$

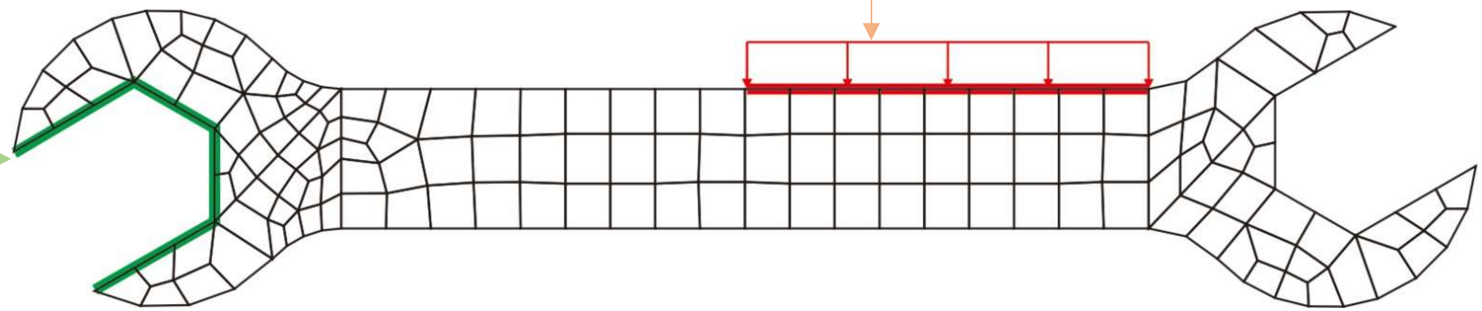
- quadratic cover

$$\hat{\mathbf{h}}_i=\hat{h}_i[\xi_i \quad \eta_i \quad \xi_i^2 \quad \xi_i\eta_i \quad \eta_i^2]$$
$$\hat{\mathbf{u}}_i^u=[u_i^{\xi} \quad u_i^{\eta} \quad u_i^{\xi^2} \quad u_i^{\xi\eta} \quad u_i^{\eta^2}]$$
$$\hat{\mathbf{u}}_i^v=[v_i^{\xi} \quad v_i^{\eta} \quad v_i^{\xi^2} \quad v_i^{\xi\eta} \quad v_i^{\eta^2}]$$

New enriched 4-node 2D element

❖ Force and boundary condition

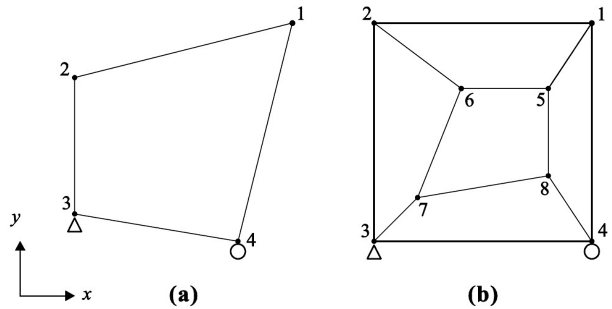
▪ Force vector : $R_S = \sum_{m=1}^{e} \int_{S_f^{(m)}} \hat{\mathbf{H}}^{(m)^{\mathrm{T}}} \mathbf{f}^S \, dV$



▪ Applying boundary condition : Suppressing enriched DOFs ($\hat{\mathbf{u}}_i$)

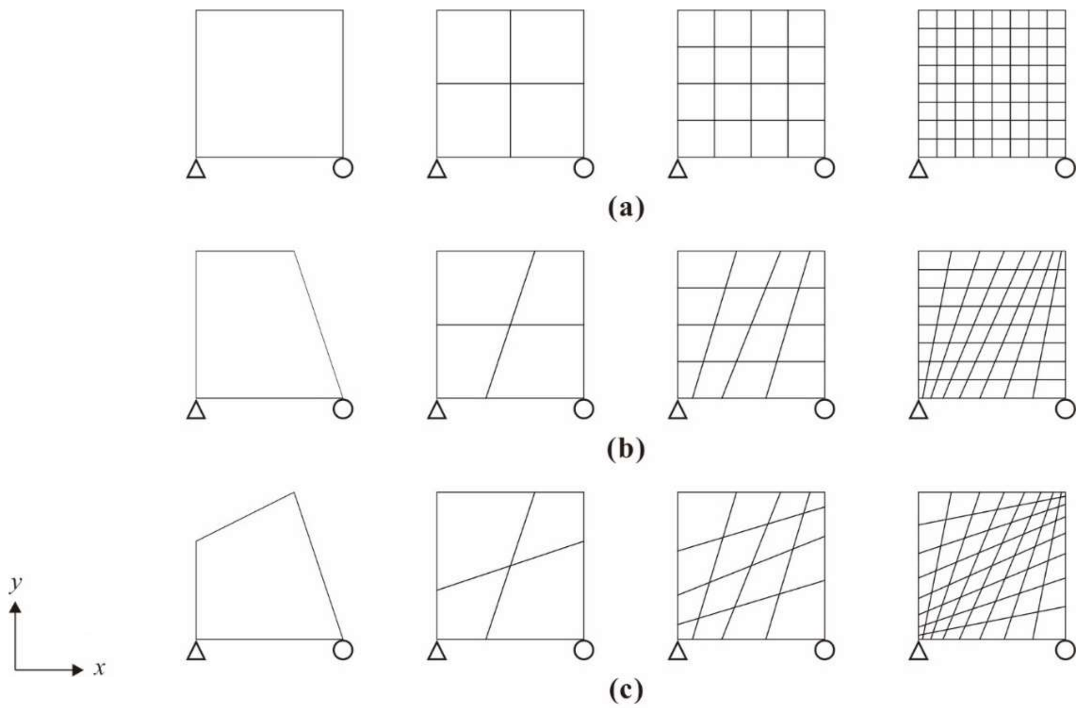
Requirements

- ❖ Resolving the LD problem
- ❖ **<u>Spatially isotropic behavior</u>**
- ❖ **<u>Pass the patch test and zero energy mode test</u>**
- ❖ Good convergence behavior
- ❖ **<u>Adaptive use of the cover function</u>**



Finite element models for basic tests
(a) Isotropy and zero energy mode tests
(b) Patch tests

Investigation of the LD problem



PU function
- Previous: Bilinear
- New: Piecewise linear

Mesh

(a) : square mesh
(b) : trapezoidal mesh
(c) : distorted mesh.

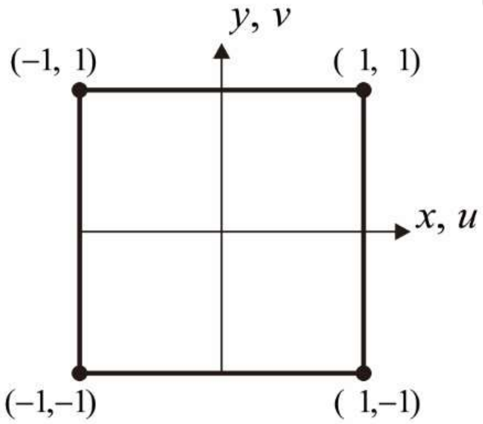
RD : # of zero eigenvalues

d : order of cover function

<table>
<tr><th rowspan="3">Element</th><th rowspan="3"># of element layer</th><th colspan="6">RD / Total DOFs</th></tr>
<tr><th colspan="2">Square mesh</th><th colspan="2">Trapezoidal mesh</th><th colspan="2">Distorted mesh</th></tr>
<tr><th>$d = 1$</th><th>$d = 2$</th><th>$d = 1$</th><th>$d = 2$</th><th>$d = 1$</th><th>$d = 2$</th></tr>
<tr><td rowspan="4">Previous</td><td>1</td><td colspan="4" rowspan="4">LD problem exists. (RD is observed)</td><td>0/13</td><td>0/25</td></tr>
<tr><td>2</td><td>0/43</td><td>0/85</td></tr>
<tr><td>4</td><td>0/139</td><td>0/277</td></tr>
<tr><td>8</td><td>0/475</td><td>0/949</td></tr>
<tr><td rowspan="4">New</td><td>1</td><td colspan="6" rowspan="4">LD problem is resolved. (RD is <u>NOT</u> observed)</td></tr>
<tr><td>2</td></tr>
<tr><td>4</td></tr>
<tr><td>8</td></tr>
</table>

Numerical examples

❖ Ad-hoc problem



- **Body force** (m=5)

$$f_x^B = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right),\ f_y^B = -\left(\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x}\right)$$

$$u = (1-x^2)^2(1-y^2)^2 e^{my} \cos mx$$

$$v = (1-x^2)^2(1-y^2)^2 e^{my} \sin\ mx$$

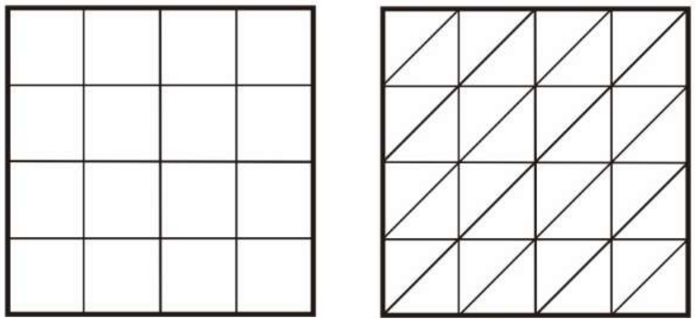
- **Boundary condition**

$u = v = 0$ at $y = -1$

- **Material property**

$E = 1.0\times10^7,\ \nu = 0.3$

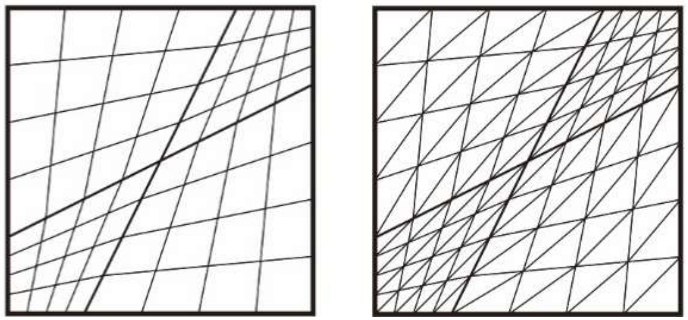
Regular meshes



Quadrilateral (N=4) Triangular (N=4)

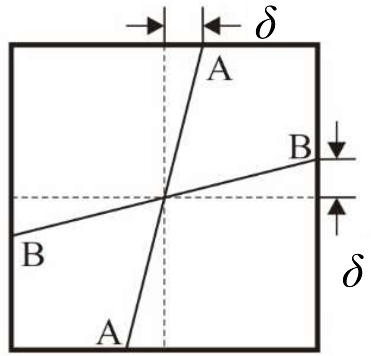
(N : the number of element layers along an edge)

Distorted meshes



Quadrilateral (N=8) Triangular (N=8)

Distortion type

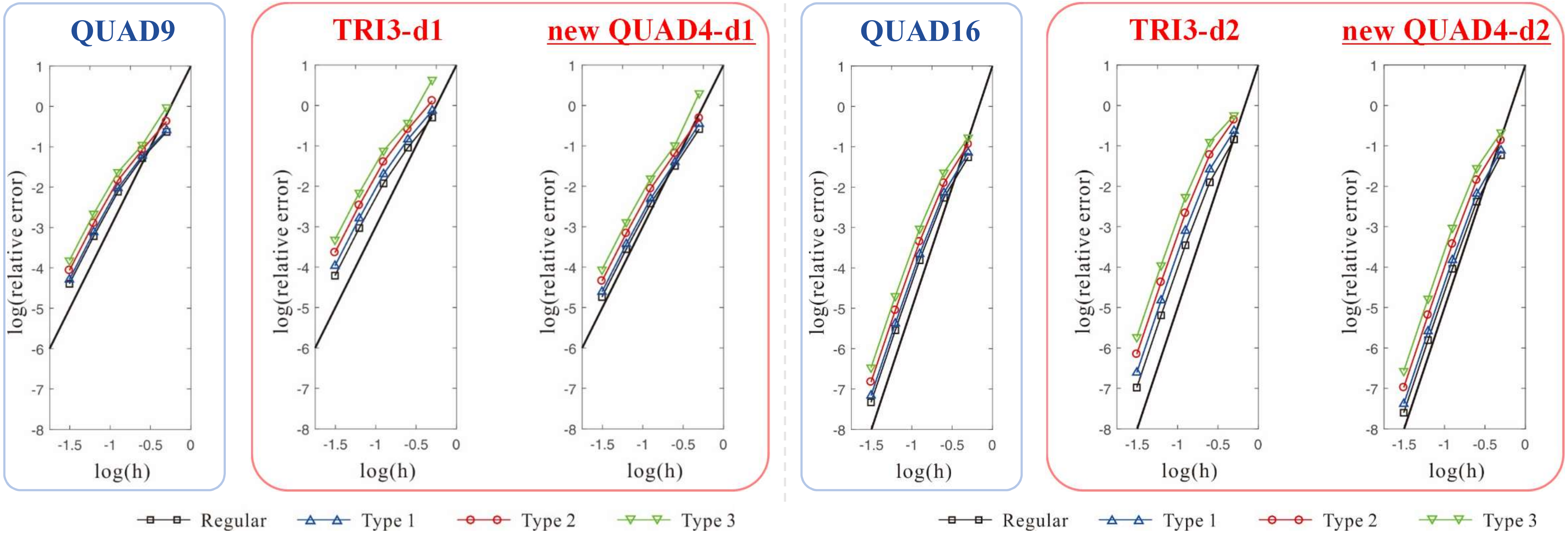


Type 1 ($\delta = 0.25$)
Type 2 ($\delta = 0.50$)
Type 3 ($\delta = 0.75$)

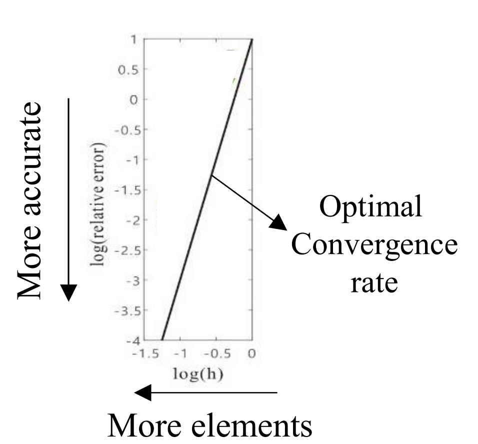
Numerical examples

Quadratic elements | Cubic elements

Standard | Enriched | Standard | Enriched

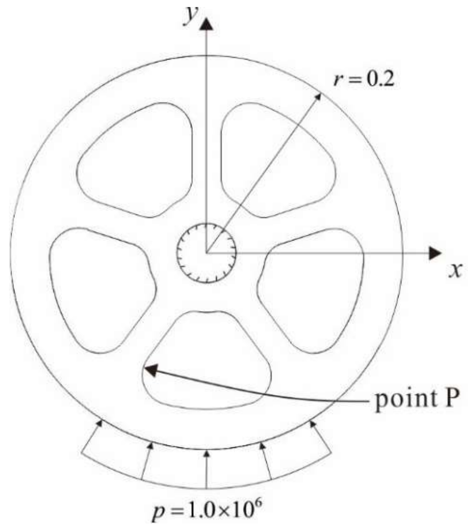


- **Quadratic elements**
  - **QUAD9** : standard 9-node quadrilateral element
  - **TRI3-d1** : 3-node triangular element enriched by linear covers
  - **QUAD4-d1** : new 4-node quadrilateral element enriched by linear covers
- **Cubic elements**
  - **QUAD16** : standard 16-node quadrilateral element
  - **TRI3-d2** : 3-node triangular element enriched by linear covers
  - **QUAD4-d2** : new 4-node quadrilateral element enriched by linear covers

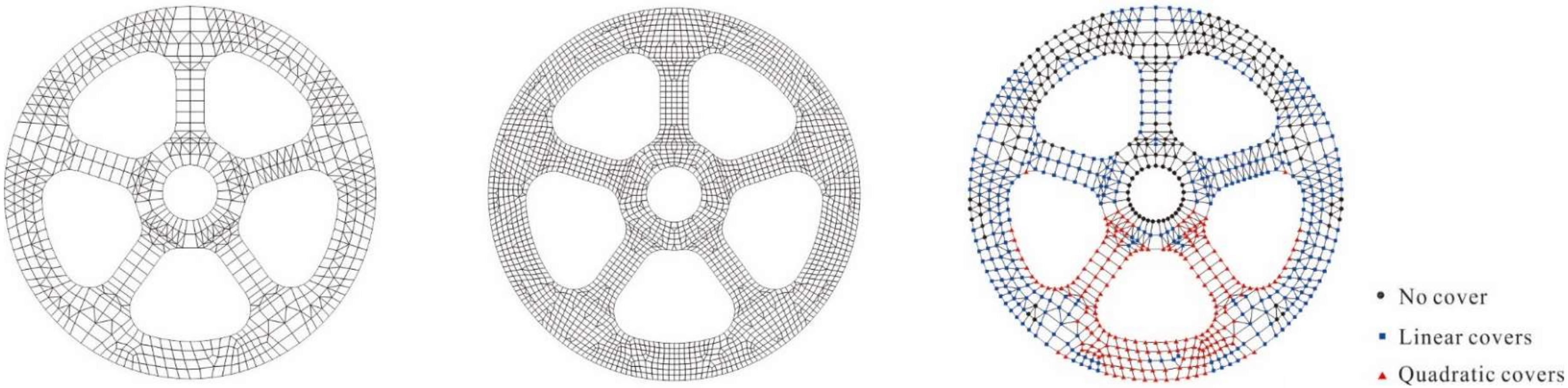


Numerical examples

❖ Wheel problem



- Material properties : $E = 7.2 \times 10^9,\ \nu = 0.3$
- **Case 1** : Standard QUAD4, TRI3 (coarse mesh) [1,532 DOFs]
- **Case 2** : Standard QUAD4, TRI3 (fine mesh) [5,268 DOFs]
- **Case 3** : **Adaptively enriched** (coarse mesh) [5,148 DOFs]



Case 1
(TRI3: 546 elements)
(QUAD4: 360 elements)

Case 2
(TRI3: 198 elements)
(QUAD4: 2289 elements)

Case 3
(TRI3: 546 elements)
(QUAD4: 360 elements)

Numerical examples

Reference solution

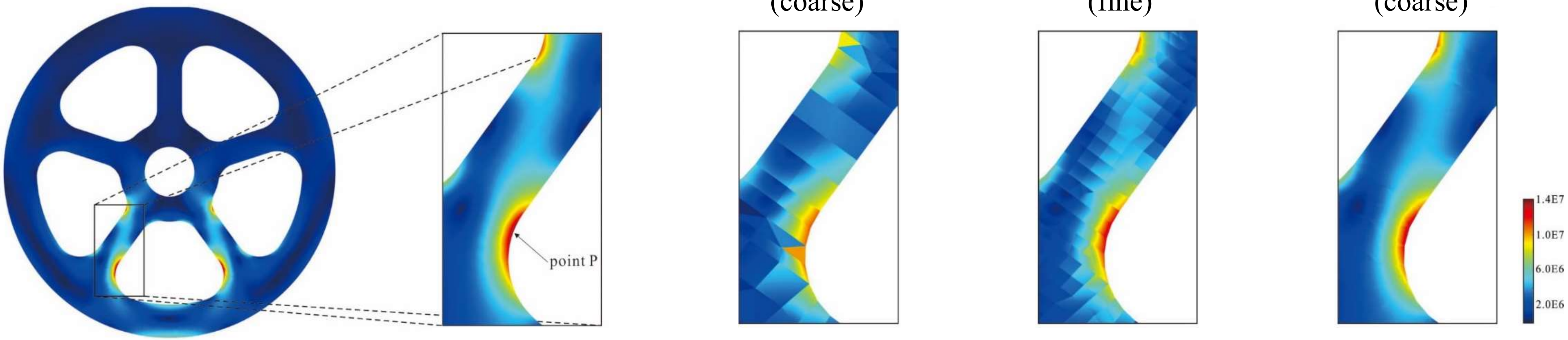
Standard linear elements

Case 1
(coarse)

Case 2
(fine)

Adaptively enriched

Case 3
(coarse)



Reference
DOFs = 74,444

Error = <u>19.59%</u>
DOFs = 1,532

Error = <u>3.75%</u>
DOFs = 5,268

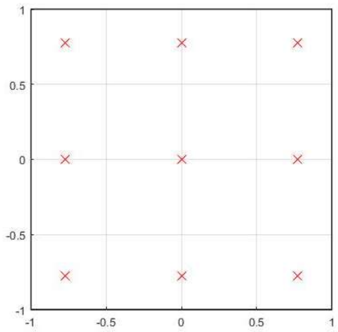
Error = <u>0.66%</u>
DOFs = 5,148

- The solution accuracy is improved by using finer mesh or by applying the cover functions.
- **<u>The adaptive use of the cover function is very effective</u>** in accurately predicting stress.

Computational cost

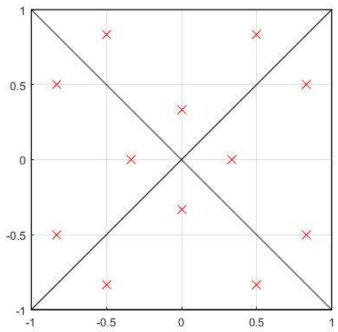
- ❖ **Quadratic elements**
  - ➢ QUAD9 : the standard 9-node element
  - ➢ QUAD4-d1 : the enriched 4-node element with linear covers
- ❖ **Cubic elements**
  - ➢ QUAD16 : the standard 16-node element
  - ➢ QUAD4-d1 : the enriched 4-node element with quadratic covers
- ❖ **Consideration**
  - ➢ Numerical integration points (new enriched elements require approximately 1.3 and 1.5 times)
  - ➢ Stiffness matrix information
  - ➢ Actual computational time of the Ad-hoc problem (assembling stiffness matrix, solving equations)

QUAD9



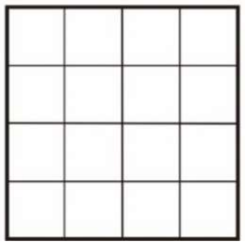
(3 x 3 points)

QUAD4-d1



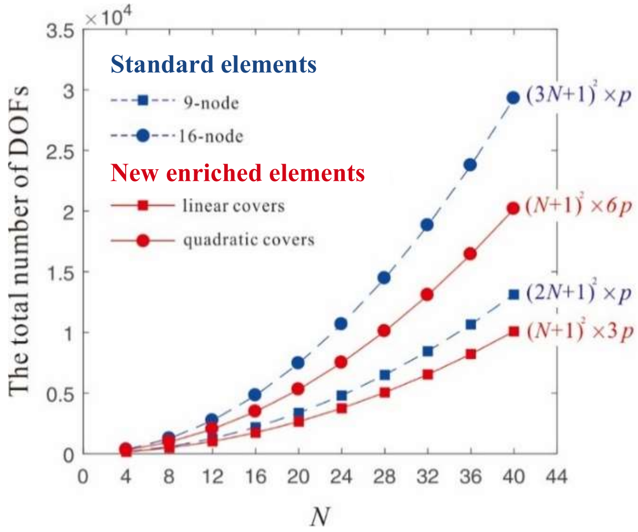
(3 x 4 points)

Ad-hoc problem



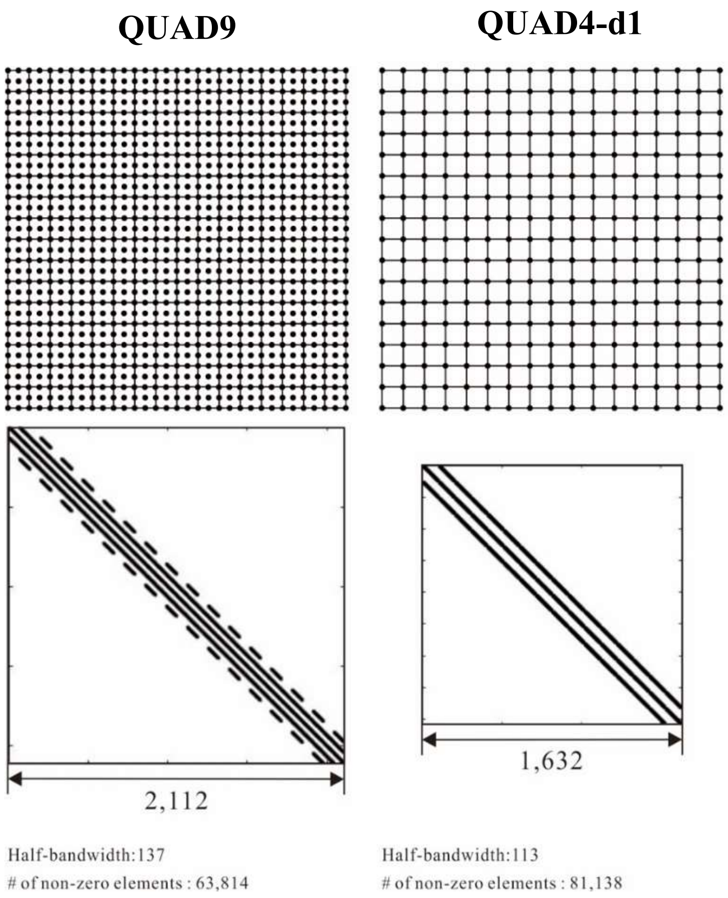
Regular mesh (N=4)

N : # of element layer



Computational cost

❖ Stiffness matrix structures



❖ Computational time

[sec]

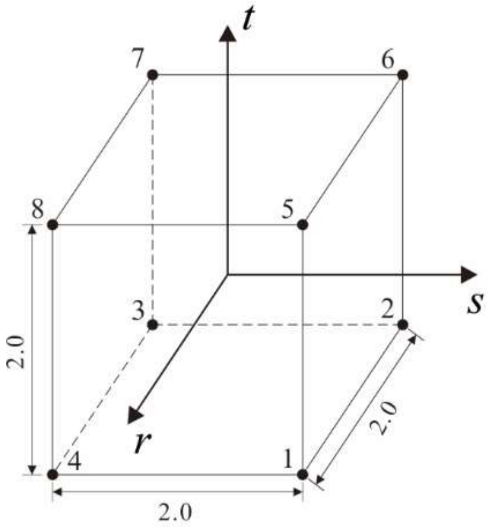
N	Standard 9-node finite element			New enriched 4-node element with linear covers		
	Stiffness construction	Equation solver	Total	Stiffness construction	Equation solver	Total
8	0.02	0.00	0.02	0.02	0.02	0.03
16	0.05	0.06	0.11	0.09	0.05	0.14
32	0.39	0.81	1.20	0.48	0.56	1.05
64	1.09	12.33	13.42	1.58	8.38	9.95
128	3.66	175.50	179.16	6.56	115.30	121.86

- The new enriched elements take more time to construct the stiffness matrix
- Solving the linear equations generally takes less time.
- As the number of elements used increases, the solving time becomes dominant.

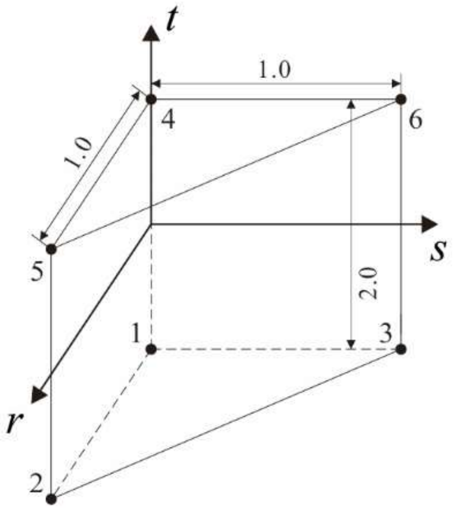
Topic 1-2. Enriched 3D solid elements

Enriched 3D solid finite elements formulation

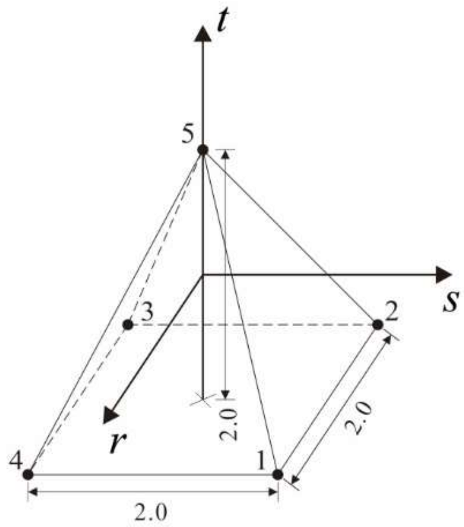
8-node hexahedral element $(n=8)$



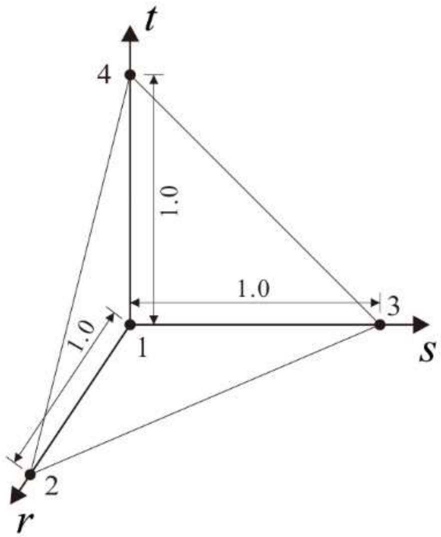
6-node prismatic element $(n=6)$



5-node pyramidal element $(n=5)$



4-node tetrahedral element $(n=4)$

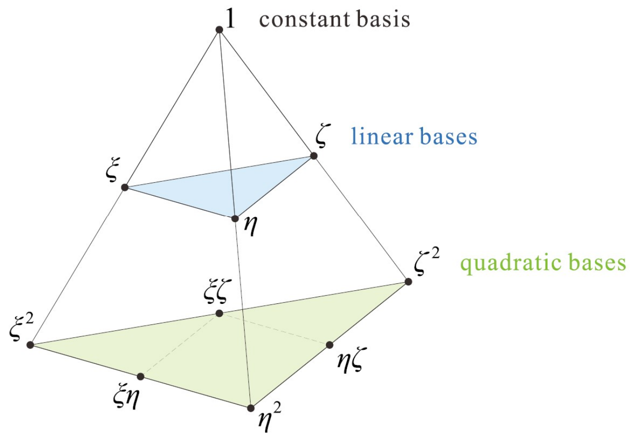


- **The geometry and displacement interpolations**

➢ $$\mathbf{x}(r,s,t)=\sum_{i=1}^{n} h_i(r,s,t)\mathbf{x}_i$$

➢ $$\mathbf{u}(r,s,t)=\bar{\mathbf{u}}+\hat{\mathbf{u}}=\sum_{i=1}^{n} \hat{h}_i(r,s,t)\bar{\mathbf{u}}_i+\sum_{i=1}^{n} \hat{\mathbf{H}}_i(r,s,t)\hat{\mathbf{u}}_i$$

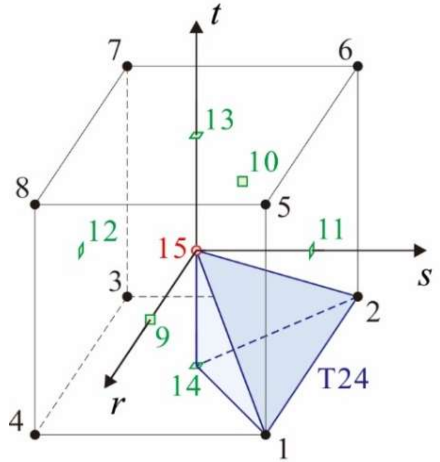
- **Bases of complete polynomials**



Sets of piecewise linear shape functions

❖ Tetrahedral sub-domains for each element

8-node hexahedral element $(n=8)$

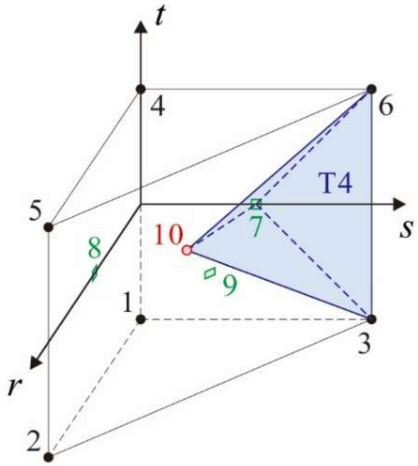


- • 1~8
- ▫ 9~14
- ∘ 15

$(n=8,\ n_{quad}=6)$

$\Omega_{14}=\{1, 2, 3, 4\}$

6-node prismatic element $(n=6)$

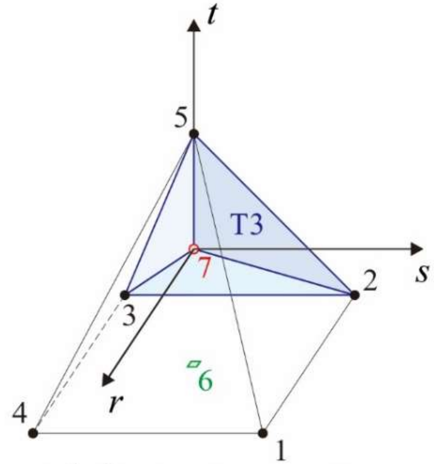


- • 1~6
- ▫ 7~9
- ∘ 10

$(n=6,\ n_{quad}=3)$

$\Omega_{7}=\{1, 3, 4, 6\}$

5-node pyramidal element $(n=5)$



- • 1~5
- ▫ 6
- ∘ 7

$(n=5,\ n_{quad}=1)$

$\Omega_{6}=\{1, 2, 3, 4\}$

- • node
- ▫ quadrilateral face center
- ∘ element center

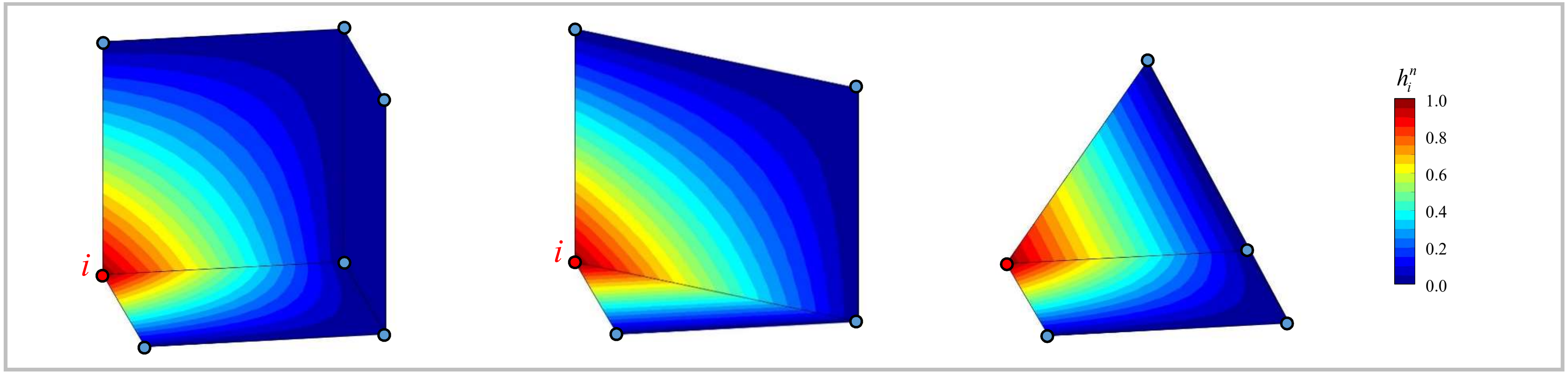
➢ **Linear shape function on each sub-domain :** $\hat{h}_i(r,s,t)=(\hat{a}_i+\hat{b}_i\,r+\hat{c}_i\,s+\hat{d}_i\,t)/n$

➢ **Shape function requirements**

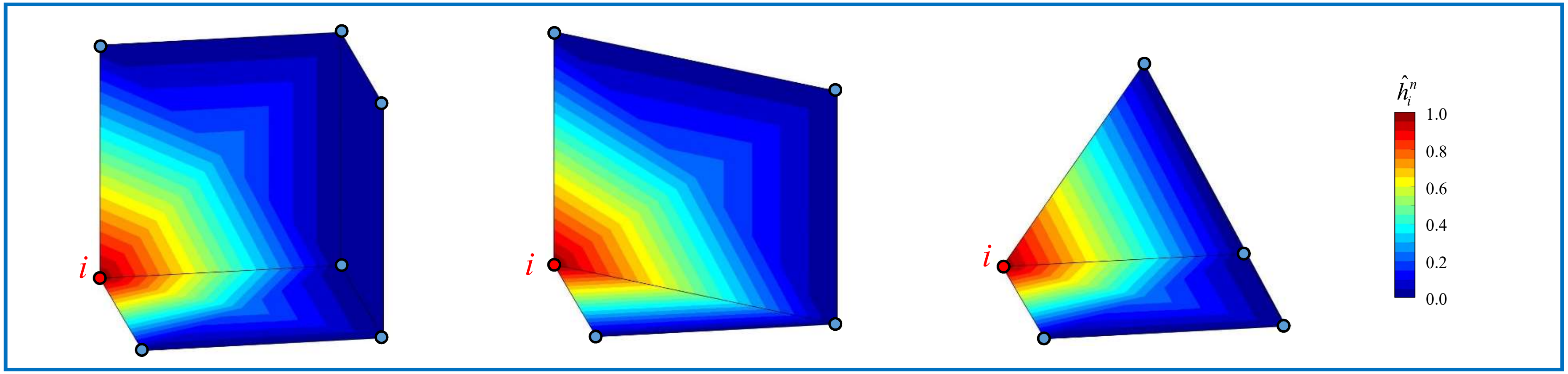
- Kronecker delta property ($\hat{h}_i(r_j,s_j,t_j)=\delta_{ij}$ with $i,j=1,\cdots,n$)
- Partition of unity: $\sum_{i=1}^{n}\hat{h}_i=1$
- Compatibility (continuous displacement interpolation across the element boundaries)
- Completeness (able to represent rigid body modes and constant strain states)

Sets of piecewise linear shape functions

➢ Shape functions of the **standard finite element method**

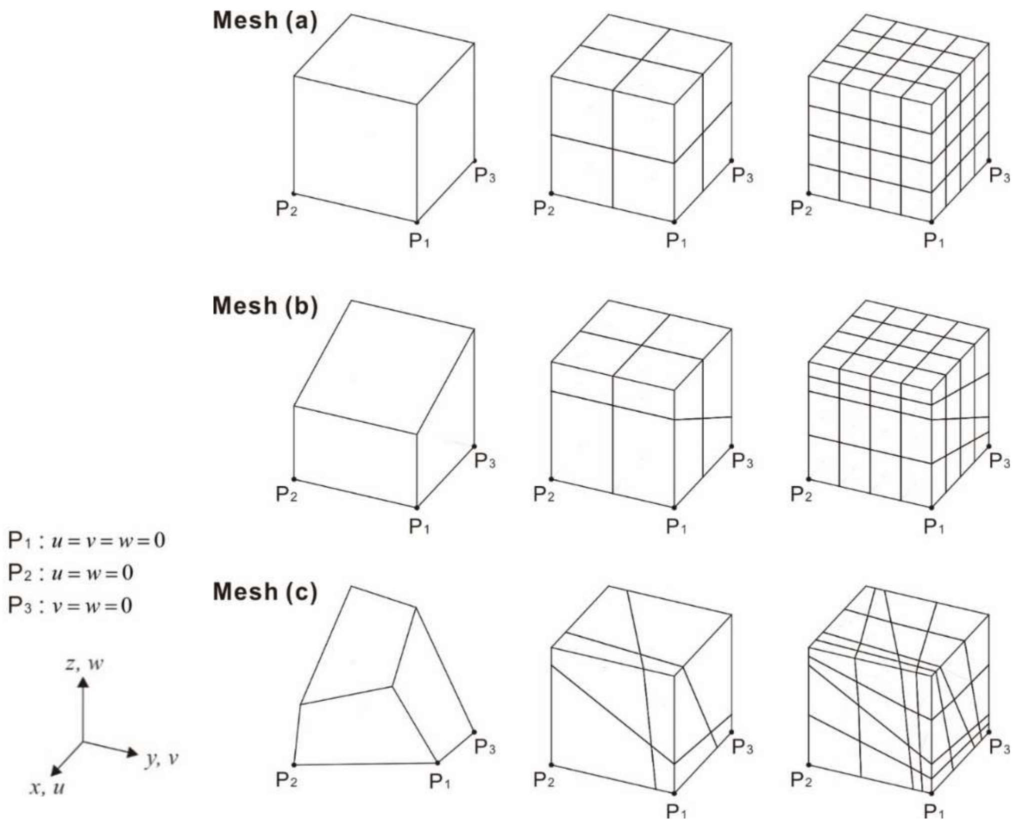


➢ **Piecewise linear shape functions**

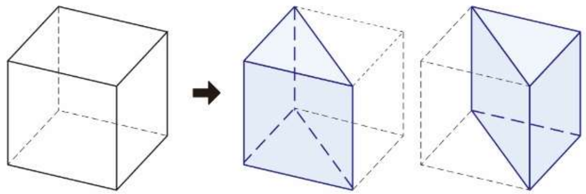


- Triangular face: linear variation, quadrilateral face: piecewise linear variation.
- New enriched elements and enriched tetrahedral element are compatible with each other.

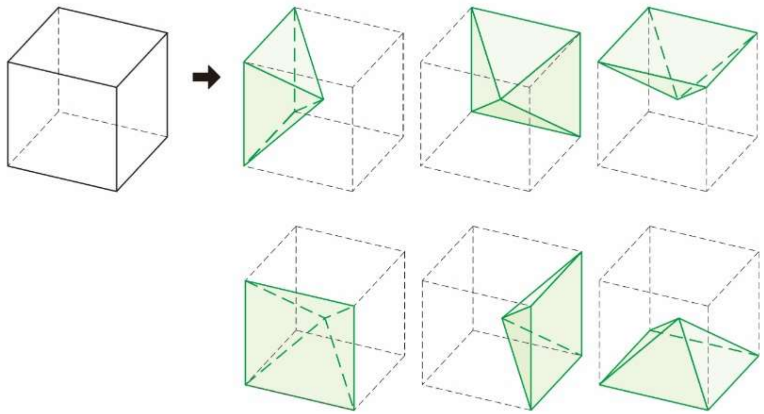
Investigation of the LD problem



- **Division of hexahedron into two prisms**



- **Division of hexahedron into six pyramids**



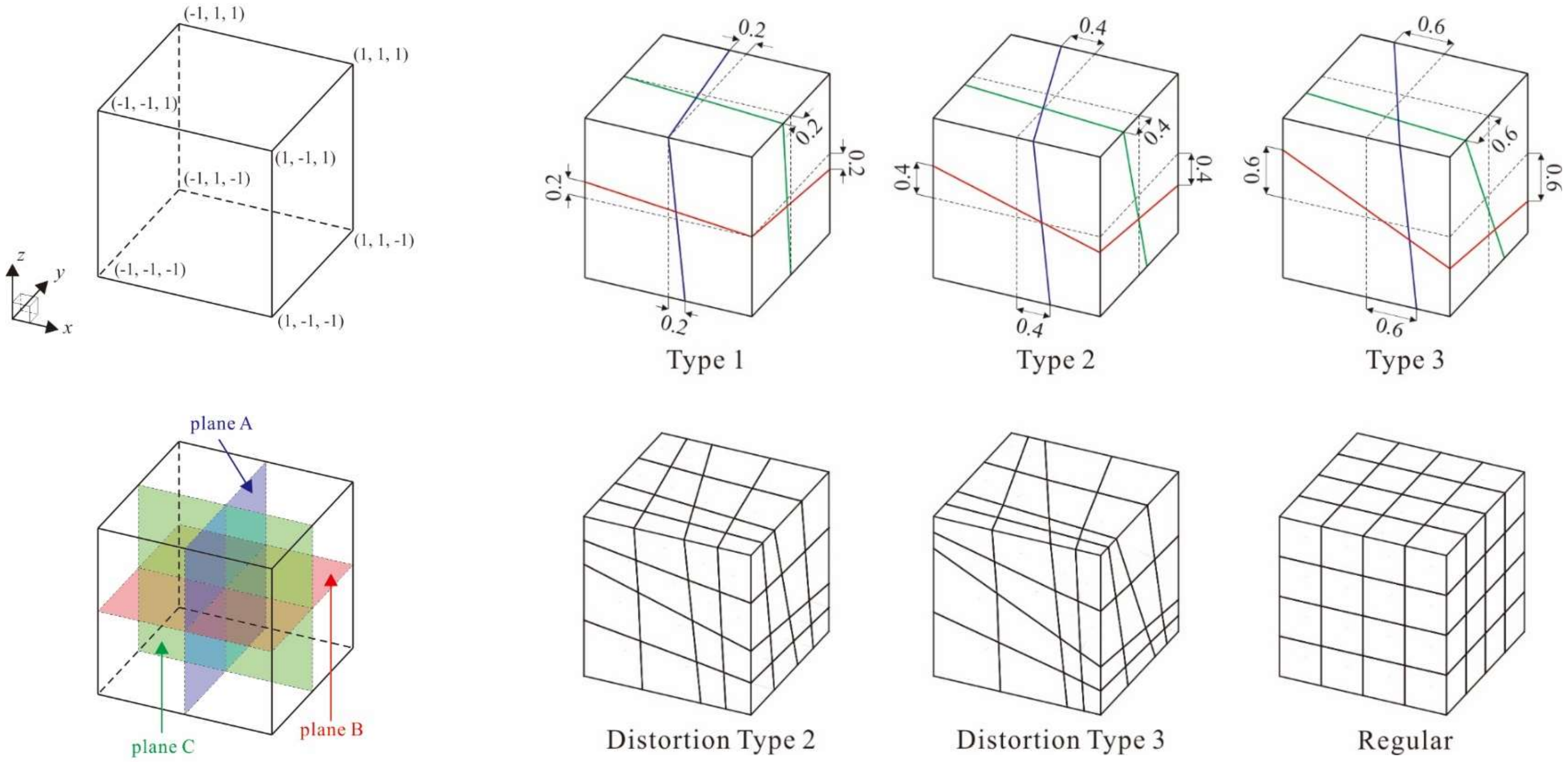
- **Hexahedral element results**

Element	Number of element layers	RD / Total DOFs Mesh (a) $d = 1$	Mesh (a) $d = 2$	Mesh (b) $d = 1$	Mesh (b) $d = 2$	Mesh (c) $d = 1$	Mesh (c) $d = 2$
Previous	1	LD problem exists. (RD is observed)				0/62	0/152
	2					0/290	0/722
	4					0/1466	0/3662
New	1	LD problem is resolved. (RD is NOT observed)					
	2						
	4						

- **PU function**
  - Previous : Shape function of the standard FEM
  - New : Piecewise linear shape function
- **RD** : # of zero eigenvalues
- ***d*** : order of cover function
- **New enriched prismatic and pyramidal elements are also free from the LD problem.**

Numerical examples

❖ Ad-hoc problem



- **Body force** (m=5)

$$u = g(x,y,z)\cos mx \sin my \cos mz$$
$$v = g(x,y,z)\sin mx \cos my \cos mz$$
$$w = g(x,y,z)\cos mx \cos my \sin mz$$

$$g(x,y,z) = (1-x^2)^2(1-y^2)^2(1-z^2)^2 e^{my}$$

$$f_x^B = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)$$
$$f_y^B = -\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right)$$
$$f_z^B = -\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

- **Boundary condition**

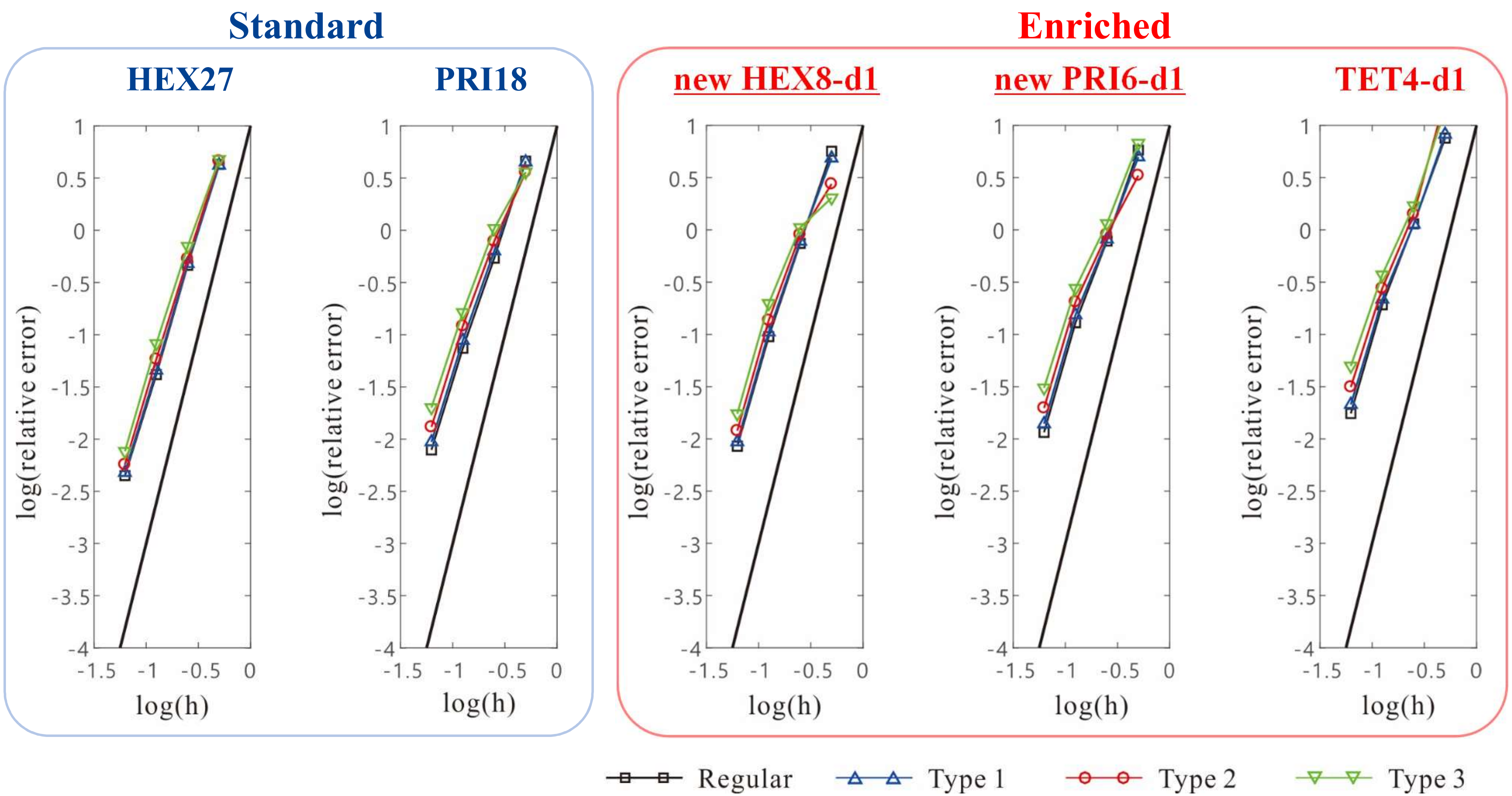
$u = v = w = 0$ at $y = -1$

- **Material property**

$E = 1.0\times 10^7,\ \nu = 0.3$

Numerical examples

➢ **Quadratic elements**

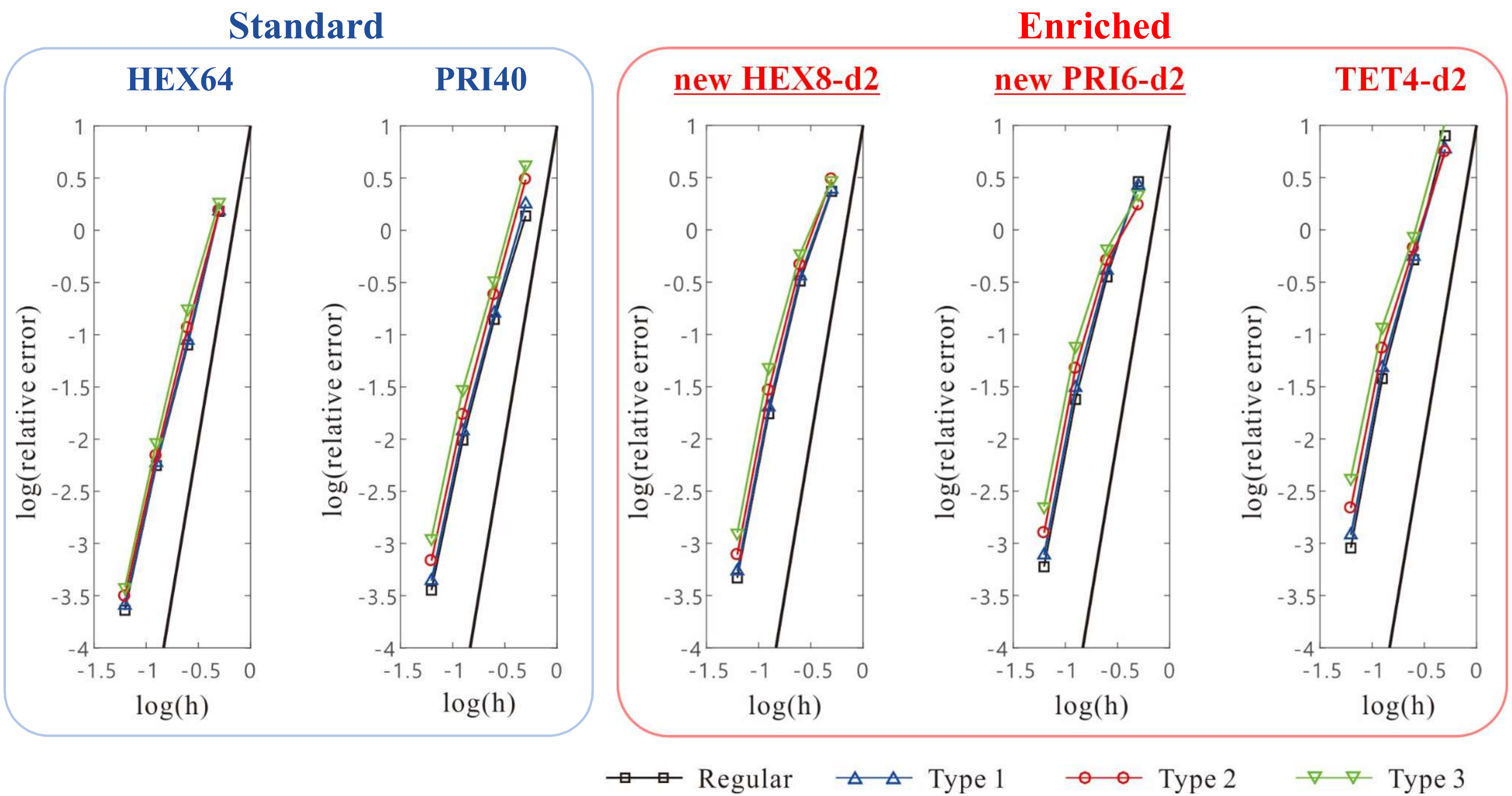


Quadratic elements (**HEX27**, **PRI18**, **HEX8-d1**, **PRI6-d1**, **TET4-d1**)

- **HEX27** : standard 27-node hexahedral element
- **PRI18** : standard 18-node prismatic element
- **HEX8-d1** : new 8-node hexahedral element enriched by linear covers
- **PRI6-d1** : new 8-node hexahedral element enriched by linear covers
- **TET4-d1** : 4-node tetrahedral element enriched by linear covers

Numerical examples

➢ **Cubic elements**

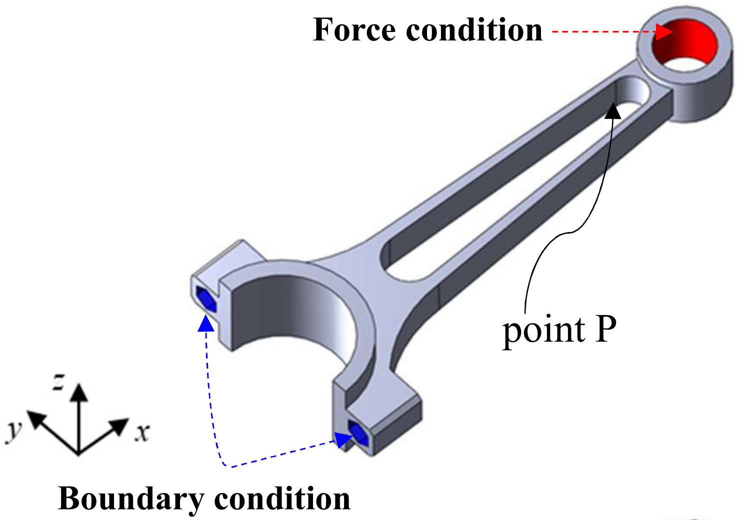


Cubic elements (**HEX64**, **PRI40**, **HEX8-d2**, **PRI6-d2**, **TET4-d2**)

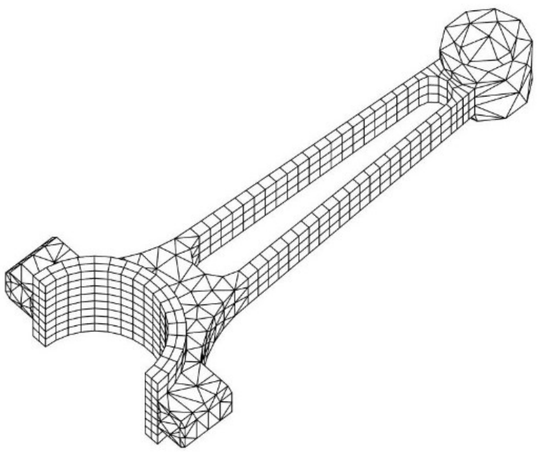
- **HEX64** : standard 64-node hexahedral element
- **PRI40** : standard 40-node prismatic element
- **HEX8-d2** : new 8-node hexahedral element enriched by quadratic covers
- **PRI6-d2** : new 8-node hexahedral element enriched by quadratic covers
- **TET4-d2** : 4-node tetrahedral element enriched by quadratic covers

Numerical examples

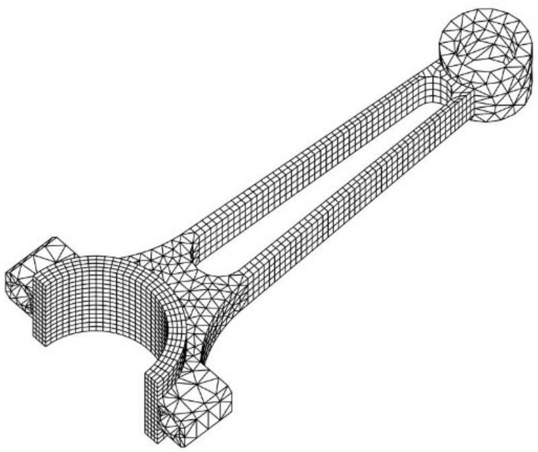
❖ Connecting rod problem



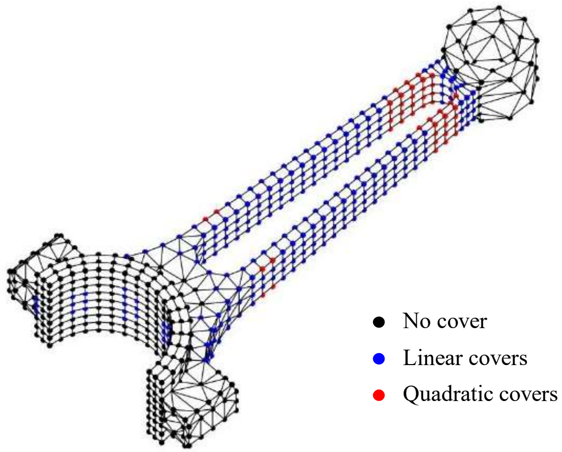
- Material properties : $E = 2.0 \times 10^{11}$, $\nu = 0.3$, $\rho = 7650$
- **Case 1** : Standard 3D solid elements (coarse mesh) [3,435 DOFs]
- **Case 2** : Standard 3D solid elements (fine mesh) [13,854 DOFs]
- **Case 3** : **Adaptively enriched** (coarse mesh) [9,924 DOFs]



Case 1
(HEX8: 428, PRI6: 86,
PYR5: 0, TET4:1044)



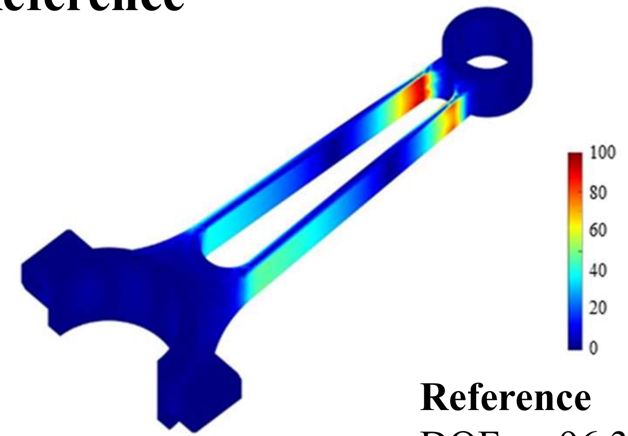
Case 2
(HEX8: 2540, PRI6: 8,
PYR5: 246, TET4:3369)



Case 3
(HEX8: 428, PRI6: 86,
PYR5: 0, TET4:1044)

Numerical examples

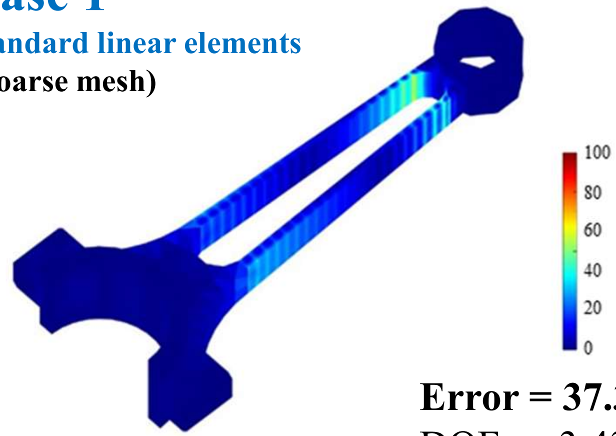
Reference



Reference
DOFs = 96,393

Case 1

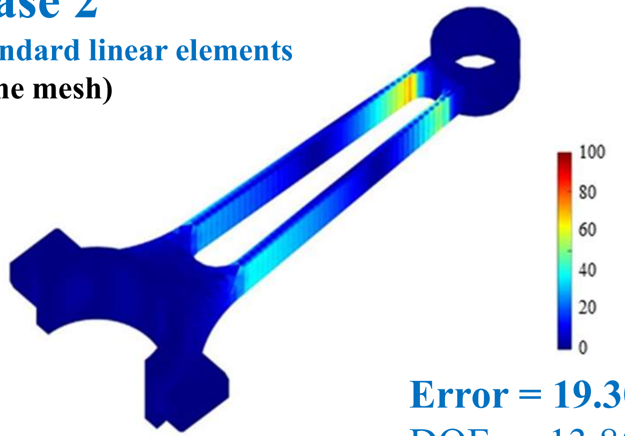
Standard linear elements
(Coarse mesh)



Error = 37.30%
DOFs = 3,435

Case 2

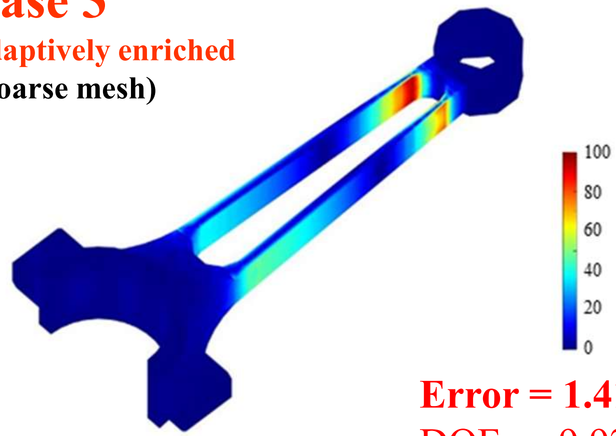
Standard linear elements
(Fine mesh)



Error = 19.30%
DOFs = 13,854

Case 3

Adaptively enriched
(Coarse mesh)



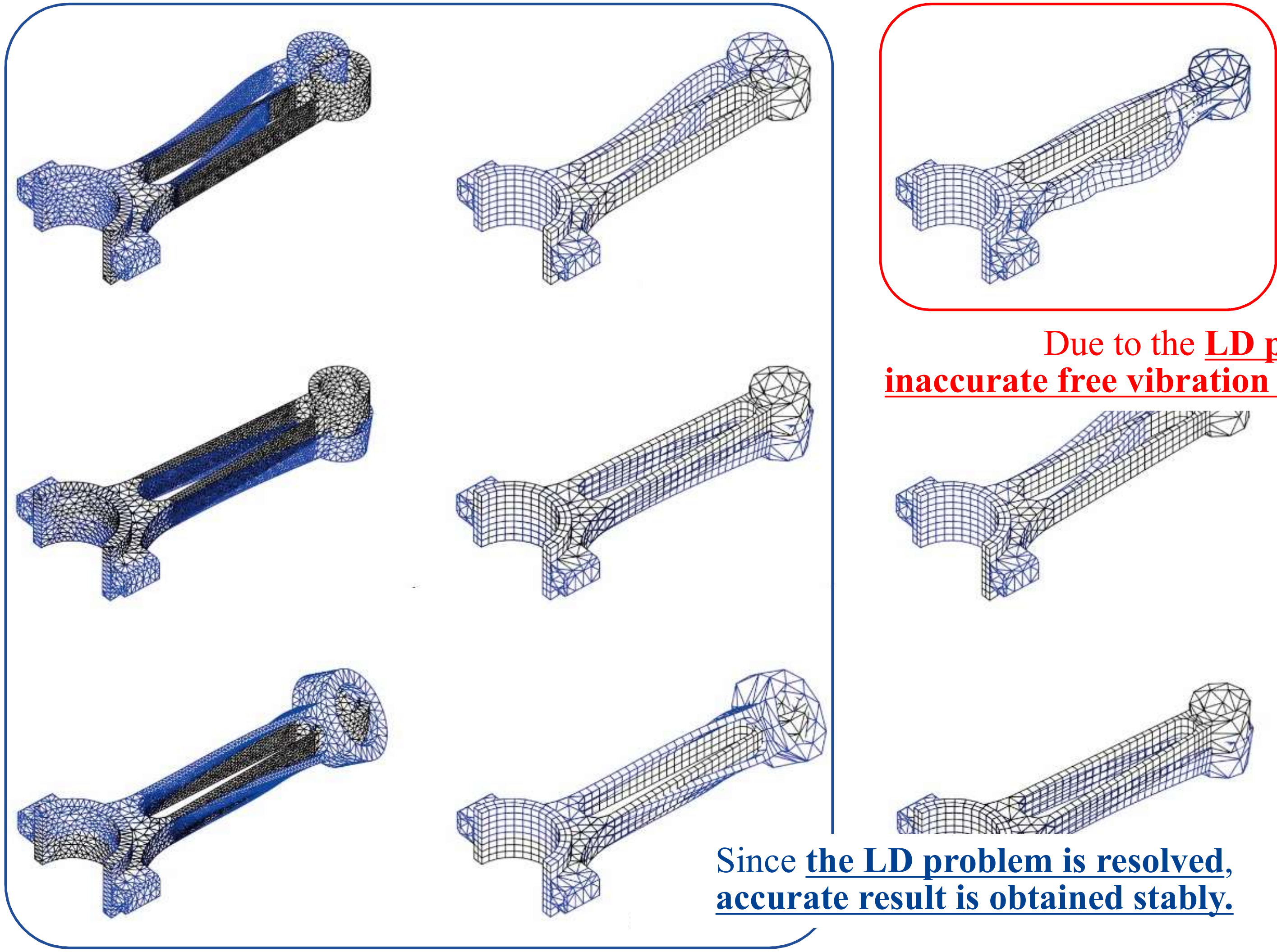
Error = 1.41%
DOFs = 9,924

- The solution accuracy is improved by using finer mesh or by applying the cover functions.
- **The adaptive use of the cover function is very effective** in accurately predicting stress.

Numerical examples

➢ **Free vibration analysis**

Mode Number	Reference	Enriched 3D elements with linear covers: New	Enriched 3D elements with linear covers: Previous
1			
2			
3			



Due to the **LD problem**, **inaccurate free vibration mode** is calculated.

Since **the LD problem is resolved**, **accurate result is obtained stably.**

Closure (Topic 1)

- **New enriched 2D and 3D solid elements** are proposed.
- **Piecewise linear shape functions** are adopted for geometry and displacement interpolations
- The elements are **free from the linear dependence problem**
- Through numerical examples, its convergence and effectiveness are demonstrated. **Adaptive use of the cover functions** improves the solution accuracy effectively.

Topic 2. Automatic procedure to improve FE solutions

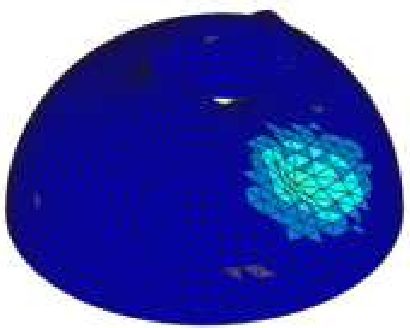
Automatic procedure to improve FE solutions

❖ Three steps of the automatic procedure

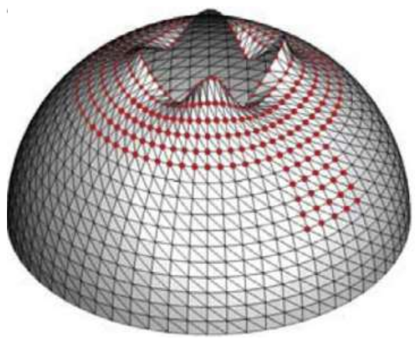
Step-1 (1^{st} FE analysis)
- Construct FE model
- Perform FE analysis

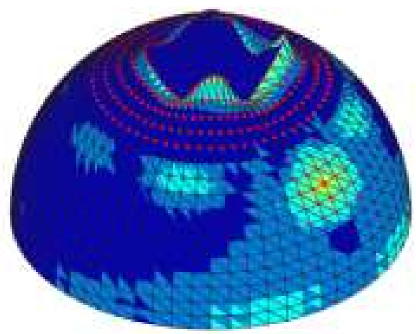
Step-2
- Calculate error indicator
- Select cover functions

Step-3 (2^{nd} FE analysis)
- Update FE model
- Perform FE analysis

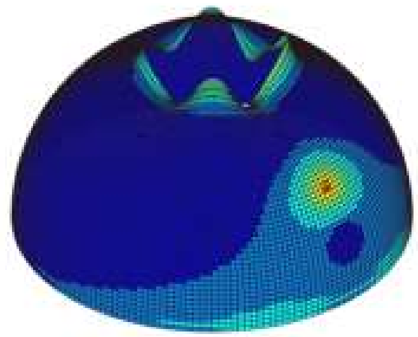


MITC3 shell FE
(Error = 75.8%)





Adaptively enriched MITC3
(Error = 6.49%)



Reference

Error indicator & scheme for the adaptive use of cover functions are needed.

Requirements for error indicator and scheme

- Simple and computationally efficient
- Asymptotically converge as the error converges
- Appropriate for selecting cover order for each node
- Minimum artificial parameters

Error indicators

❖ Previous research (2014, Kim and Bathe)

- Only consider 3-node 2D element.
- Based on stress jump at each node.
- Proposed for the adaptive use of cover functions.
- Several artificial parameters in the error indicator.

- $\bar{M}_i^\tau = \dfrac{J_i^\tau}{\gamma_e \tau_{mean}} \left(\dfrac{h}{L_c} \right)^\beta$

$$d(i) = \begin{cases} 0 & if \quad \bar{M}_i^\tau < \gamma_0 \\ 1 & if \quad \gamma_0 \le \bar{M}_i^\tau < \gamma_1 \\ 2 & if \quad \gamma_1 \le \bar{M}_i^\tau < \gamma_2 \\ 3 & if \quad \gamma_2 \le \bar{M}_i^\tau \end{cases}$$

❖ Proposed error indicators

- Consider both 3-node and 4-node 2D element.
- Minimum artificial parameters.
- Based on both stress jump and stress value.
- Type - 1

$$M_i^\tau = \left\{ \frac{J_i^\tau}{\tau_{mean}} + \left(\frac{J_{mean}}{\tau_{mean}} \right) \frac{\tau_i}{\tau_{mean}} \right\} \left(\frac{\chi_i}{L_c} \right)^{1/2}$$

- Type - 2

$$M_i^\tau = \left(\frac{J_i^\tau}{\tau_{mean}} \right) \left(\frac{\chi_i}{L_c} \right)^{1/2}$$

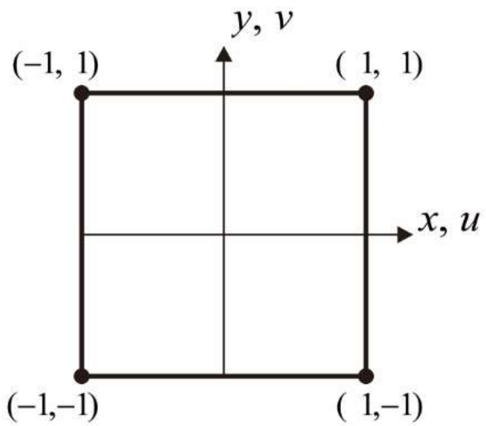
- J_{mean} : mean value of J_i^τ over FE model
- τ_{mean} : mean stress over FE model
- χ_i : diameter of node i

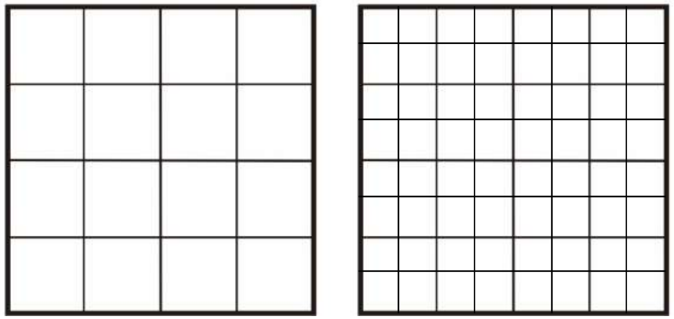
Related to stress jump

Related to stress value

Error indicators

❖ Ad-hoc problem





(N=4) (N=8)

(N : the number of element layers along an edge)

- **Body force** (m=5)

$$f_x^B = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right), \; f_y^B = -\left(\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x}\right)$$

$$u = (1-x^2)^2(1-y^2)^2 e^{my} \cos mx$$

$$v = (1-x^2)^2(1-y^2)^2 e^{my} \sin mx$$

- **Boundary condition**

$$u = v = 0 \;\text{ at }\; y = -1$$

- **Stress error [ ]**

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i$$

- **Stress jump [ ]**

$$\bar{J} = \frac{1}{N}\sum_{i=1}^{N} J_i$$

- **Error indicator (type 1) [ ]**

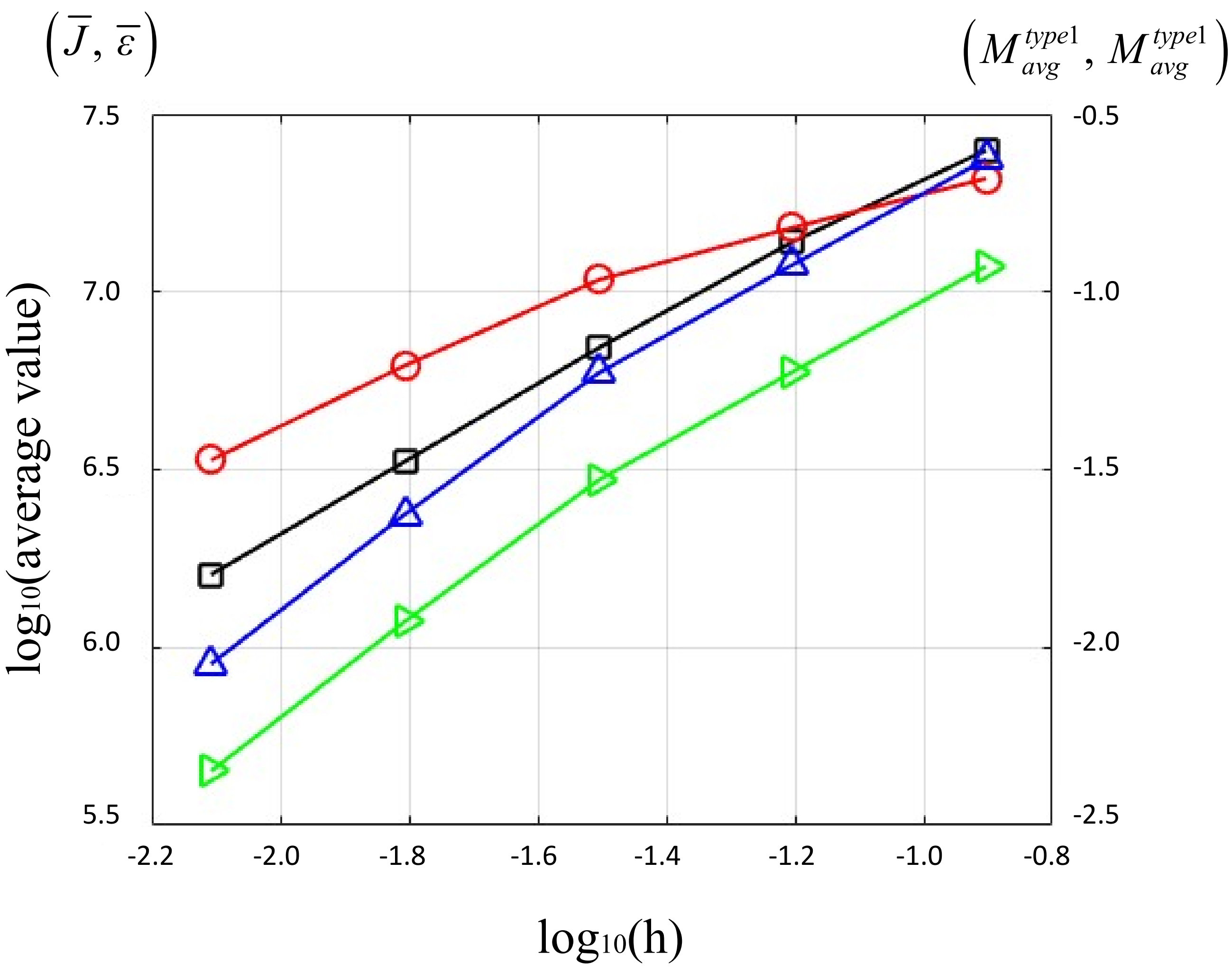
$$M_{avg}^{type1} = \frac{1}{N}\sum_{i=1}^{N} M_i^{type1}$$

- **Error indicator (type 2) [ ]**

$$M_{avg}^{type2} = \frac{1}{N}\sum_{i=1}^{N} M_i^{type2}$$

Error indicators

❖ **Ad-hoc problem**



- **Stress error [ ]**

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i$$

- **Stress jump [ ]**

$$\bar{J} = \frac{1}{N}\sum_{i=1}^{N}J_i$$

- **Error indicator (type 1) [ ]**

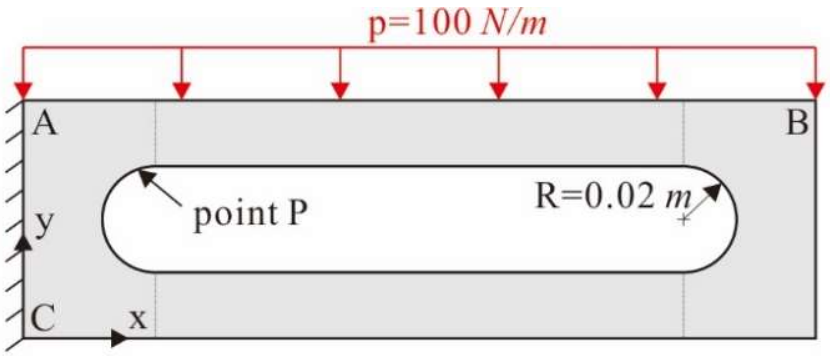
$$M_{avg}^{type1} = \frac{1}{N}\sum_{i=1}^{N}M_i^{type1}$$

- **Error indicator (type 2) [ ]**

$$M_{avg}^{type2} = \frac{1}{N}\sum_{i=1}^{N}M_i^{type2}$$

Error indicators

❖ Type-1 & -2 comparison (tool jig problem)

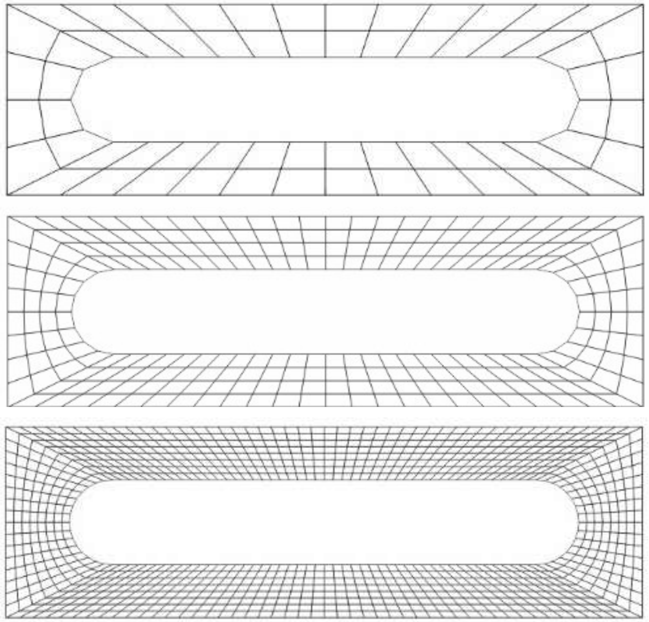


< Problem description >

Mesh 1
(64 elements)

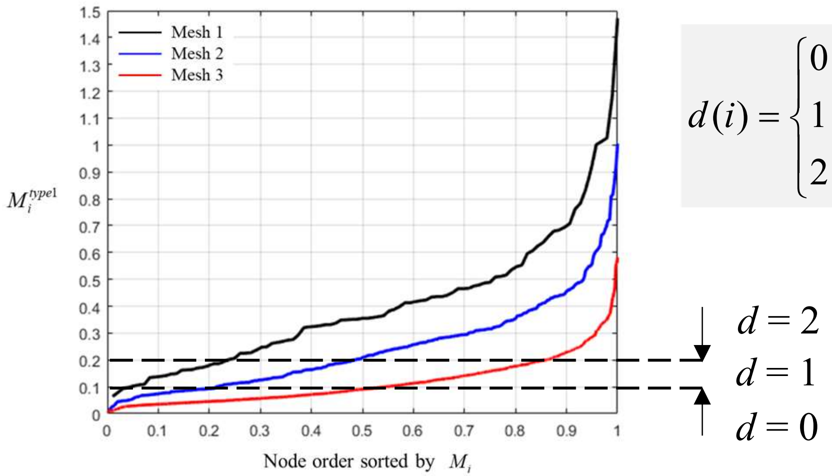
Mesh 2
(256 elements)

Mesh 3
(1024 elements)



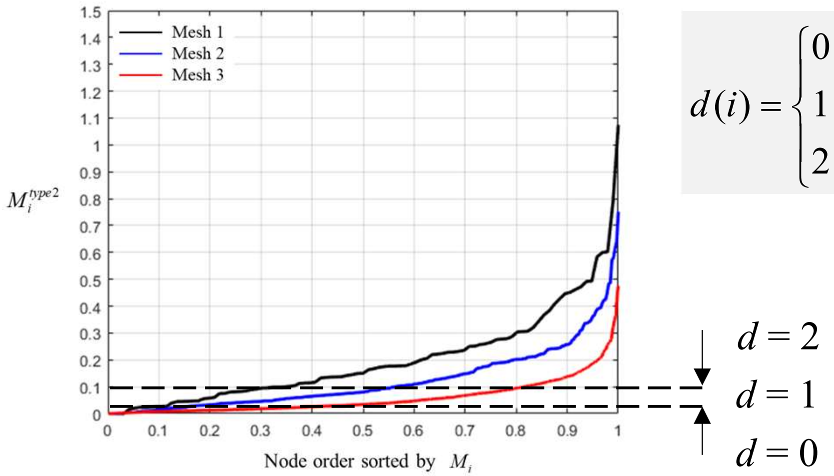
< Finite element models>

- **Type-1**



$$d(i) = \begin{cases} 0 & if \quad M_i^{\tau} < 0.10 \\ 1 & if \quad 0.10 \le M_i^{\tau} < 0.20 \\ 2 & if \quad 0.20 \le M_i^{\tau} \end{cases}$$

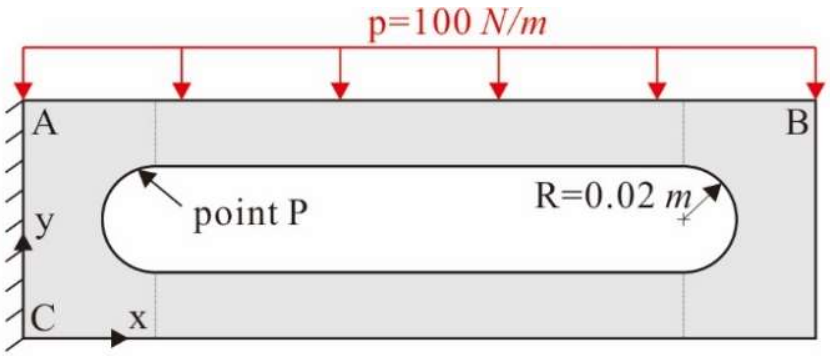
- **Type-2**



$$d(i) = \begin{cases} 0 & if \quad M_i^{\tau} < 0.02 \\ 1 & if \quad 0.02 \le M_i^{\tau} < 0.10 \\ 2 & if \quad 0.10 \le M_i^{\tau} \end{cases}$$

Error indicators

❖ **Type-1 & -2 comparison** (tool jig problem)

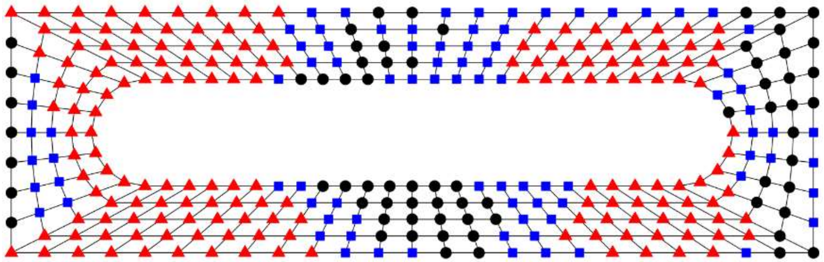


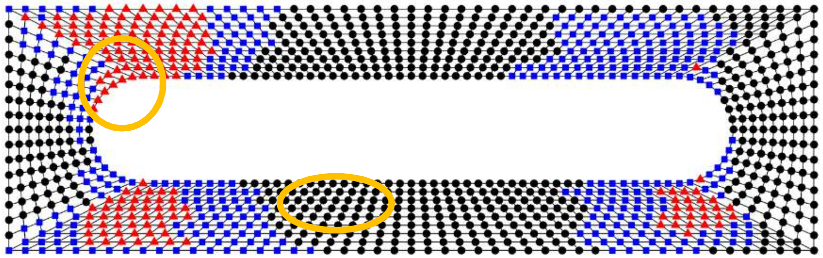
< Problem description >

< Reference solution>

Mesh 2 **Mesh 3**

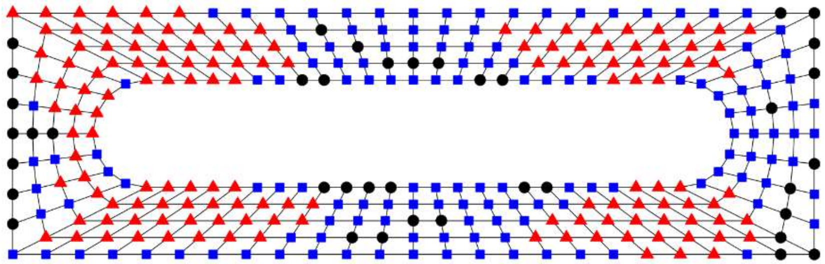
Type-1

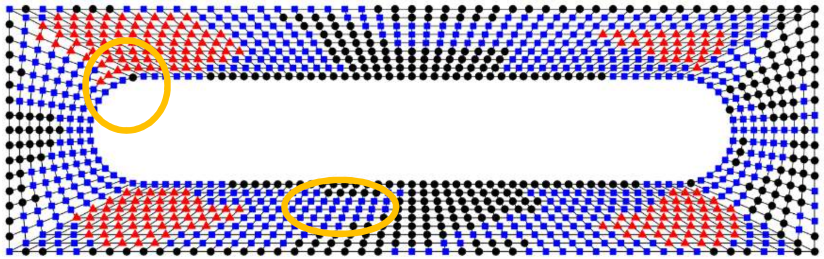




Stress error = 1.80%
Energy error = 5.13%
DOFs = 5324

Type-2



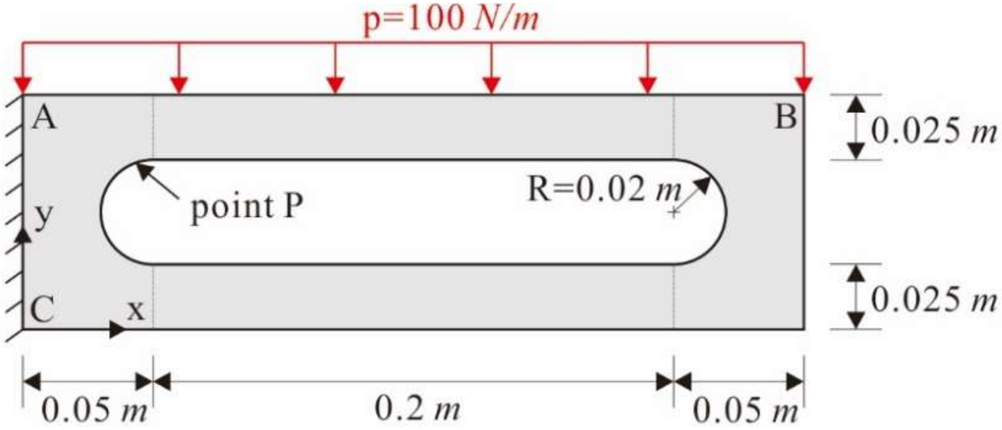


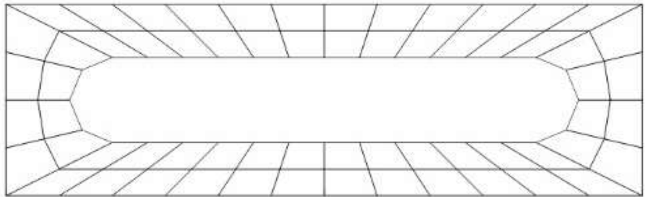
Stress error = 3.85%
Energy error = 5.20%
DOFs = 6598

● **No cover** ■ **Linear cover** ▲ **Quadratic covers**

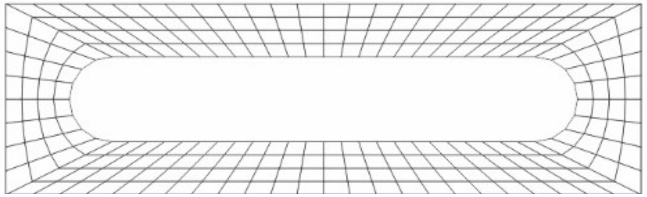
Numerical examples

❖ Tool jig problem

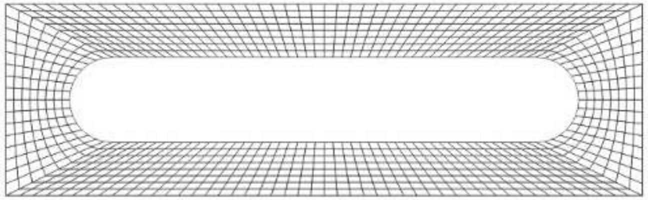




Mesh 1
(64 elements)

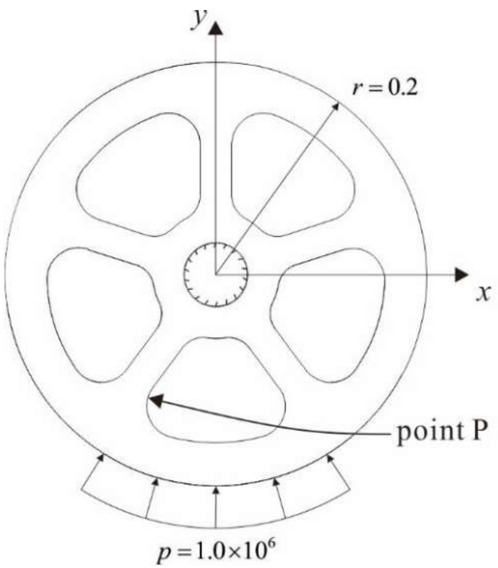


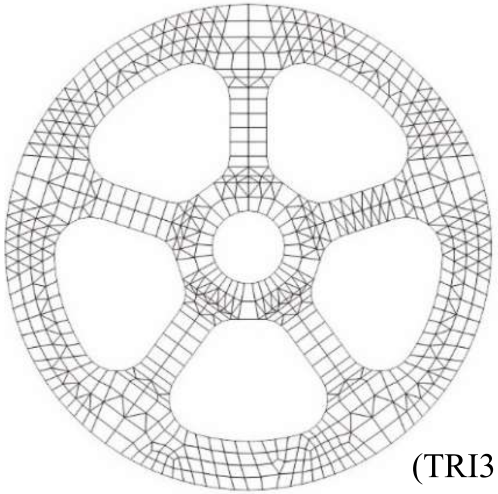
Mesh 2
(256 elements)



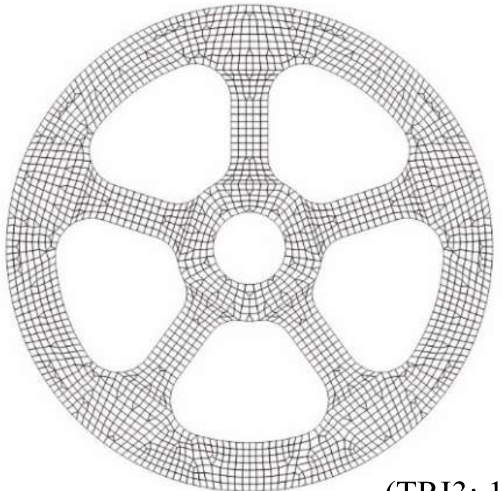
Mesh 3
(1024 elements)

❖ Wheel problem





Mesh 1
(TRI3: 546 elements)
(QUAD4: 360 elements)

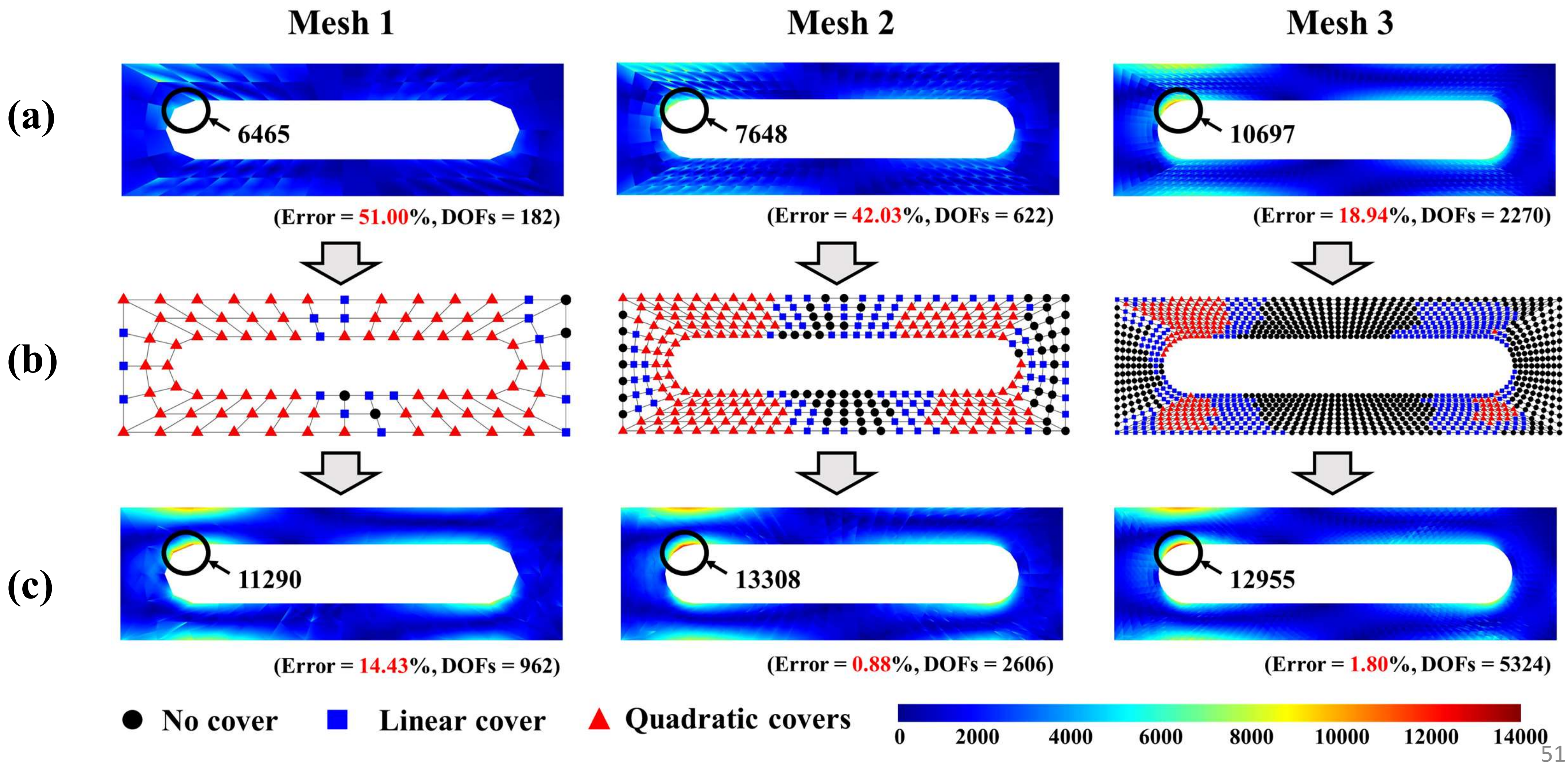


Mesh 2
(TRI3: 198 elements)
(QUAD4: 2289 elements)

Tool jig problem

❖ 4-node quadrilateral element

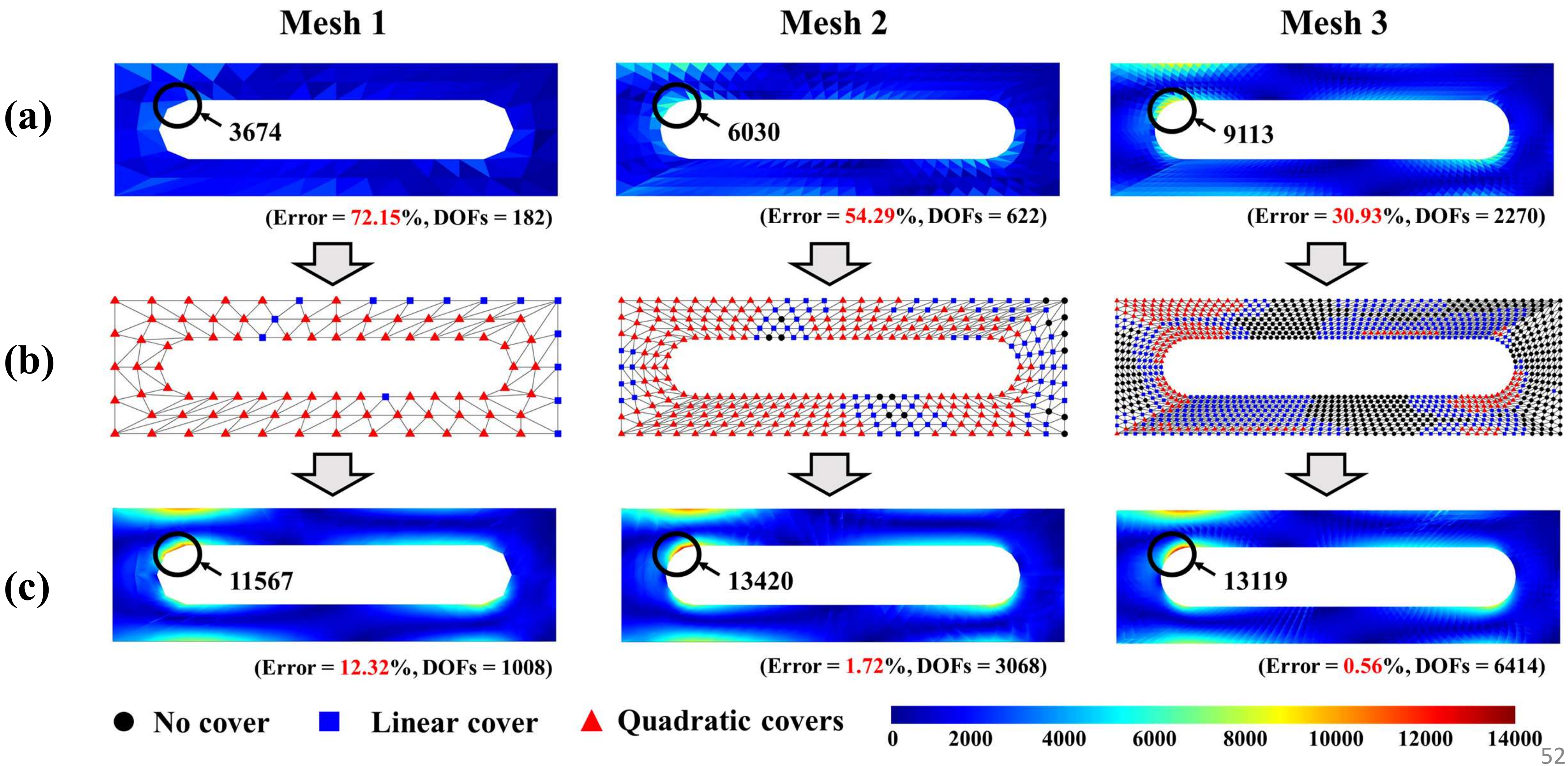
(a) Results of the standard 4-node quadrilateral element
(b) How cover functions are applied
(c) Results of the adaptive use of cover functions



Tool jig problem

❖ **3-node triangular element**

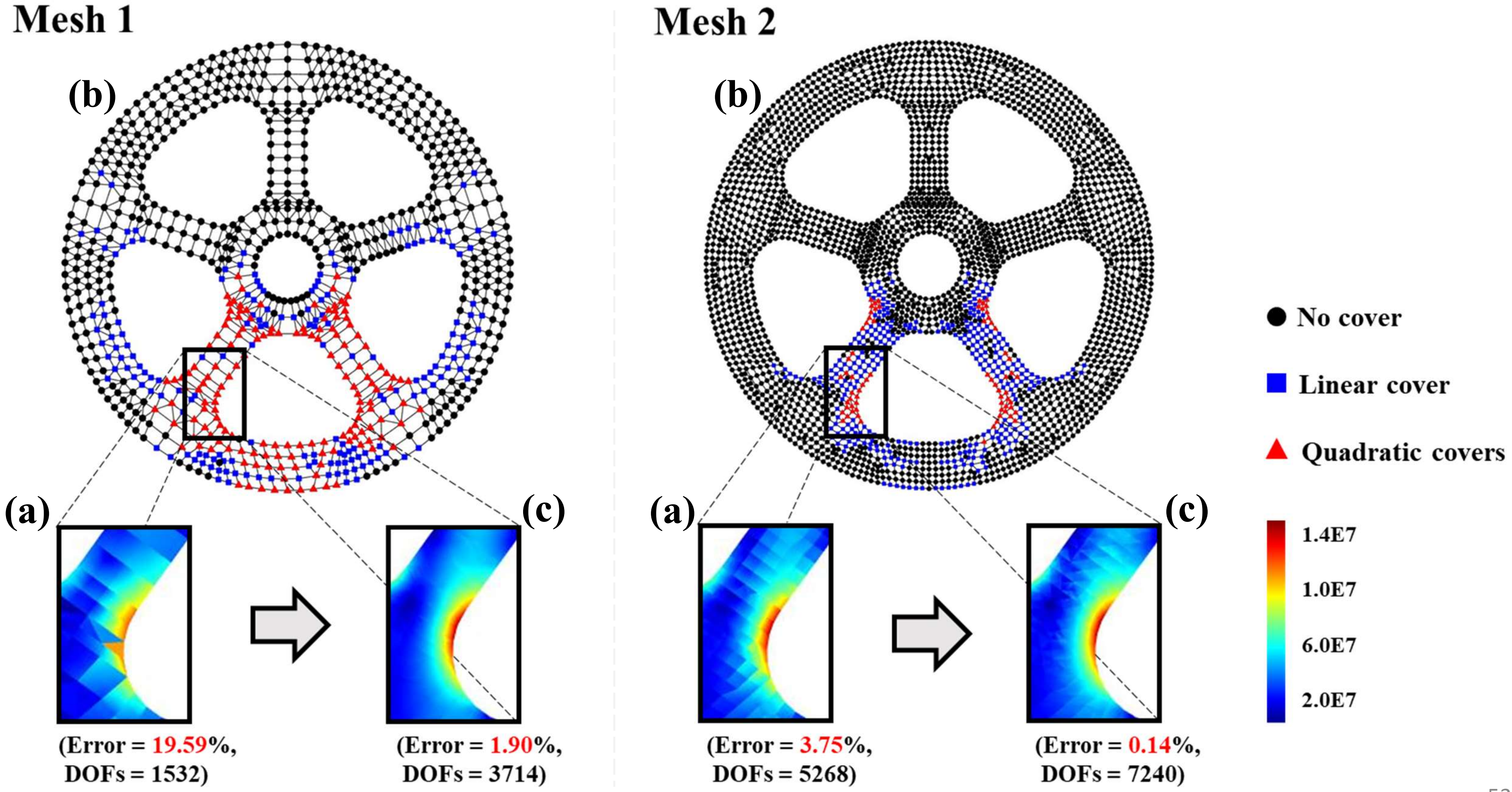
(a) Results of the standard 3-node triangular element
(b) How cover functions are applied
(c) Results of the adaptive use of cover functions



Automotive wheel problem

❖ 3-node triangular & 4-node quadrilateral elements

(a) The standard elements
(b) How cover functions are applied
(c) the adaptive use of cover functions



Closure (Topic 2)

- **Feasibility of the adaptive used of cover functions** to automatically improve finite element solutions is demonstrated.
- **New error indicator based on stress jump** and **scheme that select the appropriate order of cover function for each node** are used for the automatic procedure.
- The automatic procedure provides **significantly improved solution accuracy**.
- Further researches on large finite element models and 3D problems are necessary.

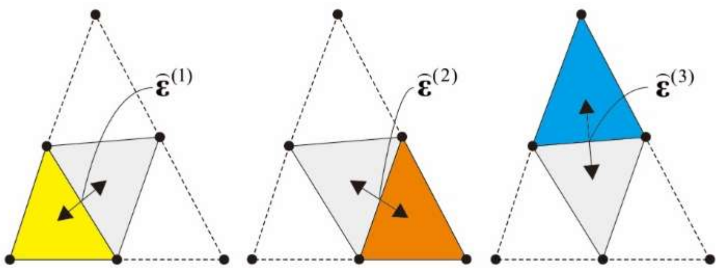
Topic 3. Enriching strain-smoothed 3-node 2D solid element

Strain-smoothed element method for 3-node triangular element

❖ **Strain smoothing process** [Lee & Lee, 2018]

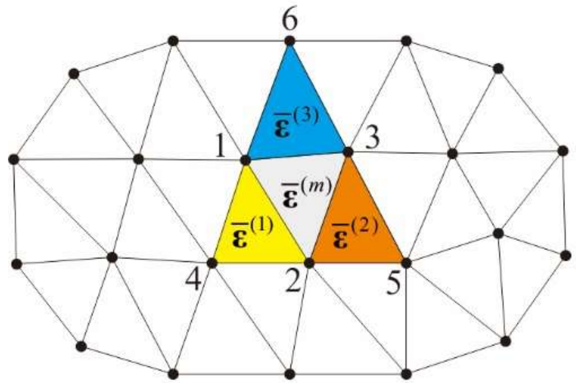
- Utilize the strains of elements adjacent to edge of target element.
- Special smoothing domain is not necessary.
- Linear strain fields within the strain-smoothed element.

Step 1. Obtain smoothed strains between the target and neighboring elements



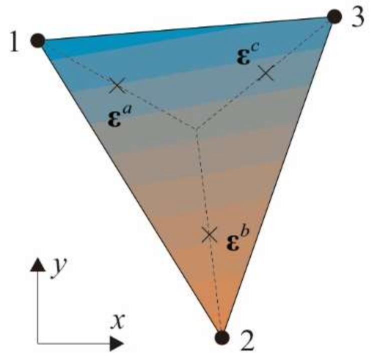
- $\hat{\boldsymbol{\varepsilon}}^{(k)} = \dfrac{1}{A^{(m)} + A^{(k)}}\left(A^{(m)}\bar{\boldsymbol{\varepsilon}}^{(m)} + A^{(k)}\bar{\boldsymbol{\varepsilon}}^{(k)}\right)$

  with $k = 1, 2, 3$

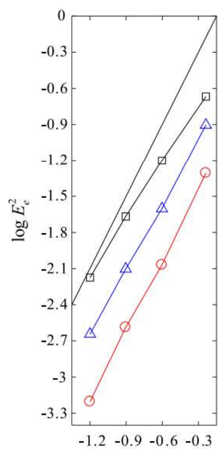


Strains of target & neighboring elements

Step 2. Construct linear strain field within the target element using smoothed strains between the target and neighboring elements

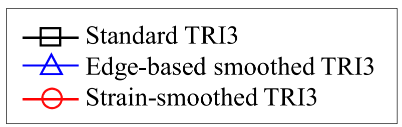


- $\boldsymbol{\varepsilon}^a = \dfrac{1}{2}\left(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(3)}\right),\ \boldsymbol{\varepsilon}^b = \dfrac{1}{2}\left(\hat{\boldsymbol{\varepsilon}}^{(1)} + \hat{\boldsymbol{\varepsilon}}^{(2)}\right),\ \boldsymbol{\varepsilon}^c = \dfrac{1}{2}\left(\hat{\boldsymbol{\varepsilon}}^{(2)} + \hat{\boldsymbol{\varepsilon}}^{(3)}\right)$
- $\tilde{\boldsymbol{\varepsilon}}^{(m)} = \left[1 - \dfrac{1}{q-p}(r + s - 2p)\right]\boldsymbol{\varepsilon}^a + \dfrac{r-p}{q-p}\boldsymbol{\varepsilon}^b + \dfrac{s-p}{q-p}\boldsymbol{\varepsilon}^c$



Convergence curve

- Cook's skew beam problem
- Improved convergence behavior



Applying strain-smoothing scheme for enriched element

❖ Strain vector of the enriched 3-node element (TRI3d-1)

- $\boldsymbol{\varepsilon}(r,s) = \bar{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\varepsilon}} = \sum_{i=1}^{3} \bar{\mathbf{B}}(r,s)\bar{\mathbf{u}}_i + \sum_{i=1}^{3} \hat{\mathbf{B}}(r,s)\hat{\mathbf{u}}_i$ with $\bar{\mathbf{u}}_i = \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \end{bmatrix}$, $\hat{\mathbf{u}}_i = \begin{bmatrix} \hat{\mathbf{u}}_i^u \\ \hat{\mathbf{u}}_i^v \end{bmatrix}$

Standard term

Enrichment term

- $\bar{\mathbf{u}}_i$: standard DOFs vector of node i
- $\hat{\mathbf{u}}_i$: enriched DOFs vector of node i
- $\bar{\mathbf{B}}$: strain-displacement matrix of standard element
- $\hat{\mathbf{B}}$: strain-displacement matrix of enriched element corresponding to $\hat{\mathbf{u}}_i$

Applying strain-smoothing scheme for strain vector corresponding to standard DOFs, $\hat{\mathbf{u}}_i$

❖ Strain vector of the enriched strain-smoothed 3-node element (SS-TRI3-d1)

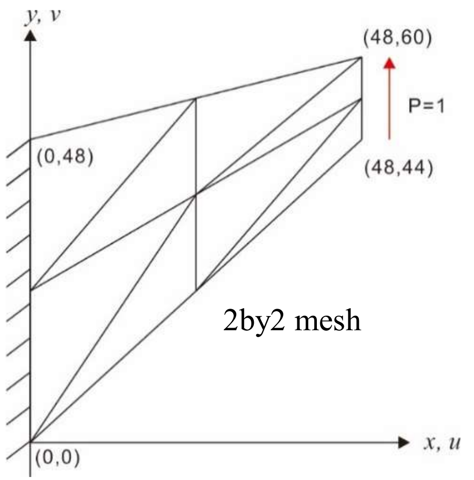
- $\boldsymbol{\varepsilon}(r,s) = \tilde{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\varepsilon}} = \sum_{i=1}^{m} \tilde{\mathbf{B}}(r,s)\bar{\mathbf{u}}_i + \sum_{i=1}^{3} \hat{\mathbf{B}}(r,s)\hat{\mathbf{u}}_i$

- $\tilde{\mathbf{B}}$: strain-displacement matrix of strain-smoothed element
- m : # of nodes belonging to the target and neighboring elements

Convergence

❖ Cook's skew beam problem

- Problem description

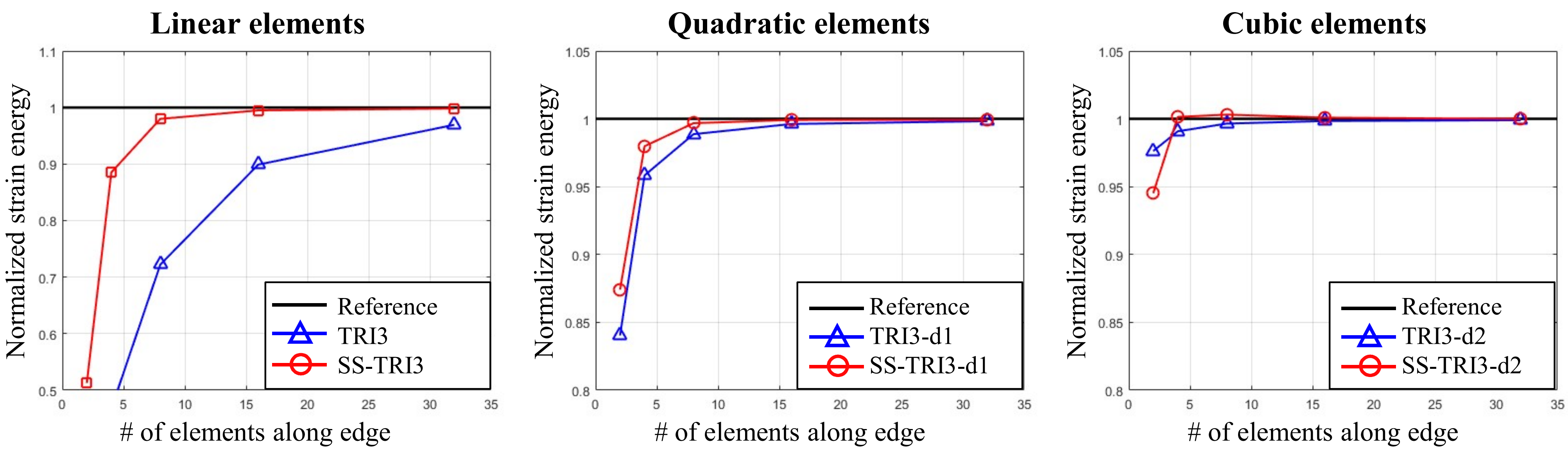


> **Linear elements**
- Standard 3-node element (**TRI3**)
- Strain-smoothed 3-node element (**SS-TRI3**)

> **Quadratic elements**
- Enriched 3-node element by linear covers (**TRI3-d1**)
- Enriched strain-smoothed 3-node element by linear covers (**SS-TRI3d-1**)

> **Cubic elements**
- Enriched 3-node element by quadratic covers (**TRI3-d2**)
- Enriched strain-smoothed 3-node element by quadratic covers (**SS-TRI3d-2**)

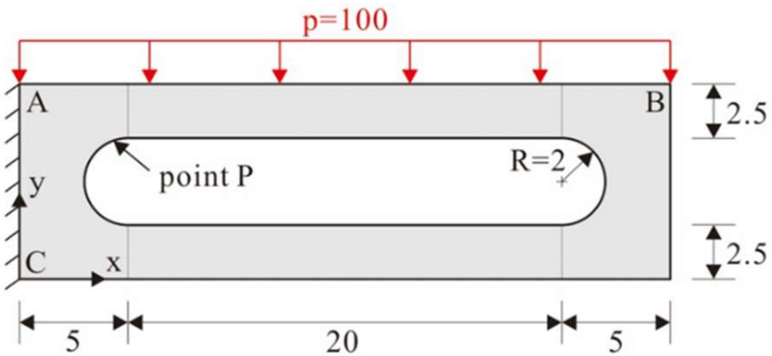
- Convergence behaviors



Adaptive use of cover functions

❖ Tool jig problem

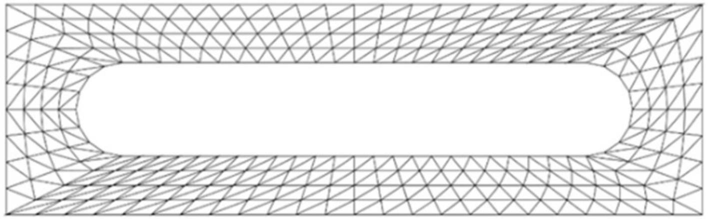
- Problem description



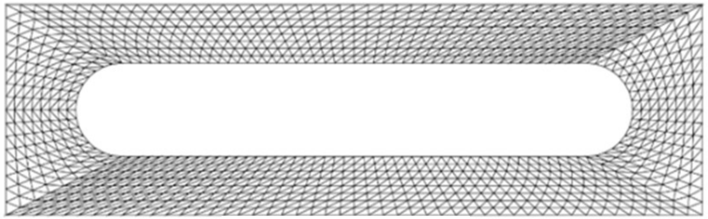
- Four different meshes are considered.
- Apply the automatic procedure.
- Compare results of TRI3 and SS-TRI3.

- Meshes

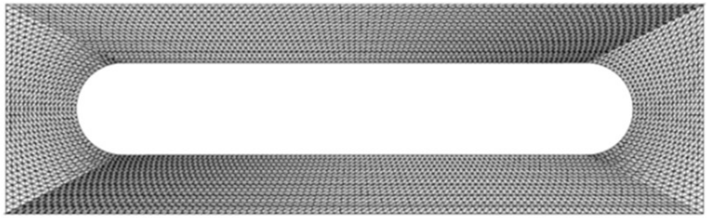
Mesh 1
(128 elements)



Mesh 2
(512 elements)



Mesh 3
(2048 elements)

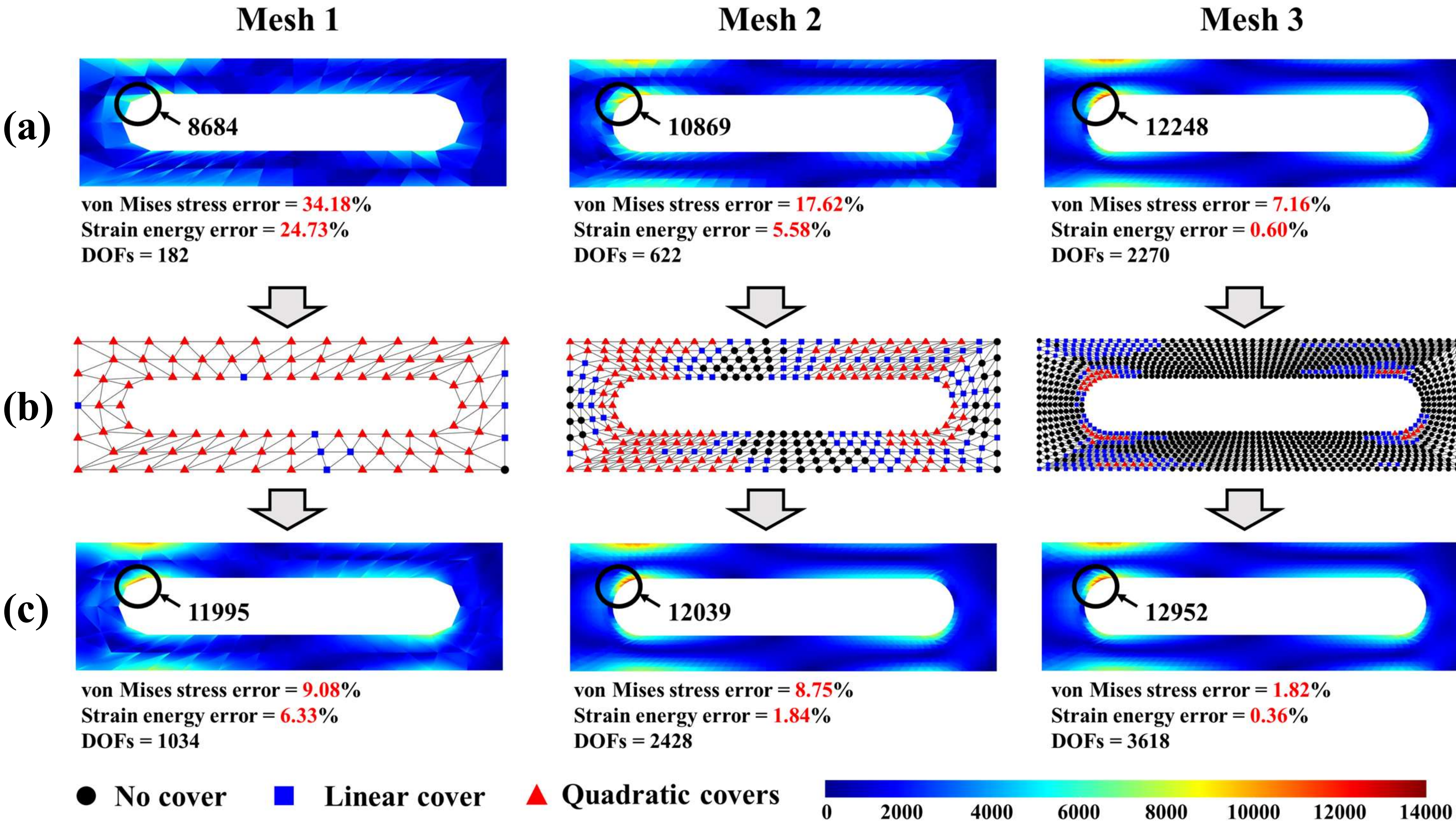


Mesh 4
(8192 elements)

Adaptive use of cover functions

❖ 3-node triangular element

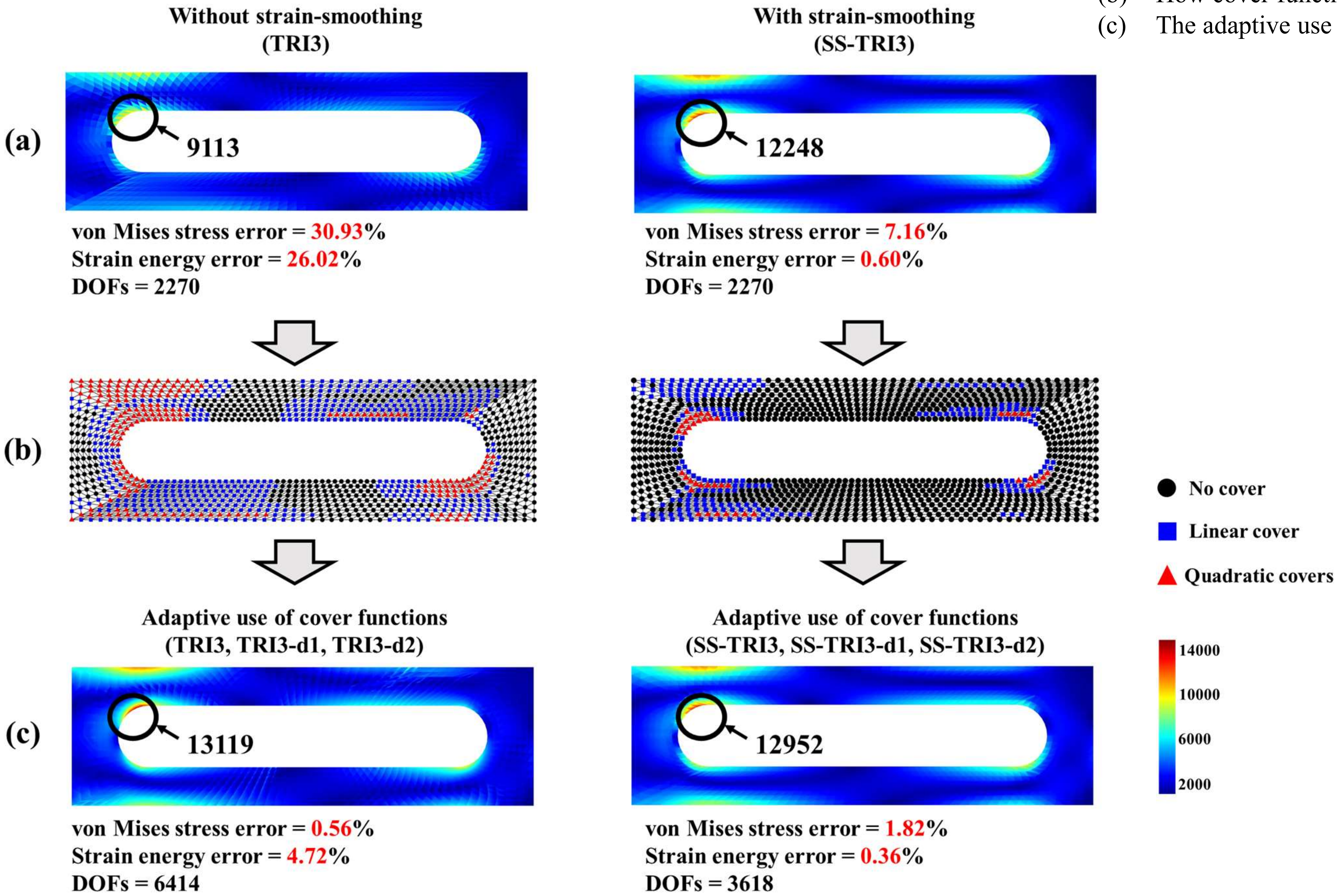
(a) Results of the strain-smoothed 3-node triangular element
(b) How cover functions are applied
(c) Results of the adaptive use of cover functions



Adaptive use of cover functions

❖ Mesh 3

(a) TRI3 & SS-TRI3 elements
(b) How cover functions are applied
(c) The adaptive use of cover functions



Adaptive use of cover functions

❖ Mesh 4

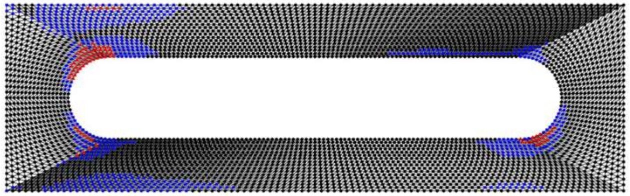
(a) TRI3 & SS-TRI3 elements
(b) How cover functions are applied
(c) The adaptive use of cover functions

Without strain-smoothing (TRI3)

(a)

von Mises stress error = 13.81%
Strain energy error = 10.15%
DOFs = 8638

(b)



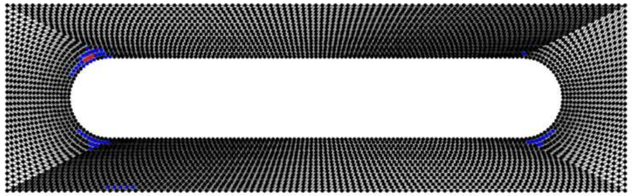
Adaptive use of cover functions (TRI3, TRI3-d1, TRI3-d2)

(c)

von Mises stress error = 2.88%
Strain energy error = 7.37%
DOFs = 11928

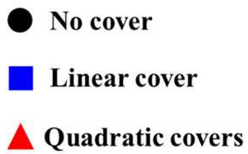
With strain-smoothing (SS-TRI3)

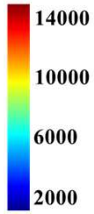
von Mises stress error = 3.99%
Strain energy error = 0.01%
DOFs = 8638



Adaptive use of cover functions (SS-TRI3, SS-TRI3-d1, SS-TRI3-d2)

von Mises stress error = 2.92%
Strain energy error = 0.06%
DOFs = 8872





Closure (Topic 3)

- **Enriched strain-smoothed 3-node triangular element** is proposed.
- The strain-smoothed method is applied for strain part corresponding to the standard DOFs.
- The enriched strain-smoothed element shows good convergence behavior.
- **When the automatic procedure is applied, the strain-smoothed element uses smaller degrees of freedom** than the standard element and **provides sufficiently accurate solution.**
- The development of enriched strain-smoothed 4-node tetrahedral element is possible in a similar way.

4. Conclusions & Future works

Conclusions & Future works

❖ New enriched finite elements

- Topic 1. Development of the enriched 2D and 3D solid finite elements
  - ✓ The new enriched elements free from the linear dependence problem are presented.
  - ✓ The LD problem is avoided in a simple and effective way.
  - ✓ The new enriched elements pass basic tests and show good convergence behaviors.
  - ✓ Development of enriched plate & shell free from the linear dependence problem.
  - ✓ Nonlinear analysis.

❖ Adaptive use of cover functions

- Topic 2. Automatic procedure to improve FE solutions
  - ✓ Feasibility of automatically improving FE solutions is shown.
  - ✓ Consideration of large FE models and 3D problems to verify and improve the automatic procedure.

❖ Enrichment scheme for smoothed finite element

- Topic 3. Enriching strain-smoothed 3-node solid element
  - ✓ Strain-smoothing method is applied for constant strain part of the enriched 3-node element.
  - ✓ When the automatic procedure is applied, the strain-smoothed element uses smaller degrees of freedom than the standard element and provides sufficiently accurate solution.
  - ✓ Development of enriched strain-smoothed 4-node tetrahedral element.

경청해 주셔서 감사드립니다.